

WEAK POSITIVITY AND THE ADDITIVITY OF THE KODAIRA  
DIMENSION, II : THE LOCAL TORELLI MAP

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Let  $f: V \longrightarrow W$  be a fibre space of projective varieties, defined over the field of complex numbers  $\mathbb{C}$ . That means that  $f$  is a surjective morphism between projective varieties with connected general fibre  $V_w = V \times_W \text{Spec}(\overline{\mathbb{C}}(\overline{w}))$ . The canonical sheaves of  $V$  and  $W$  are denoted by  $\omega_V$  and  $\omega_W$ , and  $\omega_{V/W} = \omega_V \otimes f^* \omega_W^{-1}$ .

In [13] and [15] we started to study the "positivity" of the sheaves  $f_* \omega_{V/W}^\mu$ , building up on Y. Kawamata's results [5] and using several algebraic constructions. Most of the results of [15] are proved for all fibre spaces. Just in § 8 we had to restrict ourselves to the special case, that  $V_w$  is a curve or surface of general type, in order to apply methods coming from "Geometric Invariant Theory".

The aim of this article is to replace the arguments of [15], § 8, by a careful study of local Torelli maps for cyclic coverings of varieties of general type. This allows us to obtain the following generalization of [15], Theorem I :

Theorem I. Let  $f: V \longrightarrow W$  be a fibre space of projective varieties. Assume that the Kodaira dimension  $\kappa(V_w) = \dim(V_w) > 1$  and that for some birational model  $V'_w$  of  $V_w$  and some  $\mu > 0$  the sheaf  $\omega_{V'_w}^\mu$  is generated by its global sections. Then, if  $\kappa(W) \neq -\infty$ , we have  $\kappa(V) \geq \text{Max}\{\kappa(W) + \kappa(V_w), \text{Var}(f) + \kappa(V_w)\}$ .

So, the generalized Iitaka conjecture  $C_{n,m}^+$  is true for this type of fibre spaces. Recall, that  $\text{Var}(f)$  is defined to be the minimal num-

ber  $k$ , such that there exists a subfield  $L$  of  $\overline{\mathbb{C}(W)}$  of transcendental degree  $k$  over  $\mathbb{C}$  and a variety  $F$  over  $L$  with

$$F \times_{\text{Spec}(L)} \text{Spec}(\overline{\mathbb{C}(W)}) \sim V_W \quad (\sim \text{ means birational }).$$

We will see in (3.8) ( or [15] ) that Theorem I follows from:

**Theorem II.** Let  $f: V \longrightarrow W$  be a fibre space of projective varieties,  $\text{Var}(f) = \dim(W)$ . Assume that  $\kappa(V_W) = \dim(V_W) > 1$  and that for some birational model  $V'_W$  of  $V_W$  and some  $\mu > 0$  the sheaf  $\omega_{V'_W}^H$  is generated by its global sections. Then, for all  $\eta$  of the form  $\eta = a \cdot b \cdot \mu + a + 1$ ,  $b > 1$ ,  $b \cdot \mu \geq a > 0$ ,

- i)  $\kappa(\det(f_* \omega_{V'_W}^\eta)) = \dim(W)$
- ii) There exists an ample invertible sheaf  $H$ , some  $\nu \gg 0$  and an inclusion  $\bigoplus H \longrightarrow \hat{S}^\nu(f_* \omega_{V'_W}^\eta)$ , isomorphic over an open subvariety.

In fact, since the assumptions made in theorem II are compatible with replacing  $W$  by  $W'$ , generically finite over  $W$ , and  $V$  by any desingularization  $V'$  of  $V \times_W W'$ , we see from (3.5) and (3.6) that i) and ii) are equivalent. In (3.7) we discuss the statement of Theorem II and other cases for which it is true.

In (4.5) we use the assumption that  $V_W$  is of general type. This is just done to keep the argument as simple as possible and it would be possible to get along without (4.5) by modifying the construction in § 4. We have to use it again, however, together with the "nasty" condition, that  $\omega_{V'_W}^H$  is generated by its global sections, in order to apply (1.5). There we study the local Torelli map for finite coverings, and we do not know at the moment, how to obtain similar results for a wider class of fibre spaces.

Nevertheless, we hope that the methods developed in [15] and this article are a good approach toward the solution of S. Iitaka's conjecture. Also it seems, that behind theorem II there are hidden relations

between (global) moduli theory, local Torelli maps and weak positivity of direct images of powers of dualizing sheaves, which one should try to understand in a better way.

We use the definitions and notations introduced in [15]. The basic facts about the Kodaira dimension and the classification theory can be found in [11], [5] and [12]. A survey of the known results is given in [1].

§ 1 contains just an application of the Lieberman-Peters-Wilsker spectral sequence ( see [9] ), combined with some easy calculations of the behaviour of the tangent sheaf under a cyclic covering. In § 2 we consider the period map and the Kodaira-Spencer map in order to compare the variation of a certain fibre space with the dimension of the images under these maps. We use freely ideas due to Y. Kawamata, [6]. In the next section we just recall the main results from [15] in a slightly modified way, and § 4 finally contains the proof of theorem II.

The author would like to thank Mr. Taniguchi and the foundation bearing his name for promoting the symposium at Katata and for the invitation to Japan. The results described in this paper were presented during this symposium, results, which build up on the work of Japanese mathematicians, especially S. Iitaka, T. Fujita, Y. Kawamata and K. Ueno.

#### § 1 Cyclic coverings and the local Torelli map

Let  $X$  be a non singular variety of dimension  $k$  and  $L$  any invertible sheaf. Assume that  $L^N$  has a global section  $s$  for some  $N > 0$ . We write the zero-divisor of  $s$  in the form

$$D = B + \sum_{j \geq 2} \nu_j E_j,$$

where  $B$  is the reduced part of  $D$  ( the sum over all components occurring with multiplicity one ), and  $E_j$  the other irreducible components. We assume that all components of  $D$  are non singular varieties, intersecting each other transversally, or - as we are going to say shortly:  $D$  is a normal crossing divisor.

The section  $s$  defines a  $\mathcal{O}_X$ -algebra structure on  $A = \bigoplus L^{-i}$  where the sum is taken over  $i = 0, \dots, N-1$ . Choose  $\tau: Y \rightarrow X$  to be any desingularisation of  $\text{Spec } \mathcal{O}_X(A)$  such that  $\tau^{-1}(D)$  is a normal crossing divisor too. We say that  $Y$  is obtained by taking the  $N$ -th root out of  $s$ . In [14] and [15] it is shown that  $\tau_*\omega_Y = \bigoplus (\omega_X \otimes L^{(i)})$ , where  $L^{(i)}$  is an invertible subsheaf of  $L^i$ , described there.

**Lemma 1.1.** There exists a natural inclusion

$$\tau_*\omega_Y^{\mu} \longrightarrow \bigoplus (\omega_X^{\mu} \otimes \mathcal{O}_X(L^{\mu \cdot N - \mu - N + 1 + i}))$$

( where the sum is again taken over  $i = 0, \dots, N-1$  ) being an isomorphism outside of the singular locus of  $D$ .

