

A FANSY DIVISOR ON $\overline{M}_{0,n}$

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ABSTRACT. We study the relation between projective T -varieties and their affine cones in the language of the so-called divisorial fans and polyhedral divisors. As an application, we present the Grassmannian $\text{Grass}(2, n)$ as a “fancy divisor” on the moduli space of stable, n -pointed, rational curves $\overline{M}_{0,n}$.

1. INTRODUCTION

Varieties X with torus action can be described by divisors \mathcal{S} on their Chow quotients. However, this requires the use of rather strange coefficients for \mathcal{S} . These coefficients have to be polyhedra if X is affine, and they are polyhedral subdivisions of some vector space in the general case. Outside of a compact region, these subdivisions all look alike – they form a fan which is called the tail fan of \mathcal{S} . Moreover, this structure also gives \mathcal{S} its name: We will call it the “fancy divisor” of X .

This method was developed in [AlHa] and [AHS] – it provides an extension of the language of toric varieties and gives complete information about the arrangement of the T -orbits. In Section 2, we begin with a review of the basic facts of this theory.

We present our main result, how to obtain this description for projective T -varieties, in Theorem 5.4. As an application, in Theorem 4.2, we present the Grassmannian $\text{Grass}(2, n)$ as a fancy divisor $\mathcal{S} = \sum_B \mathcal{S}_B \otimes D_B$ on the moduli space $\overline{M}_{0,n}$ of stable rational curves with n marked points. Here, D_B stands for the prime divisor consisting of the two-component curves with a point distribution according to the partition B . The polyhedral subdivisions \mathcal{S}_B look like their common tail fan – only that the origin has been replaced by a line segment whose direction depends on B .

In general, when replacing $\text{Grass}(2, n)$ by an arbitrary generalized flag variety G/P , it is difficult to obtain the Chow quotient Y where the fancy divisor is supposed to live. However, one can always determine the tail fan. It is a coarsified version of the system of Weyl chambers associated to the semi-simple algebraic group G .

2. POLYHEDRAL AND FANSY DIVISORS

Let T be an affine torus over an algebraically closed field \mathbb{K} of characteristic 0. It gives rise to the mutually dual free abelian groups $M := \text{Hom}_{\text{algGrp}}(T, \mathbb{K}^*)$ and

$N := \text{Hom}_{\text{algGrp}}(\mathbb{K}^*, T)$, and, via $T = \text{Spec } \mathbb{K}[M]$, the torus can be recovered from them. In [AlHa], we have provided a method to describe affine T -varieties X by so-called pp-divisors on lower-dimensional algebraic varieties Y , i.e. by divisors having polyhedra from $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ as their coefficients. We would like to recall some details.

Definition 2.1. If $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone, then we denote by $\text{Pol}(N_{\mathbb{Q}}, \sigma)$ the Grothendieck group of the semigroup

$$\text{Pol}^+(N_{\mathbb{Q}}, \sigma) := \{\Delta \subseteq N_{\mathbb{Q}} \mid \Delta = \sigma + [\text{compact polytope}]\}$$

with respect to Minkowski addition. We will call $\text{tail}(\Delta) := \sigma$ the tail cone of the elements of $\text{Pol}(N_{\mathbb{Q}}, \sigma)$.

Let Y be a normal and semiprojective (i.e. $Y \rightarrow Y_0$ is projective over an affine Y_0) \mathbb{K} -variety. We will call \mathbb{Q} -Cartier divisors on Y *semiample* if a multiple of them becomes base point free.

Definition 2.2. An element $\mathcal{D} = \sum_i \Delta_i \otimes D_i \in \text{Pol}(N_{\mathbb{Q}}, \sigma) \otimes_{\mathbb{Z}} \text{CaDiv}(Y)$ with prime divisors D_i is called a *pp-divisor* on (Y, N) with tail cone σ if $\Delta_i \in \text{Pol}^+(N_{\mathbb{Q}}, \sigma)$ and if the evaluations $\mathcal{D}(u) := \sum_i \min\langle \Delta_i, u \rangle D_i$ are semiample for $u \in \sigma^\vee \cap M$ and big for $u \in \text{int } \sigma^\vee \cap M$. (Note that the membership $u \in \sigma^\vee := \{u \in M_{\mathbb{Q}} \mid \langle \sigma, u \rangle \geq 0\}$ guarantees that $\min\langle \Delta_i, u \rangle > -\infty$.)

The common tail cone of the coefficients Δ_i will be denoted by $\text{tail}(\mathcal{D})$. Via $\mathcal{O}_Y(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$, a pp-divisor \mathcal{D} gives rise to the affine scheme $X := X(\mathcal{D}) := \text{Spec } \Gamma(Y, \mathcal{O}(\mathcal{D}))$ over Y_0 . The M -grading of its regular functions translates into an action of the torus T on X , and $\text{tail}(\mathcal{D})^\vee$ becomes the cone generated by the weights. Note that the modification $\tilde{X} := \tilde{X}(\mathcal{D}) := \text{Spec}_Y \mathcal{O}(\mathcal{D})$ of $X(\mathcal{D})$ is a fibration over Y with $\mathbb{T}\mathbb{V}(\text{tail}(\mathcal{D}), N) := \text{Spec } \mathbb{K}[\text{tail}(\mathcal{D})^\vee \cap M]$ as general fiber.