**Proof:** The left hand side is torsion free, the right hand side locally free and hence we may assume that  $Y$  is finite over  $X$ . Using the projection formula, it is enough to obtain an inclusion  $\omega_{Y/X} \rightarrow \tau^*L^{N-1}$  isomorphic over the regular locus of  $D$ . In fact, the sheaf  $L^{(i)}$  mentioned above, is isomorphic to  $L^i$  on the regular locus of  $D$ . In order to obtain this inclusion we might use "duality for finite morphisms" as in [15], (3.3), or:

$$\tau^{-1}(D) = N \cdot B' + \sum v_j^1 \cdot N \cdot E_j^1 \quad \text{and} \quad L^{N-1} = \mathcal{O}_Y \left( \frac{N-1}{N} \cdot \tau^{-1}(D) \right)$$

where  $B' = (\tau^{-1}(B))_{\text{red}}$  and  $E_j^1 = (\tau^{-1}(E_j))_{\text{red}}$ . By ramification theory

$$\omega_{Y/X} = \mathcal{O}_Y(\tau^{-1}(B + \sum E_j) - B' - \sum E_j^1).$$

Comparing the multiplicities we get (1.1).

(1.2) From now on we restrict ourselves to the case, that  $D$  is non singular ( especially reduced ). We might choose  $Y = \text{Spec } \mathcal{O}_X(A)$ .

**Lemma 1.3.**

- i)  $(\tau^{-1}(D))_{\text{red}} = D'$  is non singular and  $\tau^*\Omega_X^p\langle D \rangle = \Omega_Y^p\langle D' \rangle$ , where  $\Omega_X^p\langle D \rangle$  denotes the sheaf of differential forms with logarithmic poles along  $D$ .
- ii)  $\tau_*\Omega_Y^p = \Omega_X^p \oplus \bigoplus_{i=1}^{N-1} \Omega_X^p\langle D \rangle \otimes L^{-i}$ . Especially, if  $G = \langle \sigma \rangle$  is the Galois-group of  $Y$  over  $X$ , acting on  $\Omega_Y^p$  in the natural way (i.e.:  $\sigma(f \cdot dx_1 \wedge \dots \wedge dx_p) = \sigma f \cdot d(\sigma x_1) \wedge \dots \wedge d(\sigma x_p)$ ), we have  $(\tau_*\Omega_Y^p)^G = \Omega_X^p$ .
- iii) If  $T_Y = (\Omega_Y^1)^{\vee}$  is the tangent-sheaf and  $T_X\langle -D \rangle = (\Omega_X^1\langle D \rangle)^{\vee}$ , then  $(\tau_*T_Y)^G = T_X\langle -D \rangle$ .

**Proof:** i) and ii) are just a simple exercise and proved in [14], § 1, in a much more general situation. Especially from i) we obtain for  $p = k$  again that  $\omega_{Y/X} = \tau^*L^{N-1}$ . The automorphism  $\sigma$  acts on  $\tau_*\Omega_Y^1\langle D' \rangle$  the same way as  $\text{id} \times \sigma$  on  $\Omega_X^1\langle D \rangle \otimes \tau_*\mathcal{O}_Y$ . Hence  $(\tau_*T_Y\langle -D' \rangle)^G = (\tau_*(\Omega_Y^1\langle D' \rangle)^{\vee})^G = (\text{Hom}(\Omega_X^1\langle D \rangle, \tau_*\mathcal{O}_Y))^G = T_X\langle -D \rangle$ .

The inclusion of  $\Omega_Y^1$  in  $\Omega_Y^1\langle D' \rangle$  induces  $T_Y\langle -D' \rangle \rightarrow T_Y$ , compatible with the  $G$ -action. The description  $F^{\vee} = \text{Hom}(\omega_{Y/X} \otimes F, \omega_{Y/X})$  and duality theory for finite morphisms give

$$\tau_*T_Y\langle -D' \rangle = \text{Hom}(L^{N-1} \otimes \tau_*\Omega_Y^1\langle D' \rangle, \mathcal{O}_X) = \bigoplus_{i=0}^{N-1} T_X\langle -D \rangle \otimes L^{-N+1+i} \quad \text{and}$$

$$\tau_*T_Y = \text{Hom}(L^{N-1} \otimes \tau_*\Omega_Y^1, \mathcal{O}_X) = T_X \otimes L^{-N+1} \oplus \bigoplus_{i=1}^{N-1} T_X\langle -D \rangle \otimes L^{-N+1+i}.$$

Hence the inclusion  $T_X\langle -D \rangle \rightarrow \tau_*T_Y$  splits and  $T_X\langle -D \rangle$  must be the  $G$ -invariant part.

(1.4) In addition to (1.2) we assume from now on that  $X$  and  $Y$  are projective of dimension  $k > 1$ . Recall that a problem of local Torelli type consists in studying the cup-product map

$$\lambda: H^1(Y, \mathcal{T}_Y) \longrightarrow \text{Hom}(H^0(Y, \Omega_Y^k), H^1(Y, \Omega_Y^{k-1})) .$$

Of course  $\Omega_Y^k$  and  $\omega_Y$  just denote the same sheaf. The image of

$H^1(Y, \mathcal{T}_Y)^G = H^1(X, \mathcal{T}_X \langle -D \rangle)$  under  $\lambda$  is contained in

$$\text{Hom}(H^0(Y, \Omega_Y^k), H^1(Y, \Omega_Y^{k-1}))^G = \text{Hom}(H^0(X, \Omega_X^k), H^1(X, \Omega_X^{k-1})) \oplus \bigoplus_{i=1}^{N-1} \text{Hom}(H^0(X, \Omega_X^k \langle D \rangle \otimes L^{-i}), H^1(X, \Omega_X^{k-1} \langle D \rangle \otimes L^{-i})) .$$

Hence  $\lambda^G = \lambda|_{H^1(Y, \mathcal{T}_Y)^G}$  can be described as direkt sum of  $\{\lambda_i\}_{i=0, \dots, N-1}$

$$\text{where } \lambda_0: H^1(X, \mathcal{T}_X \langle -D \rangle) \longrightarrow \text{Hom}(H^0(X, \Omega_X^k), H^1(X, \Omega_X^{k-1})) \quad \text{and}$$

$$\lambda_i: H^1(X, \mathcal{T}_X \langle -D \rangle) \longrightarrow \text{Hom}(H^0(X, \Omega_X^k \langle D \rangle \otimes L^{-i}), H^1(X, \Omega_X^{k-1} \langle D \rangle \otimes L^{-i}))$$

are the cup-product maps. The main result in this section is:

**Theorem 1.5.** Using the assumptions and notations introduced above, we

assume that for some  $i \in \{1, \dots, N-1\}$  we have

$\kappa(X, \omega_X^{-1} \otimes L^{2i-N}) = \dim(X) = k$  and that  $L^{-i} \otimes \omega_X^k \langle D \rangle = \omega_X \otimes L^{N-i}$  is generated by its global sections. Then  $\lambda_i$  and hence  $\lambda^G$  are injective.

**Proof:** We just have to apply the spectral-sequence described in [9], § 2.

Using the notations introduced there, we take

$$W = \Omega_X^k \langle D \rangle \otimes L^{-i} = \omega_X \otimes L^{N-i}, \quad F = \mathcal{T}_X \langle -D \rangle \quad \text{and} \quad V = H^0(X, \Omega_X^k \langle D \rangle \otimes L^{-i}) .$$

The spectral-sequence of hypercohomology

$$H_{Koz WV}^p(H^q(X, \mathcal{T}_X \langle -D \rangle)) \Rightarrow \underline{H}^{p+q}$$

induces the five-term exact sequence

$$0 \longrightarrow H_{Koz WV}^1(H^0(X, \mathcal{T}_X \langle -D \rangle)) \longrightarrow \underline{H}^1 \longrightarrow H_{Koz WV}^0(H^1(X, \mathcal{T}_X \langle -D \rangle)) \longrightarrow \longrightarrow H_{Koz WV}^2(H^0(X, \mathcal{T}_X \langle -D \rangle)) \longrightarrow \underline{H}^2 .$$

From our second assumption and [9], 2.7, we get  $\underline{H}^1 = 0$  ( and in fact  $\underline{H}^2 = 0$  ). Moreover, since  $\mathcal{T}_X \langle -D \rangle \otimes \Omega_X^k \langle D \rangle \otimes L^{-i} = \Omega_X^{k-1} \langle D \rangle \otimes L^{-i}$  we obtain

$$H_{Koz WV}^0(H^1(X, \mathcal{T}_X \langle -D \rangle)) = \text{Ker}(H^1(X, \mathcal{T}_X \langle -D \rangle) \longrightarrow H^1(X, \Omega_X^{k-1} \langle D \rangle \otimes L^{-i}) \otimes V^V) =$$

$= \text{Ker}(\lambda_i)$  and  $H_{Koz WV}^2(H^0(X, \mathcal{T}_X \langle -D \rangle)) = k^2 / I^2$  where  $k^2$  is contained in  $H^0(X, \mathcal{T}_X \langle -D \rangle \otimes \omega_X^2 \otimes L^{2N-2i}) \otimes \wedge^2 V^V$ . If  $\text{Ker}(\lambda_i)$  was not zero, we would obtain an inclusion  $\omega_X^{-1} \otimes L^{2i-N} \longrightarrow \mathcal{T}_X \langle -D \rangle \otimes \omega_X \otimes L^N = \Omega_X^{k-1} \langle D \rangle$ .

Using our first assumption, we would obtain an invertible subsheaf of  $\Omega_X^{k-1}$  of L-dimension  $k$ . This is not possible by the vanishing theorem of F. Bogomolov ( see for example [14], § 2 ).

**Remark 1.6.** In § 4 we are going to apply (1.5) in the following situation:

Let  $X$  be a projective variety, such that  $\kappa(X) = \dim(X)$ , and such that for some  $\mu > 0$  the sheaf  $\omega_X^\mu$  is generated by its global sections.

Let  $N \geq 2 \cdot \mu$  and  $\omega_X^N = \mathcal{O}_X(D)$  for a non singular divisor  $D$ . For  $L = \omega_X$  and  $i = N - \mu + 1$  the assumptions of (1.5) are verified. In fact,  $\omega_X^{-1} \otimes L^{2i-N} = \omega_X^{N-2\mu+1}$ ,  $N-2\mu+1 > 0$ , and  $L^{N-i} \otimes \omega_X = \omega_X^\mu$ .

(1.7) In order to understand the geometric meaning of (1.5) we consider quasi-projective varieties and sheaves:  $X_0, L_0, L_0^N = \mathcal{O}_{X_0}(D_0)$  and  $Y_0$  as in (1.2). In addition we assume that there exists a non singular quasi-projective variety  $S_0$  and a projective smooth morphism  $g_0: X_0 \longrightarrow S_0$  with connected general fibre, such that  $g_0|_{D_0}$  is also smooth. Hence  $h_0 = g_0 \cdot \tau_0: Y_0 \longrightarrow S_0$  is again a smooth family of projective varieties.

Let  $s \in S_0$  be a closed point and  $X_s, D_s, Y_s$  the fibres over  $s$ . Let  $t_{S_0, s} = t_s$  be the Zariski tangent space and

$$\rho_s: t_s \longrightarrow H^1(Y_s, \mathcal{T}_{Y_s})$$

the Kodaira-Spencer map, corresponding to the deformation of  $Y_s$ .

The deformation  $X_{0-D_0}$  of the "open" variety  $X_s - D_s$  defines in the same way (see [7])  $\rho'_s: t_s \longrightarrow H^1(X_s, \mathcal{T}_{X_s \langle -D_s \rangle})$ .

Lemma 1.8. The two maps  $\rho_s$  and  $\rho'_s$  coincide if we identify  $H^1(X_s, \mathcal{T}_{X_s}(-D_s))$  with the subspace  $H^1(Y_s, \mathcal{T}_{Y_s})^G$  of  $H^1(Y_s, \mathcal{T}_{Y_s})$ .

Proof: An element of  $\mathcal{t}_s$  is a morphism  $\phi: \text{Spec}(\mathbb{C}[\epsilon]) \rightarrow S_0$ . By base change we obtain infinitesimal extensions of  $X_{s-D_s}$ , of  $Y_s$  and of  $\tau_0|_{Y_s}$ . If  $\xi \in H^1(X_s, \mathcal{T}_{X_s}(-D_s))$  and  $\zeta \in H^1(Y_s, \mathcal{T}_{Y_s})$  denote the corresponding cohomology classes, then  $\tau_s^* \xi = \zeta$  and  $\zeta$  is invariant under  $G$ .

Corollary 1.9. Using the notations introduced in (1.7) we assume that  $X_s, D_s, L_s = L_0|_{X_s}$  fulfill the assumptions made in (1.5). Let  $\lambda_s: H^1(Y_s, \mathcal{T}_{Y_s}) \rightarrow \text{Hom}(H^0(Y_s, \Omega_{Y_s}^k), H^1(Y_s, \Omega_{Y_s}^{k-1}))$  denote the local Torelli map. Then  $\rho_s(\mathcal{t}_s) \cap \text{Ker}(\lambda_s) = 0$ .

§ 2 The period map

In order to interpret (1.9) in terms of "positivity-properties", we have to consider the period map. Let  $h_0: Y_0 \rightarrow S_0$  be any smooth projective morphism of quasi-projective varieties with connected general fibre,  $n = \dim(Y_0)$ ,  $m = \dim(S_0)$  and  $k = n - m$ . For a closed point  $s \in S_0$  and  $Y_s = h_0^{-1}(s)$  we denote again the Kodaira-Spencer map by  $\rho_s$  and the local Torelli map by  $\lambda_s$ .

Lemma 2.1. Assume for all  $s \in S_0$  in sufficiently general position we have  $\rho_s(\mathcal{t}_s) \cap \text{Ker}(\lambda_s) = 0$ . Then there exist an étale open set  $\pi: U \rightarrow S_0$  (i.e.:  $U$  is étale over the open subvariety  $\pi(U)$ ) and dominant morphisms  $\sigma: U \rightarrow Z$  and  $\phi: F \rightarrow Z$ , such that:

- i)  $U \times_{S_0} Y_0 \cong U \times_Z F$ , compatible with the projections to  $U$ .
- ii) For  $u \in U$ , in  $\mathcal{t}_{U,u} = \pi^* \mathcal{t}_{S_0, \pi(u)}$  the subspaces  $\mathcal{t}_{\sigma^{-1}(\sigma(u)), u}$  and

$\pi^* \text{Ker}(\rho_{\pi(u)})$  coincide.

Proof: Let  $S$  be a projective variety, containing  $S_0$  as an open subvariety, such that  $S-S_0$  is a normal crossing divisor. The sheaf  $R^k h_{0*} \mathcal{C}_{Y_0}$  carries a variation of the Hodge-structure. As in [2], § 9, we take  $\mathcal{D}$  to be the corresponding period domain and  $\Gamma$  to be the image of  $\Pi_1(S)$  in  $\text{Aut}(\mathcal{D})$ . By (9.3) and (9.10) of [2] we have the period map  $p: S_0 \rightarrow \mathcal{D}/\Gamma$ . There exists a subvariety  $\tilde{S}$  in  $S$ , such that  $p$  can be extended to a proper analytic morphism  $\tilde{p}: \tilde{S} \rightarrow \mathcal{D}/\Gamma$  (loc. cit.).