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y_0 \end{array}$$

This construction is, in some sense, functorial. If $f \in \mathbb{K}(Y) \otimes_{\mathbb{Z}} N$ is a ‘‘polyhedral, rational function’’, then \mathcal{D} and $\mathcal{D} + \text{div}(f)$ define isomorphic graded \mathcal{O}_Y -algebras. Up to this manipulation with ‘‘polyhedral principal divisors’’, morphisms of affine varieties with torus action are exclusively induced by the following morphisms $\mathcal{D} \rightarrow \mathcal{D}'$ of pp-divisors (cf. [AlHa, §8] for details):

Let $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ and $\mathcal{D}' = \sum_j \Delta'_j \otimes D'_j$ be pp-divisors on (Y, N) and (Y', N') with tail cones σ and σ' , respectively. If $\psi : Y \rightarrow Y'$ is such that none of the supports of the D'_i contains $\psi(Y)$, and if $F : N \rightarrow N'$ is a linear map with $F(\sigma) \subseteq \sigma'$, then the relation

$$\sum_i \Delta'_i \otimes \psi^*(D'_i) =: \psi^*(\mathcal{D}') \leq F_*(\mathcal{D}) := \sum_i (F(\Delta_i) + \sigma') \otimes D_i$$

inside $\text{Pol}(N'_{\mathbb{Q}}, \sigma') \otimes_{\mathbb{Z}} \text{CaDiv}(Y)$ gives rise to an equivariant (with respect to $T = N \otimes_{\mathbb{Z}} \mathbb{K}^* \xrightarrow{F \otimes \text{id}} N' \otimes_{\mathbb{Z}} \mathbb{K}^* = T'$) morphism $X(\mathcal{D}) \rightarrow X(\mathcal{D}')$. In particular, there are adjunction maps $\psi^*(\mathcal{D}') \rightarrow \mathcal{D}'$ and $\mathcal{D} \rightarrow F_*(\mathcal{D})$.

In [AHS], polyhedral divisors on (Y, N) have been glued together to obtain a description of non-affine T -varieties in terms of so-called divisorial fans. To define them, we first have to broaden our previous notion of pp-divisors by allowing the empty set \emptyset as a possible coefficient for \mathcal{D} :

Definition 2.3. Let Y be a normal and semiprojective variety over \mathbb{K} .

- (1) $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ is called an (enhanced) pp-divisor if $Z(\mathcal{D}) := \bigcup_{\Delta_i \neq \emptyset} D_i$ is the support of some effective, semiample divisor on Y , and if $\mathcal{D}|_{Y \setminus Z} := \sum_{\Delta_i \neq \emptyset} \Delta_i \otimes D_i|_{Y \setminus Z}$ is a pp-divisor in the usual sense.
- (2) If $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ and $\mathcal{D}' = \sum_i \Delta'_i \otimes D_i$ are (enhanced) pp-divisors on (Y, N) , then we denote $\mathcal{D} \cap \mathcal{D}' := \sum_i (\Delta_i \cap \Delta'_i) \otimes D_i$. It is here that empty coefficients have their natural appearance.
- (3) A finite set $\mathcal{S} = \{\mathcal{D}^\nu\}$ is called a divisorial fan on (Y, N) if its elements and all their mutual intersections are (enhanced) pp-divisors on Y , and if $X(\mathcal{D}^\mu \cap \mathcal{D}^\nu) \rightarrow X(\mathcal{D}^\mu)$ is always an open embedding.

Note that in (1), one still needs to know about the tail cone of an enhanced pp-divisor. The easiest possibility to keep it as part of the data is to ask for at least one non-empty coefficient Δ_i . This is not a restriction at all, since one can always add additional summands with the neutral element σ as coefficient.

The conditions of (3) can alternatively be presented in an explicit way. However, they then turn out to be very technical, cf. [AHS, 5.1 and 5.3]. Hence, we will restrict ourselves to a special case:

First, if $\mathcal{D}, \mathcal{D}'$ are pp-divisors on (Y, N) with coefficients $\Delta_i \subseteq \Delta'_i$, then, by [AHS, 3.5], a necessary condition for $X(\mathcal{D}) \rightarrow X(\mathcal{D}')$ becoming an open embedding is that the Δ_i are (possibly empty) faces of Δ'_i , implying that $\text{tail}(\mathcal{D}) \leq \text{tail}(\mathcal{D}')$. In particular, the coefficients Δ_i^ν of the elements \mathcal{D}^ν of a divisorial fan \mathcal{S} form polyhedral subdivisions \mathcal{S}_i of $N_{\mathbb{Q}}$, their cells are labeled by the ν 's or \mathcal{D}^ν 's, and their tail cones fit into the so-called tail fan $\text{tail}(\mathcal{S})$.