From [6], § 3, or [10], we know that  $\tilde{p}: \tilde{S} \rightarrow \tilde{p}(\tilde{S})$  is bimeromorphically equivalent to a rational map  $p': S_0 \rightarrow Z$  between algebraic non singular varieties. Making  $S_0$  smaller, if necessary, we may assume that  $p'$  is a morphism, dominant, and that  $Z$  is bimeromorphic to  $p(S_0)$ .

Claim 2.2. For  $s$  in sufficiently general position in  $S_0$

$$\mathcal{t}_{p'^{-1}(p'(s)), s} = \text{Ker}(\rho_s) .$$

Proof: By our assumption  $\text{Ker}(\rho_s) = \text{Ker}(\lambda_s \cdot \rho_s)$ . The tangent map from  $\mathcal{t}_{S_0, s}$  in the tangent space of  $\mathcal{D}/\Gamma$  is given by the cup-product with the images under  $\rho_s$  in  $H^1(Y_s, \mathcal{T}_{Y_s})$ . Hence  $\text{Ker}(\rho_s)$  must be contained in  $\mathcal{t}_{p'^{-1}(p'(s)), s}$ . On the other hand, if  $\xi \in \mathcal{t}_{p'^{-1}(p'(s)), s}$ , the cup-product with  $\rho_s(\xi)$  is trivial on the whole cohomology of  $Y_s$ , especially on  $H^0(Y_s, \Omega_{Y_s}^k)$ .

The statements of (2.1) are compatible with replacing  $S_0$  by any étale open set and  $Y_0$  by the fibre-product. Therefore we may assume:

- $\dim(H^1(Y_s, \mathcal{T}_{Y_s}))$  is constant for all  $s \in S_0$ .
- (2.2) is true for all  $s \in S_0$ .

-  $p': S_0 \longrightarrow Z$  has a section  $i: Z \longrightarrow S_0$ .

Define  $\phi = pr_2: F = Y_0 \times_{S_0} Z \longrightarrow Z$ , where the fibre-product is taken with respect to  $i$ . The functor  $\text{Isom}_{S_0}(Y_0, F \times_Z S_0)$  is represented by an algebraic scheme  $j: I \longrightarrow S_0$  (see [3]), where the fibre-product is taken with respect to  $\phi$ .

Claim 2.3.  $j$  is surjective.

Proof: For all points  $s$  of  $S_0$  we assumed that  $\rho_s$  restricted to  $t_{p'^{-1}(p'(s)), s}$  is trivial. From [8], Theorem 6.2, we know that for all  $x$  in  $p'^{-1}(p'(s))$  the fibres  $Y_x = h_0^{-1}(x)$  are isomorphic. Hence  $j^{-1}(s) = \text{Isom}_{S_0}(Y_s, \phi^{-1}(p'(s))) \neq \emptyset$ .

Now we may choose  $\pi: U \longrightarrow S_0$  such that  $I \times_{S_0} U$  has a section and  $\sigma = p' \cdot \pi$ . Now i) of (2.1) is verified by definition of  $I$  and ii) follows from (2.2).

(2.4) Let  $h_0: Y_0 \longrightarrow S_0$  be as above and  $h: Y \longrightarrow S$  a projective fibre space, such that  $Y_0$  and  $S_0$  are open subvarieties of  $Y$  and  $S$  respectively and  $h_0 = h|_{Y_0}$ .

Lemma 2.5. If for one point  $s$  in  $S_0$  the map  $\lambda_s \cdot \rho_s$  is injective, then  $\kappa(\det(h_* \omega_{Y/S})) = \dim(S)$ .

Proof: (see also [6])

The assumption and the statement of (2.5) are compatible with blowing up  $Y$  and also with blowing up  $S$ . In fact, if  $h = \delta \cdot h'$  for some fibre space  $h': Y \longrightarrow S'$  and a sequence of blowing up's  $\delta: S' \longrightarrow S$ , then  $\delta_* \det(h'_* \omega_{Y/S'})$  and  $\det(h_* \omega_{Y/S})$  coincide outside of a closed subvariety of  $S$  of codimension bigger than or equal to two.

Therefore we may assume  $S-S_0$  and  $Y-Y_0$  to be normal crossing divisors. In [15], 3.2, we have seen that we may replace  $S$  by any non singular finite covering  $S' \longrightarrow S$  and  $Y$  by any desingularization of  $Y \times_S S'$ . Using the "covering lemma" of Kawamata (see [5], Theorem 17, or [13], 2.1) we are able to "kill" the semisimple part of the monodromy around the components of  $S-S_0$ . Hence we may assume that the local monodromies of  $R^k h_{0*} \omega_{Y_0}$  around the components of  $S-S_0$  are unipotent.

Under these assumptions Y. Kawamata proved in [5], Theorem 5, that  $h_* \omega_{Y/S}$  is locally free and semi-positive, and therefore  $\det(h_* \omega_{Y/S})$  is an arithmetically positive invertible sheaf (i.e.: for every curve  $C$  in  $S$  the intersection number  $C \cdot c_1(\det(h_* \omega_{Y/S})) = 0$ ). From [14], 3.2, or [6] we know that, in order to obtain (2.5), it is enough to show the following Lemma, due to Fujita and Kawamata:

Lemma 2.6. Assume that for some point  $s$  in  $S_0$  the map  $\lambda_s \cdot \rho_s$  is injective, then  $c_1(\det(h_* \omega_{Y/S}))^m > 0$ .

Sketch of the proof (See [6], Theorem 3, or [5], Lemma 21):

Let  $h$  be the Hermitian metric on  $h_{0*} \omega_{Y_0/S_0}$ , induced by the intersection form of the Hodge-structure. For any tubular neighbourhood  $U_\epsilon$  of  $S-S_0$  of radius  $\epsilon > 0$  we can extend  $h|_{S-U_\epsilon}$  to a  $C^\infty$  Hermitian metric  $h_\epsilon$  of  $h_* \omega_{Y/S}$ . Let  $\theta_\epsilon$  and  $\theta$  be the curvature forms corresponding to  $h_\epsilon$  and  $h$  respectively. Then

$$\begin{aligned} c_1(\det(h_* \omega_{Y/S}))^m &= \int_S ((\sqrt{-1}/2\pi) \text{Tr} \theta_\epsilon)^m = \\ &= \int_{U_\epsilon} ((\sqrt{-1}/2\pi) \text{Tr} \theta_\epsilon)^m + \int_{S-U_\epsilon} ((\sqrt{-1}/2\pi) \text{Tr} \theta)^m \end{aligned}$$

Taking the limit  $\epsilon \longrightarrow 0$ , the first term goes to zero since  $h$  has only logarithmic growth along  $S-S_0$ . The second term however is positive. In fact,  $(\sqrt{-1}/2\pi) \theta$  is positive semi-definite. Our assumption and [2], (2.15) and (5.2), show that  $(\sqrt{-1}/2\pi) \text{Tr} \theta$  is in  $s$  positive definite.

### § 3 Weak positivity

Let  $W$  be a non singular quasi-projective variety and  $F$  a torsion free coherent sheaf on  $W$ . Let  $i: \hat{W} \rightarrow W$  be the biggest open subvariety, such that  $i^*F$  is locally free. Recall:

**Definition 3.1.**  $F$  is weakly positive, if for every ample invertible sheaf  $H$  on  $W$  and every positive number  $\alpha$  there exists some positive number  $\beta$ , such that  $S^{\alpha \cdot \beta}(i^*F) \otimes i^*H^\beta$  is generated over an open subvariety by its global sections. That means that for  $\hat{S}^{\alpha \cdot \beta}(F) = i_*S^{\alpha \cdot \beta}(i^*F)$  we have a map  $\otimes \mathcal{O}_W \rightarrow \hat{S}^{\alpha \cdot \beta}(F) \otimes H^\beta$ , surjective in the general point of  $W$ .

Of course, we may always consider  $i_*F$  instead of  $F$  and assume that  $\hat{W} = W$ . In [15], § 1, we listed several properties of weakly positive sheaves. We need in addition:

**Lemma 3.2.** Let  $F$  and  $G$  be torsion free coherent sheaves on  $W$ .

- i) If  $F$  is weakly positive, then for every invertible ample sheaf  $H$  and  $\alpha > 0$  there exists  $\beta_0$  such that for all  $\beta > \beta_0$  the sheaf  $\hat{S}^{\alpha \cdot \beta}(F) \otimes H^\beta$  is generated over an open sub-variety by its global sections.
- ii)  $F$  is weakly positive if and only if for every non singular finite covering  $\pi: W' \rightarrow W$  and every ample invertible sheaf  $H'$  on  $W'$  the sheaf  $\pi^*F \otimes H'$  is weakly positive.
- iii) If  $F$  and  $G$  are weakly positive, then  $F \otimes G$ ,  $\det(F) = i_*\det(i^*F)$  and  $\hat{S}^\nu(F)$ , for all  $\nu > 0$ , are weakly positive.

**Proof:** We just have to copy the corresponding arguments for "ampleness". We may assume, that  $F$  and  $G$  are locally free sheaves.

- i) By definition there exists  $\beta > 0$ , such that for all  $\gamma > 0$

$S^{2 \cdot \alpha \cdot \beta \cdot \gamma}(F) \otimes H^{\beta \cdot \gamma}$  is generated by its global sections over an open subvariety.  $H$  being ample, we can find  $c > 0$  such that  $S^{\alpha \cdot 1}(F) \otimes H^{\gamma \cdot \beta + 1}$  is generated by global sections for  $\iota = 1, \dots, 2 \cdot \beta$  and  $\gamma \geq c$ . Hence  $S^{\alpha(2 \cdot \beta \cdot \gamma + 1)}(F) \otimes H^{2 \cdot \beta \cdot \gamma + 1}$  is generated by its global sections over an open subvariety for all  $\gamma > c$ . We may take  $\beta_0 = 2 \cdot c \cdot \beta$ .

- ii) If  $F$  is weakly positive,  $\pi^*S^{\alpha \cdot \beta}(F) \otimes \pi^*H^\beta$  is generated by its global sections for  $\beta \gg 0$ . If  $\alpha'$  and  $H'$  are given, we may choose  $\gamma$  such that  $\pi^*H$  is contained in  $H'^\gamma$ . For  $\alpha = \alpha' \cdot \gamma$  we get that  $S^{\alpha' \cdot \gamma \cdot \beta}(J) \otimes H'^{\gamma \cdot \beta}$  is generated by its global sections over an open subvariety for  $J = \pi^*F$ .

In order to obtain the other direction, we choose for given  $H$  and  $\alpha$  a non singular finite covering  $\pi: W' \rightarrow W$  (see for example [13], 2.1) such that  $H = H'^d$  for  $d = 1 + 2 \cdot \alpha$ . By assumption there exists  $\beta > 0$  such that  $S^{2 \cdot \alpha \cdot \beta}(\pi^*F \otimes H') \otimes H'^\beta = \pi^*S^{2 \cdot \alpha \cdot \beta}(F) \otimes \pi^*H^\beta$  is generated by its global sections over an open subvariety. Applying  $\pi_*$ , we obtain maps, surjective over an open subvariety  $\otimes \pi_*\pi^*H^\beta \rightarrow \pi_*\mathcal{O}_{W'} \otimes S^{2 \cdot \alpha \cdot \beta}(F) \otimes H^{2 \cdot \beta} \rightarrow S^{2 \cdot \alpha \cdot \beta}(F) \otimes H^{2 \cdot \beta}$ . For  $\beta \gg 0$  we find the sections wanted.

- iii) Let  $H$  be any ample invertible sheaf and  $J$  weakly positive. For all  $\nu > c$  the sheaf  $S^\nu(J \otimes H)$  is generated by its global sections over an open set and hence weakly positive (see [15], 1.4, 1). From [4], § 5, we know that every sheaf  $T(J \otimes H)$  corresponding to an irreducible representation of  $GL(\text{rank}(J \otimes H))$  of large weight is a direct summand of  $S^{\nu_1}(J \otimes H) \otimes \dots \otimes S^{\nu_t}(J \otimes H)$  for some numbers  $\nu_1 > c, \dots, \nu_t > c$ . Therefore every  $T(J \otimes H)$  of large weight is weakly positive. If  $T(J \otimes H)$  is just any sheaf given by an irreducible representation of positive weight, then  $S^1(T(J \otimes H))$  will be a direct sum of bundles of

large weights for  $\epsilon \gg 0$ . From [15], 1.4,3), we obtain the weak positivity of  $T(J \otimes H)$  for every bundle coming from an irreducible representation of positive weight, and hence for every positive representation. So for example  $S^2((F \otimes G) \otimes H) = (S^2(F) \otimes (F \otimes G) \otimes S^2(G)) \otimes H^2$  and  $(J \otimes G) \otimes H^2$  are weakly positive. From ii) we obtain the weak positivity of  $F \otimes G$ . Now the other sheaves in iii) are quotients of tensor products.

(3.3) Let  $f: V \rightarrow W$  be a fibre space of projective varieties. The following proposition is just a corollary of (3.2,iii) and [15], 5.4.

**Proposition 3.4.** Let  $H$  be an ample invertible sheaf and  $M$  any invertible sheaf on  $W$ . Let  $k'$  be any multiple of  $k$ , bigger than one. If one has an inclusion of  $H$  in  $f_* \omega_{V/W}^k$ , then for all  $\gamma \gg 0$  the sheaf  $(\hat{S}^\gamma(f_* \omega_{V/W}^{k'}) \otimes M^{-1})$  is weakly positive.

**Proof:** Let  $W_0 = \{x \in W \mid f \text{ smooth along } f^{-1}(x)\}$ . Our statement being compatible with replacing  $W$  by the complement of a sub-variety of codimension two, we may assume that  $W - W_0$  is a normal crossing divisor and that  $f$  is flat. [15], 5.4, says that there exist a non singular finite covering  $\pi: W' \rightarrow W$  and an ample invertible sheaf  $H'$  on  $W'$  such that  $\pi^* f_* \omega_{V/W}^{k'} \otimes H'^{-1}$  is weakly positive. For all  $\gamma \gg 0$  the sheaf  $\pi^* M^{-1} \otimes H'^\gamma$  has a section. So  $S^\gamma(\pi^* f_* \omega_{V/W}^{k'} \otimes H'^{-1})$  is a subsheaf of  $\pi^*(S^\gamma(f_* \omega_{V/W}^{k'}) \otimes M^{-1})$  and (3.4) follows from (3.2,iii) and [15], 1.4.