Definition 2.4. A “fancy divisor” is a set of pp-divisors $\mathcal{S} = \{\mathcal{D}^\nu = \sum_i \Delta_i^\nu \otimes D_i\}$ on (Y, N) such that for any μ, ν there are

- (1) a $u^{\mu\nu} \in M$ and, numbers $c_i^{\mu\nu}$ with $\max\langle \Delta_i^\mu, u^{\mu\nu} \rangle \leq c_i^{\mu\nu} \leq \min\langle \Delta_i^\nu, u^{\mu\nu} \rangle$ and $\Delta_i^\mu \cap [\langle \bullet, u^{\mu\nu} \rangle = c_i^{\mu\nu}] = \Delta_i^\nu \cap [\langle \bullet, u^{\mu\nu} \rangle = c_i^{\mu\nu}]$, and
- (2) an effective, semiample divisor $E^{\mu\nu}$ on $Y \setminus Z(\mathcal{D}^\nu)$ with $\bigcup\{D_i \mid \Delta_i^\mu \cap \Delta_i^\nu = \emptyset\} = \text{supp } E^{\mu\nu}$ and $k \mathcal{D}^\nu(u^{\mu\nu}) - E^{\mu\nu}$ being semiample for $k \gg 0$.

Proposition 2.5 ([AHS, 6.9]). *Fancy divisors generate (via taking the finitely many mutual intersections of the pp-divisors) divisorial fans.*

3. THE TAIL FAN OF G/P

Let G be a semi-simple linear algebraic group; we fix a maximal torus and a Borel subgroup $T \subseteq B \subseteq G$. Denoting by M and N the mutually dual lattices of characters and 1-parameter subgroups of T , respectively, these choices provide a system

of positive roots $R^+ \subseteq R$ with basis D . We denote by $\Lambda_R := \mathbb{Z}R := \text{span}_{\mathbb{Z}} R \subseteq M$ the root lattice, by $W := N(T)/T = \langle s_\alpha \mid \alpha \in D \rangle$ the Weyl group, and by $\mathfrak{f} := (\mathbb{N}R^+)^\vee \subseteq N_{\mathbb{Q}}$ the cone of fundamental weights of the *dual* root system R^* . Note that $(R^+)^\vee := (\mathbb{N}R^+)^\vee$ means the dual of the cone generated by R^+ , but, alternatively, we use the symbol $\alpha^\vee \in R^* \subseteq N$ to denote the co-root assigned to α . See [Spr] or [FuHa] for the basic facts concerning root systems and algebraic groups or Lie algebras; see Section 4 for the special case of $G = \text{SL}(n, \mathbb{K})$.

We would like to describe the T -action on generalized flag varieties. Hence, we fix a subset $I \subseteq D$ and denote by $P_I \supseteq B$ the corresponding parabolic subgroup, cf. [Spr, §8.4]. Its weights are $R^+ \cup (-R_I^+)$ with $R_I^+ := R^+ \cap \mathbb{Z}I$ and $R_I := R \cap \mathbb{Z}I$; in particular, $P_\emptyset = B$. Considering the subgroup $W_I := \langle s_\alpha \mid \alpha \in I \rangle \subseteq W$, the set $W^I := \{w \in W \mid w(I) > 0\}$ provides nice representatives of the left cosets of W_I , hence giving a bijection $W^I \times W_I \xrightarrow{\sim} W$. This splitting satisfies $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$ with $\ell(w)$ referring to the length of a minimal representation of $w \in W$ as a product of s_α with $\alpha \in D$. Both W and W_I contain longest elements w^0 and w_I^0 , respectively. This gives rise to $w^I := w^0 w_I^0 \in W^I$.

Eventually, each $w \in W$ provides a set $R(w) := \{\alpha \in R^+ \mid w(\alpha) < 0\}$ and $U_w := \prod_{\alpha \in R(w)} X_\alpha \xrightarrow[\sim]{\prod x_\alpha^{-1}} \mathbb{K}^{\#R(w)}$ where X_α is the 1-parameter subgroup characterized by the isomorphisms $x_\alpha : \mathbb{K}^1 \xrightarrow{\sim} X_\alpha \subseteq G$ with $x_\alpha(\alpha(t) \cdot \xi) = t x_\alpha(\xi) t^{-1}$ for $t \in T$. Note that $\#R(w) = \ell(w)$. After choosing, for each $w \in W$, a representative $w \in N(T) \subseteq G$, this leads to the Bruhat decomposition $G = \bigcup_{w \in W^I} U_{w^{-1}} w P_I$. The special choice $w := w^I$ yields the dense, open cell, and we denote by $\mathbb{A} := (w^I)^{-1} U_{(w^I)^{-1}} w^I P$ a shift of it – considered as a T -invariant, open subset $\mathbb{A} \subseteq G/P$. Note that \mathbb{A} is then an affine space.

Lemma 3.1. *The open subsets $\{w \cdot \mathbb{A} \mid w \in W^I\}$ cover G/P_I .*

Proof. It is sufficient to have the inclusion $w^{-1} U_{w^{-1}} w \subseteq (w^I)^{-1} U_{(w^I)^{-1}} w^I$ for arbitrary $w \in W^I$, and this means to check that $vR(w^{-1}) \subseteq R((w^I)^{-1})$ with $v := w^I w^{-1}$. If $\alpha \in R(w^{-1})$, then, the essential part is to see that $v(\alpha) \in R^+$. If this were not true, then w^I would preserve the positivity of $-w^{-1}(\alpha)$, i.e. $-w^{-1}(\alpha) \in R^+ \cap (w^I)^{-1} R^+$. However, since $W_I(R^+ \setminus R_I^+) = R^+ \setminus R_I^+$ and $w^0(R^+) = -R^+$, this would mean that $-w^{-1}(\alpha) \in R_I^+$, i.e. that $-\alpha \in w(R_I^+)$. On the other hand, $w \in W^I$ means that w preserves the positivity of I , hence that of R_I^+ . Hence, the negative root $-\alpha$ cannot belong to $w(R_I^+)$. \square

Note that, since $w^I \in W^I$, the original open Bruhat cell $w^I \mathbb{A}$ belongs to the open covering provided by the previous lemma. Moreover, the fact that W acts via conjugation on T implies that this canonical covering is T -invariant. Hence, it makes sense to ask for the fancy divisor \mathcal{S} on some Y describing the T -variety G/P_I . As usual, the underlying variety Y is not uniquely determined, but there is a minimal choice – the Chow quotient of G/P_I by T .

Proposition 3.2. *Assume that $I \cap (R^+ \setminus R_I^+)^\perp = \emptyset$. Then, the tail fan of the T -variety G/P_I is a coarsened system of negative Weyl chambers:*

$$\text{tail}(\mathcal{S}) = -W^I(W_I\mathfrak{f}) := \{-w W_I\mathfrak{f} \mid w \in W^I\}$$

in $N_{\mathbb{Q}}$. The lattice associated to \mathcal{S} is $\Lambda_R^* \supseteq N$; it equals the weight lattice of the dual root system R^* .

Proof. The torus T acts on sets like U_w via the weights $R(w)$. Hence, the weights of the affine T -space \mathbb{A} are

$$(w^I)^{-1}R((w^I)^{-1}) = (w^I)^{-1}R^+ \cap (-R^+) = -(R^+ \setminus (w^I)^{-1}R^+) = -(R^+ \setminus R_I^+),$$

where the last equality follows from $R^+ \cap (w^I)^{-1}R^+ = R_I^+$ which was already used and shown in the proof of Lemma 3.1. Thus, to get $W_I\mathfrak{f}$ as the tail cone of the pp-divisor describing \mathbb{A} , we have to make sure that $W_I\mathfrak{f} = (R^+ \setminus R_I^+)^\vee$, which, in the special case $I = \emptyset$, is exactly the definition of \mathfrak{f} .

Since, for an element $w \in W_I$, one has $\langle w(\mathfrak{f}), R^+ \setminus R_I^+ \rangle = \langle \mathfrak{f}, w^{-1}(R^+ \setminus R_I^+) \rangle = \langle \mathfrak{f}, R^+ \setminus R_I^+ \rangle \geq 0$, we easily obtain the inclusion $W_I\mathfrak{f} \subseteq (R^+ \setminus R_I^+)^\vee$. On the other hand, let $c \in N$ with $\langle c, R^+ \setminus R_I^+ \rangle \geq 0$. If an $\alpha \in I \subseteq D$ fulfills $\langle c, \alpha \rangle < 0$, then, for any $\beta \in D$,

$$\langle s_\alpha(c), \beta \rangle = \langle c, s_\alpha(\beta) \rangle = \begin{cases} \langle c, -\alpha \rangle > 0 & \text{if } \beta = \alpha \\ \text{one out of } \langle c, D \setminus \{\alpha\} \rangle & \text{if } \beta \neq \alpha. \end{cases}$$

Hence, $s_\alpha(c)$ has a better performance on D than the original c on D . Since $s_\alpha \in W_I$, induction shows that there is a $w \in W_I$ such that $w(c) \in \mathfrak{f}$.

Eventually, to describe the whole tail fan for G/P , we just have to apply the elements of W^I on the tail cone for \mathbb{A} . However, we should also remark that the cones of $\text{tail}(\mathcal{S})$ do not contain any linear subspaces. This, together with the claim concerning the lattice structure, follows from the fact that $\mathbb{Z}(R^+ \setminus R_I^+) = \Lambda_R$. To see this, let us take an arbitrary $\alpha \in I$. By the assumption of the proposition, there is a $\beta \in (R^+ \setminus R_I^+)$ such that $\langle \beta, \alpha^\vee \rangle \neq 0$. Using [Spr, Lemma 9.1.3], we obtain that then at least one of the possibilities $\gamma := \beta \pm \alpha$ belongs to R , hence to $R \setminus R_I$. Thus, we may conclude that $\alpha \in \text{span}_{\mathbb{Z}}(\beta, \gamma) \subseteq \mathbb{Z}(R^+ \setminus R_I^+)$. \square

Remark. If the assumption of Proposition 3.2 is not satisfied, i.e. if $I' := I \cap (R^+ \setminus R_I^+)^\perp \neq \emptyset$, then the T -action admits a non-discrete kernel. This is reflected by the fact that the weight cones are lower-dimensional or, equivalently, that their duals $W_I\mathfrak{f}$ contain a common linear subspace, namely $\text{span}_{\mathbb{Q}}\{\alpha^\vee \mid \alpha \in I'\} \subseteq N_{\mathbb{Q}}$. Then, dividing this out, i.e. replacing Λ_R^* by the corresponding quotient, the claim of Proposition 3.2 remains true.