**Theorem 3.5.** Let  $f: V \rightarrow W$  be a fibre space. Assume that for every non singular  $W'$ , generically finite over  $W$  and for every desingularization  $V'$  of  $V \times_W W'$  the induced fibre space  $f': V' \rightarrow W'$  has the property:  $\kappa(W', \det(f'_* \omega_{V'/W'}^k)) = \dim(W)$  for fixed  $k > 0$ . Let  $M$  be any invertible sheaf on  $W$  and  $k'$  any multiple of  $k$ , bigger than one. Then for all  $\gamma \gg 0$  the sheaf  $\hat{S}^\gamma(f_* \omega_{V/W}^{k'}) \otimes M^{-1}$  is weakly positive.

For some  $k' \gg k$  this follows from (3.4) and [15], 6.2. The proof given in [15] was more complicated than necessary and we sketch a more direct proof.

**Proof:** By a wellknown lemma of K. Kodaira (see for example [15], 6.3) the assumption in (3.5) means that for all  $H'$ , ample on  $W'$ , we find  $\delta > 0$  such that  $H'$  is contained in  $\det(f'_* \omega_{V'/W'}^k)^{\delta}$ . Using this formulation, we may replace  $W$  by the complement of a sub-variety of codimension bigger than or equal to two. Especially we may assume  $f$  to be flat. Moreover, from [15], 1.4,5), we know that we may replace  $W$  by any non singular finite covering. Using [15], 6.1, we may hence assume that  $f$  is flat with semi-stable fibres. Now let  $H$  be an ample invertible sheaf on  $W$  and  $\delta > 0$  such that  $H$  is contained in  $\det(f_* \omega_{V/W}^k)^{\delta}$ . For  $r = \text{rk}(f_* \omega_{V/W}^k)$  we have an inclusion of  $\det(f_* \omega_{V/W}^k)^{\delta}$  into  $(f_* \omega_{V/W}^k)^{\otimes r}$ . Hence for  $s = r \cdot \delta$  we have  $H$  as a subsheaf of  $(f_* \omega_{V/W}^k)^{\otimes s}$ . Let  $f(s): V(s) \rightarrow W$  be a desingularization of  $V \times_W V \times_W \dots \times_W V$  ( $s$  times). In [15], 3.5, we obtained (making  $W$  smaller) the equality  $f(s)_* \omega_{V(s)/W}^k = (f_* \omega_{V/W}^k)^{\otimes s}$ . Applying (3.4) to the fibre space  $f(s)$  we know that for some  $\nu > 0$  the sheaf  $S^\nu((f_* \omega_{V/W}^{k'})^{\otimes s}) \otimes M^{-1}$  is weakly positive. Hence the quotient sheaf  $S^{\nu \cdot s}(f_* \omega_{V/W}^{k'}) \otimes M^{-1}$  must also be weakly positive, and we obtain (3.5).

In order to give a simpler interpretation of the statement in (3.5) we can use:

**Lemma 3.6.** Let  $F$  be a torsion free coherent sheaf on  $W$ . Then the following two statements are equivalent:

- i) There exist an ample invertible sheaf  $H$  on  $W$ , some  $\nu \gg 0$  and an inclusion  $\oplus H \rightarrow \hat{S}^\nu(F)$ , being an isomorphism over an open subvariety.

ii) For all invertible sheaves  $M$  on  $W$  there exists some  $\gamma \gg 0$  such that  $\hat{S}^\gamma(F) \otimes M^{-1}$  is weakly positive.

Proof: From i) it follows that for all  $\beta \gg 0$  there exists a map from  $H^\beta \otimes M^{-1}$  to  $\hat{S}^{\beta \cdot \gamma}(F) \otimes M^{-1}$ , surjective over an open subvariety. Choosing  $\beta$  big enough, we may assume that  $H^\beta \otimes M^{-1}$  is very ample. ii) follows directly from the definition of weak positivity.

On the other hand, if  $\hat{S}^\gamma(F) \otimes M^{-1}$  is weakly positive for some ample invertible sheaf  $M$  on  $W$ , then  $\hat{S}^{2 \cdot \beta \cdot \gamma}(F) \otimes M^{-2 \cdot \beta} \otimes M^\beta$  is generated by its global sections over an open subvariety for some  $\beta \gg 0$  and we get a map  $M^\beta \longrightarrow \hat{S}^{2 \cdot \beta \cdot \gamma}(F)$ , surjective over an open set. Of course we may choose  $\text{rk}(\hat{S}^{2 \cdot \beta \cdot \gamma}(F))$  copies of  $M^\beta$  such that the corresponding sections generate the sheaf  $\hat{S}^{2 \cdot \beta \cdot \gamma}(F) \otimes M^{-\beta}$  in the general point of  $W$  and we obtain i) for  $H = M^\beta$  and  $\nu = 2 \cdot \gamma \cdot \beta$ .

Remark 3.7. Theorem II, (3.5) and (3.6) give some hope that the answer to the following question might be positive:

Question  $Q_{n,m}$ : Let  $f: V \longrightarrow W$  be a fibre space,  $n = \dim(V)$  and  $m = \dim(W)$ . Assume that  $\text{Var}(f) = \dim(W)$ . Is it true that there exist an ample invertible sheaf  $H$  on  $W$ , some numbers  $\eta \gg 0$  and  $\nu \gg 0$  and an inclusion  $\bigoplus H \longrightarrow \hat{S}^\nu(f_* \omega_{V/W}^\eta)$ , being an isomorphism over an open subvariety.

We know, that the answer to  $Q_{n,m}$  is positive in the following cases:

- 1)  $V_W$  is a curve ( (3.5) and [15] or [12] ).
- 2)  $W$  is a curve and  $V_W$  a surface ( (3.5) and [12] ).
- 3) For some  $V'_W$ , birationally equivalent to  $V_W$ , and some  $\mu > 0$   $\omega_{V'_W}^\mu = \mathcal{O}_{V'_W}$  ( [6] ).
- 4) For surfaces of general type ( (3.5) and [15] or [12] ) or more generally under the assumptions made in Theorem II.

Moreover (3.5) and (3.6) say that in order to verify  $Q_{n,m}$  we just have to consider  $\det(f_* \omega_{V/W}^\eta)$ .

Maybe  $Q_{n,m}$  is the right way to study the generalized Iitaka conjecture  $C_{n,m}^+$  ( see Theorem I ). In any case  $C_{n,m}^+$  follows from a positive answer to  $Q_{n,m}$ . In fact, Theorem II of [15] can be formulated in the following way:

Theorem 3.8. Let  $f: V \longrightarrow W$  be a fibre space.

Assume that for all fibre spaces  $f'': V'' \longrightarrow W''$  with

$\mathbb{C}(W'') \subset \mathbb{C}(W)$ ,  $V_W \sim V_W'' \times_{\text{Spec}(\mathbb{C}(W''))} \text{Spec}(\mathbb{C}(W))$  and  $\text{Var}(f'') = \dim(W'')$

the answer to  $Q_{n'',m''}$  is positive (  $n'' = \dim(V'')$ ,  $m'' = \dim(W'')$  ).

Then  $C_{n,m}^+$  is true, i.e. if  $\kappa(W) \neq -\infty$

$\kappa(V) = \text{Max}(\kappa(W) + \kappa(V_W), \text{Var}(f) + \kappa(V_W))$ .