Now, there is a standard procedure (cf. [AlHa, §11]) to establish the pp-divisor \mathcal{D} for, say, the affine chart $\mathbb{A} \subseteq G/P_I$.

Recipe 3.3. Setting $\ell := \ell(w^I) = \#(R^+ \setminus R_I^+)$, one has the two exact sequences

$0 \rightarrow M' \rightarrow \mathbb{Z}^\ell \xrightarrow{\text{deg}_{\text{loc}}} \Lambda_R \rightarrow 0$ and, more important, its dual

$$\begin{array}{ccccccc} & & N & & & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \Lambda_R^* & \xrightarrow{-\langle \bullet, R^+ \setminus R_I^+ \rangle} & \mathbb{Z}^\ell & \xrightarrow{\pi'} & N' \longrightarrow 0. \end{array}$$

The scheme Y' carrying the pp-divisor \mathcal{D} for the standard big cell is the Chow quotient $\mathbb{A} //^{\text{ch}} T$. By [KSZ], it equals the toric variety associated to the fan Σ' arising from the coarsest common refinement of the $\pi'_\mathbb{Q}(\mathbb{Q}_{\geq 0}^\ell\text{-faces})$ in $N'_\mathbb{Q} := N' \otimes_\mathbb{Z} \mathbb{Q}$. To determine \mathcal{D} itself, we need to know, for every (first integral generator of a) one-dimensional cone $c \in \Sigma'^{(1)}$, the polyhedron $\Delta(c) := (\pi'_\mathbb{Q}^{-1}(c) \cap \mathbb{Q}_{\geq 0}^\ell) - s(c)$, where $s : N' \rightarrow \mathbb{Z}^\ell$ is a pre-chosen section of π' which does nothing but shift all the “fiber polytopes” into $\Lambda_R^* \otimes \mathbb{Q}$. Then, $\mathcal{D} = \sum_c \Delta(c) \otimes \overline{\text{orb}(c)}$ with $\text{orb}(c) \subseteq Y'$ denoting the 1-codimensional T -orbit corresponding to the ray $c \in \Sigma'^{(1)}$.

However, there are two problems. First, every chart $w\mathbb{A}$ leads to another Chow quotient $Y'_w = w\mathbb{A} //^{\text{ch}} T$. They are all birationally equivalent, with $w_i\mathbb{A} //^{\text{ch}} T$ being a blow up of $(w_i\mathbb{A} \cap w_j\mathbb{A}) //^{\text{ch}} T$, and they are dominated by $Y = (G/P_I) //^{\text{ch}} T$. Hence, one has to pull back all the pp-divisors \mathcal{D}_w from Y'_w to Y before glueing them.

Second, while the Y'_w are at least toric with an explicitly computable, but rather complicated fan, their common modification Y is not. Hence, one should look for those situations where Y is already somehow known. This leads to the case of $G/P_I = \text{Grass}(2, n)$ where the Chow quotient has been calculated in [Kap].

4. THE GRASSMANNIAN AS A T -VARIETY

Here, we describe the special case of the Grassmannian $\text{Grass}(k, n)$ for G/P_I . We begin with transferring notation and results from Section 3 to this special case. With $G = \text{SL}(n, \mathbb{K})$, the subgroups $T \subseteq B$ consist of the diagonal and upper triangular matrices, respectively. In $M = \mathbb{Z}^n / \mathbb{Z} \cdot \underline{1}$, we denote by L_i the image of the i -th basic vector $e_i \in \mathbb{Z}^n$. Then, $D = \{\alpha_i := L_i - L_{i+1} \mid i = 1, \dots, n-1\}$ and $R^+ = \{\alpha_{ij} := L_i - L_j \mid i < j\}$. For each root α_{ij} , we have the 1-parameter family $X_{ij} = \{I_n + \xi E_{ij} \mid \xi \in \mathbb{K}\}$. Analogously, if $e^i \in \mathbb{Z}^n$ denotes the dual basis, then $\alpha_{ij}^\vee := e^i - e^j \in N = \ker(\underline{1}) \subseteq \mathbb{Z}^n$ are the co-roots. Thus, root and weight lattice are

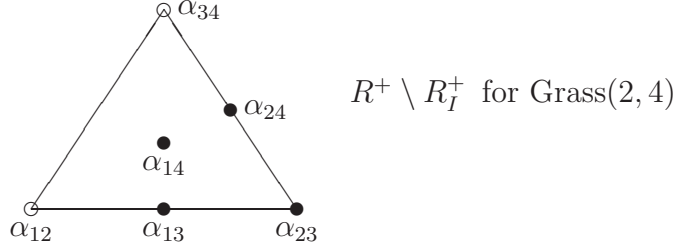
$$\Lambda_R = \langle L_i - L_j \rangle \hookrightarrow M = \langle L_i \rangle = \langle e^i - e^j \rangle^* = \Lambda_W \quad (\text{with } L_1 + \dots + L_n = 0),$$

satisfying $\Lambda_W / \Lambda_R \cong \mathbb{Z}/n\mathbb{Z}$. The root system of $\text{SL}(n)$ is self dual. Hence, if ℓ_i denotes the equivalence class of e^i in $\mathbb{Z}^n / \mathbb{Z} \cdot \underline{1}$, then

$$\Lambda_W^* = \langle \ell_i - \ell_j \rangle = N \hookrightarrow \langle \ell_i \rangle = \langle L_i - L_j \rangle^* = \Lambda_R^* \quad (\text{with } \ell_1 + \dots + \ell_n = 0)$$

looks similar to the line above. Since $\langle \alpha_i, \sum_{v=1}^p \ell_v \rangle = \delta_{ip}$, the standard Weyl chamber is $\mathfrak{f} = \langle \ell_1, \ell_1 + \ell_2, \dots, \sum_{i=1}^{n-1} \ell_i \rangle$. The embedding $\mathfrak{S}_n \hookrightarrow \text{GL}(n)$ via permutation

matrices yields an isomorphism $\mathfrak{S}_n \xrightarrow{\sim} W$ such that the W -action becomes $w(L_i) = L_{w(i)}$.



The Grassmannian $\text{Grass}(k, n)$ is obtained via $I := D \setminus \{\alpha_k\} = \{\alpha_1, \dots, \widehat{\alpha}_k, \dots, \alpha_{n-1}\}$. In particular, $P_I = \{A \in \text{SL}(n) \mid A_{ij} = 0 \text{ for } i > k, j \leq k\}$ and $W_I = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \subseteq \mathfrak{S}_n$. Moreover, $W^I = \mathfrak{S}_{k, n-k} := \{w \in \mathfrak{S}_n \mid w(1) < \dots < w(k), w(k+1) < \dots < w(n)\}$ is the set of $(k, n-k)$ -shuffles with $\#W^I = \binom{n}{k}$ and $w^I = \binom{1 \dots k}{k \dots 1} \binom{k+1 \dots n}{n \dots k+1} = (1 \ 2 \ \dots \ n)^{-k}$. Eventually, we have $-(R^+ \setminus R_I^+) = \{L_i - L_j \mid 1 \leq j \leq k < i \leq n\}$, and the $k(n-k)$ -dimensional charts $w\mathbb{A}$ turn into the usual ones with coordinates a_{ij} where $i \notin w(1, \dots, k)$, $j = 1, \dots, k$ and weights $L_i - L_{w(j)}$.

Proposition 4.1. *The tail fan of the T -variety $\text{Grass}(k, n)$ equals*

$$\text{tail}(\mathcal{S}) = \{\langle \pm \ell_1, \pm \ell_2, \dots, \pm \ell_n \rangle \subseteq N_{\mathbb{Q}} \mid \text{the negative sign occurs exactly } k \text{ times}\}$$

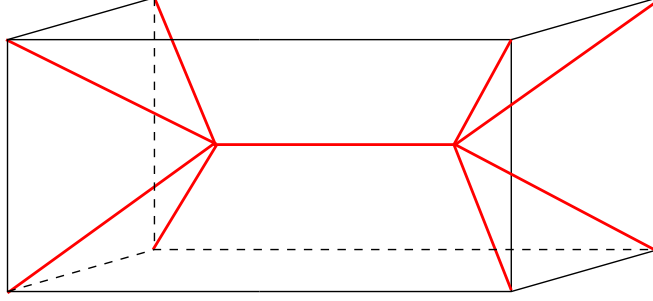
consisting of $\binom{n}{k}$ cones, and the associated lattice is $\Lambda_R^* = \langle \ell_i \rangle$.

Proof. By Proposition 3.2, it remains to show that $W_I \mathfrak{f} = \langle \ell_1, \dots, \ell_k, -\ell_{k+1}, \dots, -\ell_n \rangle$. The left hand side is (not minimally) generated by the elements $\ell_J := \sum_{j \in J} \ell_j$ with $J \subseteq \{1, \dots, k\}$ or $J \supseteq \{1, \dots, k\}$. While the first type does already fit into the claimed pattern on the right, we use $\ell_J = -\ell_{\{1, \dots, n\} \setminus J}$ to treat the second. \square

Remark. Via the projection $\mathbb{Z}^n \rightarrow \Lambda_R^*$, $e^i \mapsto \ell_i$, one obtains $\text{tail}(\mathcal{S})$ as the image of the $\binom{n}{k}$ (out of 2^n) coordinate orthants with sign pattern $(n-k, k)$.

In the case of $k = 2$, Kapranov has shown in [Kap] that $\text{Grass}(2, n) //^{\text{ch}T} = \overline{M}_{0,n}$, where the latter denotes the moduli space of n -pointed, stable, rational curves. For every partition $B = [B' \sqcup B'' = \{1, \dots, n\}]$ with $\#(B'), \#(B'') \geq 2$, there is a distinguished prime divisor D_B on $\overline{M}_{0,n}$ – it is the closure of the set of curves with two (mutually intersecting) components and point distribution according to B .