#### § 4 The proof of theorem II

Let  $f: V \longrightarrow W$  be the fibre space considered in theorem II,  $\mu > 0$  such that  $\omega_{V'_W}^\mu$  is generated by its global sections. Choose

$a > 0$  and  $b > 1$  such that  $N = b \cdot \mu \geq a$  and write  $\eta = a \cdot N + a + 1$ .

Let  $H$  be an invertible ample sheaf on  $W$ . As we have seen in (3.5)

it is enough to prove part i) of theorem II and this means, that

we have to show that for some  $\delta > 0$  we obtain  $H$  as a subsheaf of

$\det(f_* \omega_{V/W}^\eta)^\delta$  ( [15], 6.3 ).

Using this formulation we may replace  $W$  by the complement of any subvariety of codimension bigger than or equal to two and moreover by any non singular finite covering ( [15], § 3 ). Hence we can restrict ourselves to the following situation:

(4.1)  $f: V \longrightarrow W$  is a flat projective morphism of quasi-projective non singular varieties with connected general fibre  $V_W$  and  $\omega_{V/W}^N$  itself is generated by global sections.

Let  $\mathbb{P} = \mathbb{P}((f_*\omega_{V/W}^N)^\vee) \longrightarrow W$  be the projective bundle such that  $p_*\mathcal{O}_{\mathbb{P}}(1) = (f_*\omega_{V/W}^N)^\vee$  is the dual of the locally free sheaf  $f_*\omega_{V/W}^N$ . Let  $V' = V \times_W \mathbb{P}$  and  $f' = \text{pr}_2: V' \longrightarrow \mathbb{P}$ .

Proposition 4.2. If  $H$  is an ample invertible sheaf on  $W$ , then - using the notations introduced above - for all  $v \gg 0$  the sheaf

$$S^v(f'_*\omega_{V'/\mathbb{P}}^n \otimes \mathcal{O}_{\mathbb{P}}(a)) \otimes_{\mathbb{P}} H^{-1}$$

is weakly positive.

Before proving (4.2) we give the proof of theorem II:

In (3.2,iii) we have seen that the determinant of a weakly positive sheaf is also weakly positive. Hence for some  $\delta, \delta', \delta'' \gg 0$  the sheaf  $\det(f'_*\omega_{V'/\mathbb{P}}^n) \otimes \mathcal{O}_{\mathbb{P}}(\delta') \otimes_{\mathbb{P}} H^{-\delta''}$  is weakly positive. We also may assume that it has a non-trivial section. In fact,  $\mathcal{O}_{\mathbb{P}}(1) \otimes H^1$  is ample for  $v \gg 0$ . Replace  $\delta$  by  $\delta \cdot \beta$ ,  $\delta'$  by  $(\delta' + 1) \cdot \beta$  and  $\delta''$  by  $(\delta'' - 1) \cdot \beta$ .

$f'_*\omega_{V'/\mathbb{P}}^n = p^* f_*\omega_{V/W}^n$  by flat base change and hence we obtain a section of

$$p_*\mathcal{O}_{\mathbb{P}}(\delta') \otimes \det(f_*\omega_{V/W}^n) \otimes H^{-\delta''} = \text{Hom}(S^{\delta'}(f_*\omega_{V/W}^n), \det(f_*\omega_{V/W}^n) \otimes H^{-\delta''})$$

By [13] or [15], theorem III,  $f_*\omega_{V/W}^n$  is weakly positive,

and by (3.2,iii) the symmetric powers of this sheaf are weakly positive.

From [15], 1.4.1), it follows that  $\det(f_*\omega_{V/W}^n) \otimes H^{-\delta''}$  is weakly positive and - choosing  $\beta$  big enough - we obtain an inclusion of  $H^{(\delta'' - 1) \cdot \beta}$  into  $\det(f_*\omega_{V/W}^n) \otimes H^{-\delta''}$ .

Proof of (4.2): By construction of  $f': V' \longrightarrow \mathbb{P}$  and base change  $\text{Hom}(f_*\omega_{V/W}^n, f_*\omega_{V/W}^n) = p_*(f'_*\omega_{V'/\mathbb{P}}^n \otimes \mathcal{O}_{\mathbb{P}}(1))$  and the identity on the left hand side induces a global section  $s$  of  $f'_*\omega_{V'/\mathbb{P}}^n \otimes \mathcal{O}_{\mathbb{P}}(1)$ . Let  $C$  be

the zero-divisor of  $s$  in  $V'$ . We may call  $C$  the universal divisor of  $\omega_{V/W}^N$ . In fact, if locally in  $W$ ,  $s_U: \mathcal{O}_U \longrightarrow f_*\omega_{V/W}^N|_U$  is a section corresponding to a divisor  $C_U$  in  $f^{-1}(U)$ , the dual of  $s_U$

$s_U^\vee: (f_*\omega_{V/W}^N)^\vee|_U \longrightarrow \mathcal{O}_U$  determines a section  $i: U \longrightarrow \mathbb{P}$  of  $p$ .

The restriction  $C|_{f'^{-1}(i(U))}$  is just  $C_U$ . Especially - since  $N$  is a multiple of  $\mu$  - we obtain from the theorem of Bertini:

Claim 4.3. The restriction of  $C$  to a general fibre of  $f'$  is non singular.

For  $x \in \mathbb{P}$  set  $\mathbb{P}_x = p^{-1}(p(x))$  and  $V'_x = f'^{-1}(x) \simeq f^{-1}(p(x))$ .

The restriction of  $f'$  to  $f'^{-1}(\mathbb{P}_x)$  is just the second projection from  $V'_x \times \mathbb{P}_x$  and  $\bar{C}_x = C|_{f'^{-1}(\mathbb{P}_x)}$  is just an embedded deformation of

$C_x = C|_{V'_x} = \bar{C}_x|_{V'_x \times \{x\}}$ . Let  $N_{\bar{C}_x/V'_x \times \mathbb{P}_x}$  be the normal sheaf of  $\bar{C}_x$ .

We have an exact sequence

$$0 \longrightarrow \text{pr}_{2*}\mathcal{O}_{V'_x \times \mathbb{P}_x} \longrightarrow \text{pr}_{2*}\mathcal{O}_{V'_x \times \mathbb{P}_x}(\bar{C}_x) \longrightarrow \text{pr}_{2*}N_{\bar{C}_x/V'_x \times \mathbb{P}_x}$$

The cokernel of the first map is nothing but

$$\text{coker}(s|_{\mathbb{P}_x}: \mathcal{O}_{\mathbb{P}_x} \longrightarrow (\mathcal{O}_{\mathbb{P}_x}(1))^{\otimes r}) = T_{\mathbb{P}_x}$$

Hence we obtain:

Claim 4.4. The infinitesimal deformation of  $C_x$  induces an inclusion of  $T_{\mathbb{P}_x, x}$  into  $H^0(C_x, N_{C_x/V'_x})$ .

Assume now, that  $V'_x$  and  $C_x$  are non singular.  $V_x$  being of general type, we know that  $H^0(V'_x, T_{V'_x}) = 0$  (for example this follows from the vanishing theorem of Bogomolov, [14], Theorem III).