Theorem 4.2. *The T -variety $\text{Grass}(2, n)$ corresponds to the fancy divisor $\mathcal{S} = \sum_B \mathcal{S}_B \otimes D_B$ on $(\overline{M}_{0,n}, \Lambda_R^* = \langle \ell_i \rangle)$ where \mathcal{S}_B is the polyhedral subdivision arising from $\text{tail}(\mathcal{S}) = \{\langle \pm \ell_1, \dots, \pm \ell_n \rangle \subseteq N_{\mathbb{Q}} \mid \text{two negative signs}\}$ by replacing the origin with the compact edge C_B bounded by the vertices $\frac{\#B'-1}{n-2} \ell_{B'}$ and $\frac{\#B'+1-n}{n-2} \ell_{B'} = \frac{\#B'-1}{n-2} \ell_{B'} - \ell_{B'}$.*



One of the three fancy coefficients $\mathcal{S}_B \subseteq \mathbb{Q}^3$ of \mathcal{S} on $\overline{M}_{0,4} = \mathbb{P}^1$ describing $\text{Grass}(2, 4)$

Remark. 1) Replacing B' by B'' does not alter the edge C_B . Its center is $\frac{2\#B'-n}{2(n-2)} \ell_{B'} = \frac{2\#B''-n}{2(n-2)} \ell_{B''}$, and, as a vector, it equals $\pm \ell_{B'} = \mp \ell_{B''}$. In particular, it is the balanced partitions B that lead to edges being centered in the origin.

2) In Recipe 3.3, we mentioned the non-canonical choice of a section $s : N' \rightarrow \mathbb{Z}^\ell$. Different choices lead to different fancy divisors – but then their elements do only differ by a polyhedral principal divisor as was explained in Section 2, right after Definition 2.2. Nevertheless, all occurring polyhedral coefficients in the pp-divisors of \mathcal{S} are lattice polyhedra. This makes it possible to encode \mathcal{S} by even choosing a *rational* section $s : N'_\mathbb{Q} \rightarrow \mathbb{Q}^\ell$. Among those, there is one that is canonical – and this was used in the previous theorem.

The proof of Theorem 4.2 does not involve a glueing of the affine charts and their corresponding pp-divisors as mentioned at the end of Section 3. Instead, we will treat the affine cone over $\text{Grass}(k, n)$ with respect to its Plücker embedding. Hence, we will take a short break and proceed with a chapter addressing the relation between fancy divisors of projective varieties and the pp-divisors of their affine cones in general. Then, the proof of Theorem 4.2 will be given in Section 6.

5. AFFINE CONES OF PROJECTIVE T -VARIETIES

Let $Z \subseteq \mathbb{P}_\mathbb{K}^N$ be a projectively normal variety and denote by $C(Z) \subseteq \mathbb{K}^{N+1}$ its affine cone; let them be equipped with compatible actions of an n - and an $(n+1)$ -dimensional torus T and \widetilde{T} , respectively. These actions may be described by exhibiting the degrees of the homogeneous coordinates z_0, \dots, z_N or of the coordinates $z_0/z_v, \dots, z_N/z_v$ of the affine charts $U(z_v) \subseteq \mathbb{P}^N$ in the character groups \widetilde{M} and M , respectively. Eventually, denoting by $p : \widetilde{N} \rightarrow N$ the projection corresponding to

$\tilde{T} \rightarrow T$, leads to the following commutative, mutually dual diagrams:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' & \longrightarrow & \ker \underline{1} & \xrightarrow{\text{deg}} & M & \longrightarrow & 0 \\
& & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'' & \longrightarrow & \mathbb{Z}^{N+1} & \xrightarrow{\text{deg}} & \widetilde{M} & \longrightarrow & 0 \\
& & & & \downarrow \underline{1} & & \downarrow & & \\
& & & & \mathbb{Z} & \xlongequal{\quad} & \mathbb{Z} & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
\qquad
\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{N} & \longrightarrow & \mathbb{Z}^{N+1} & \xrightarrow[\widetilde{\pi}'']{s} & N'' & \longrightarrow & 0 \\
& & \downarrow p & & \downarrow p & & \parallel & & \\
0 & \longrightarrow & N & \longrightarrow & \mathbb{Z}^{N+1}/\underline{1} & \xrightarrow{\pi''} & N'' & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

The Chow quotients do not distinguish between a projective variety and its affine cone – we have $Y := Z //^{\text{ch}} T = C(Z) //^{\text{ch}} \tilde{T}$, which is a closed subvariety of $Y'' := \mathbb{P}^N //^{\text{ch}} T = \mathbb{K}^{N+1} //^{\text{ch}} \tilde{T}$. Similar to the local situation at the end of Section 3, the latter equals the toric variety $\mathbb{T}\mathbb{V}(\Sigma'', N'')$ associated to the fan Σ'' arising from the coarsest common refinement of the $\widetilde{\pi}''_{\mathbb{Q}}(\mathbb{Q}_{\geq 0}^{N+1}$ -faces) or, equivalently, of $\pi''_{\mathbb{Q}}$ (cones of \mathbb{P}^N -fan) in $N''_{\mathbb{Q}} := N'' \otimes_{\mathbb{Z}} \mathbb{Q}$. From [KSZ] and [Kap], we know that Σ'' is the normal fan of the secondary polytope $\text{Sec}(\Delta)$ of $\Delta := \text{conv}\{\text{deg } z_v \mid v = 0, \dots, N\} \subseteq \widetilde{M}_{\mathbb{Q}}$. The faces of $\text{Sec}(\Delta)$ or, equivalently, the cones of Σ'' correspond to the so-called regular subdivisions of Δ . This can be made explicit by assigning to a $c \in \tau \in \Sigma''$ the normal fan