Let  $\rho'_x: T_{\mathbb{P}_x, x} \longrightarrow H^1(V'_x, T_{V'_x}(-C_x))$  be the Kodaira-Spencer map corresponding to the (open) deformation of  $V'_x - C_x$  given by  $f'$  (see (1.7))

and [7]).  $\rho'_x |_{\mathcal{L}_{\mathbb{P},x}}$  factors over the inclusion (4.4) and the inclusion  $H^0(C_x, N_{C_x/V'_x}) \longrightarrow H^1(V'_x, T_{V'_x} \langle -C_x \rangle)$  coming from the exact sequence  $0 \longrightarrow T_{V'_x} \langle -C_x \rangle \longrightarrow T_{V'_x} \longrightarrow N_{C_x/V'_x} \longrightarrow 0$  ( see [7] ). We obtained:

Claim 4.5.  $\rho'_x |_{\mathcal{L}_{\mathbb{P},x}}$  is injective for all  $x$  with non singular  $V'_x$ .

(4.6) Let  $\theta_0: S_0 \longrightarrow \mathbb{P}$  be an étale open set such that:  $g_0 = \text{pr}_2: X_0 = V'_x \times_{\mathbb{P}} S_0 \longrightarrow \mathbb{P}$  is smooth,  $D_0 = C \times_{\mathbb{P}} S_0$  is smooth over  $S_0$  and there exists an invertible sheaf  $K_0$  on  $S_0$  such that  $\theta_0^* \mathcal{O}_{\mathbb{P}}(1) = K_0^N$ . Choose  $L_0 = \omega_{X_0/S_0} \otimes g_0^* K_0$ . We have  $L_0^N = \mathcal{O}_{X_0}(D_0)$  and as we did in (1.2) and (1.7) we may take the  $N$ -th root out of  $D_0$  in order to obtain  $h_0: Y_0 \longrightarrow S_0$ . From (1.6) and (1.9) we know that - using the notation introduced there -  $\rho_s(\mathcal{L}_{S_0,s}) \cap \text{Ker}(\lambda_s) = 0$  for all  $s \in S$ .

Claim 4.7. For all  $s \in S_0$  in sufficiently general position the Kodaira-Spencer map  $\rho_s$  ( and hence  $\lambda_s \cdot \rho_s$  ) is injective.

Proof: If not, by (2.1) - after replacing  $S_0$  by an étale open set - morphisms  $\sigma: S_0 \longrightarrow Z$  and  $\phi: F \longrightarrow Z$  would exist such that  $\dim(Z) < \dim(S_0)$  and  $Y_0 \simeq S_0 \times_Z F$ . By (2.1,ii) the fibres of  $\sigma$  are tangent to the kernel of  $\rho_s$ . If  $p_0 = p \cdot \theta_0: S_0 \longrightarrow W$ , (4.5) guarantees that the general fibres of  $p_0$  and  $\sigma$  intersect each other in a finite number of points. Especially, if  $T_0 = \sigma^{-1}(z)$  for  $z$  in sufficiently general position,  $p_0(T_0)$  has a positive dimension. By construction  $h_0^{-1}(T_0) = T_0 \times_{F_2}$  is a Galois-covering of  $g_0^{-1}(T_0)$ . The Galois-group  $G$  of  $T_0 \times_{F_2}$  is a subgroup of  $\text{Aut}_{T_0}(T_0 \times_{F_2})$  of finite order and hence comes from  $\text{Aut}(F_2)$ . Therefore

$$g_0^{-1}(T_0) = (T_0 \times_{F_2})/G = T_0 \times (F_2/G)$$

in contradiction to our assumption "  $\text{Var}(f) = \dim(W)$  " .

(4.8) Choose now a non singular finite covering  $\theta: S \longrightarrow \mathbb{P}$  such that  $\theta^* \mathcal{O}_{\mathbb{P}}(1) = K^N$  for an invertible sheaf  $K$  on  $S$  ( For example: Let  $H^m$  be an ample sheaf on  $\mathbb{P}$  such that  $H^{mN} \otimes \mathcal{O}_{\mathbb{P}}(1) = \mathcal{O}_{\mathbb{P}}(E)$  for a non singular divisor  $E$  and apply [13],2.1 to  $E$  ). Moreover we may assume that the discriminant  $\Delta(S/\mathbb{P})$  meets the set of points in  $\mathbb{P}$  over which  $f'$  is not smooth in a subvariety of codimension bigger than or equal to two. From [15], 1.4,5) , we know that (4.2) follows from:

Claim 4.9. For every invertible sheaf  $H$  on  $S$  and all  $v \gg 0$  the sheaf  $S^v(\theta^* f'^* \omega_{V'/\mathbb{P}}^{\otimes v} \otimes K^{a \cdot N}) \otimes H^{-1}$  is weakly positive.

Proof: In order to prove (4.9) we are allowed to choose  $S$  smaller, for example to leave out the points  $s \in S$  where  $\theta$  is not étale and  $f'$  not smooth along  $f'^{-1}(\theta(s))$ . Then  $X = V'_x \times_{\mathbb{P}} S$  is a non singular variety and for  $g = \text{pr}_2: X \longrightarrow S$  we obtain from [15],3.2 , ( or from flat base change ) that  $g_* \omega_{X/S}^{\delta} \simeq \theta^* f'^* \omega_{V'/\mathbb{P}}^{\delta}$  for all  $\delta > 0$ .

For  $D = C \times_{\mathbb{P}} S$  we have  $\mathcal{O}_X(D) = \omega_{X/S}^N \otimes g^* K^N$  and, as in (1.1), let  $\tau: Y \longrightarrow X$  be the morphism obtained by taking the  $N$ -th root out of  $D$  and  $h = g \cdot \tau$ . The restriction of  $D$  to the general fibre of  $g$  is non singular and from (1.1) we get an inclusion

$$\tau_* \omega_{Y/S}^{a+1} \longrightarrow \bigoplus_{i=0}^{N-1} \omega_{X/S}^{a+1} \otimes (\omega_{X/S} \otimes g^* K)^{a \cdot N - a + 1 + i}$$

being an isomorphism along the general fibre of  $g$ .

If we apply  $g_*$  and take the projection to the summand "  $i = a-1$  " ( this is possible since  $a \leq N = b \cdot \mu$  ), we get a morphism from  $h_* \omega_{Y/S}^{a+1}$  to  $g_* \omega_{X/S}^{a \cdot N + a + 1} \otimes K^{a \cdot N}$ , surjective over an open sub-variety.

Hence (4.9) and (4.2) follow from:

Claim 4.10.  $\hat{S}^v(h_* \omega_{Y/S}^{a+1}) \otimes H^{-1}$  is weakly positive for all  $v \gg 0$ .

Proof: Let  $\bar{h}: \bar{Y} \longrightarrow \bar{S}$  be any projective compactification of  $Y$  and  $S$  and  $\bar{H}$  a prolongation of  $H$ . Of course we can find  $S_0$  in  $S$  such that (4.6) is fulfilled. By (4.7), the assumptions of (2.5) are verified and hence  $\kappa(\det(\bar{h}_* \omega_{\bar{Y}/\bar{S}})) = \dim(\bar{S})$ . The same argument remains true if we replace  $\bar{S}$  by any  $\bar{S}'$  generically finite over  $\bar{S}$  and  $\bar{Y}$  by any desingularization  $\bar{Y}'$  of  $\bar{S}' \times_{\bar{S}} \bar{Y}$ . Setting  $k = 1$  and  $k' = a+1$  we obtain from (3.5) that  $\hat{S}^v(\bar{h}_* \omega_{\bar{Y}'/\bar{S}'}^{a+1}) \otimes \bar{H}^{-1}$  is weakly positive for all  $v \gg 0$  and hence (4.10) is true.

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Partially supported by Deutsche Forschungsgemeinschaft ( Heisenberg - Programm )

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