$$\mathcal{N}(\widetilde{\pi}''^{-1}(c) \cap \mathbb{Q}_{\geq 0}^{N+1}) \leq \mathcal{N}(\ker \widetilde{\pi}'' \cap \mathbb{Q}_{\geq 0}^{N+1})$$

with “ \leq ” meaning “is a subdivision of”. Since the latter fan equals the cone over Δ , the first provides a subdivision $S(\tau)$ of the original Δ , cf. [KSZ, Lemma 2.4].

Lemma 5.1. *Let $\tau \in \Sigma''$ be a cone and $y \in \text{orb}(\tau) \subseteq \mathbb{T}\mathbb{V}(\Sigma'', N'') = \mathbb{P}^N //^{\text{ch}} T$. Then, y corresponds to a cycle $\sum_{\nu} \lambda_{\nu} Z_{\nu}$ with certain T -orbits $Z_{\nu} \subseteq \mathbb{P}^N$, and their images $\mu(Z_{\nu})$ under the moment map $\mu : \mathbb{P}^N \rightarrow \widetilde{M}_{\mathbb{R}}$ yield the subdivision $S(\tau)$.*

Proof. This follows directly from [KSZ, Proposition 1.1], its reformulation in [Kap, (0.2.10)], and the Sections [Kap, (1.2.6+7)] dealing with the moment map. \square

Following Recipe 3.3, the pp-divisor $\mathcal{D} = \sum_i \Delta_i \otimes D_i$ on (Y, \widetilde{N}) that describes $C(Z)$ is built from $\Delta_i := (\widetilde{\pi}''_{\mathbb{Q}}^{-1}(c^i) \cap \mathbb{Q}_{\geq 0}^{N+1}) - s(c^i)$ and $D_i := \overline{\text{orb}(c^i)}|_Y$ with $c^i \in \Sigma''^{(1)}$ browsing through the rays of Σ'' . As before, $s : N'' \rightarrow \mathbb{Z}^{N+1}$ denotes a section of $\widetilde{\pi}''$.

Definition 5.2. Denoting by $\{E_0, \dots, E_N\}$ the canonical basis of \mathbb{Z}^{N+1} in the left diagram, we define, for $v = 0, \dots, N$, the faces $\partial_v(\Delta_i) := \Delta_i \cap (E_v^{\perp} - s(c^i))$ of Δ_i .

Remark. 1) Let $\text{face}(\Delta, E_v) := \{a \in \Delta \mid \langle a, E_v \rangle = \min \langle \Delta, E_v \rangle\} \subseteq \widetilde{N}_{\mathbb{Q}}$ denote the Δ -face minimizing the linear form E_v . Then,

$$\partial_v(\Delta_i) = \begin{cases} \text{face}(\Delta_i, E_v) & \text{if } \min \langle \Delta_i + s(c^i), E_v \rangle = 0, \text{ but} \\ \emptyset & \text{if } \min \langle \Delta_i + s(c^i), E_v \rangle > 0. \end{cases}$$

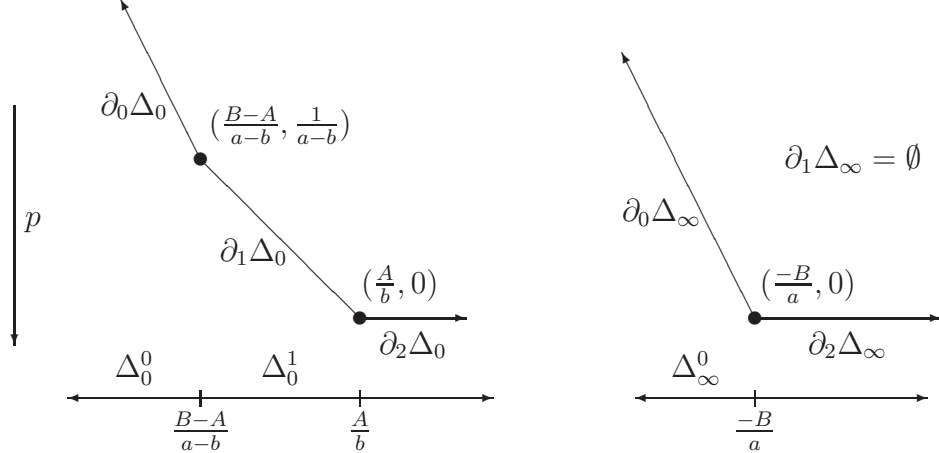
The union of these Δ_i -faces is $\bigcup_v \partial_v(\Delta_i) = \partial(\Delta_i) = (\tilde{\pi}_{\mathbb{Q}}''^{-1}(c^i) \cap \partial\mathbb{Q}_{\geq 0}^{N+1}) - s(c^i)$.

2) To decide whether $\partial_v(\Delta_i)$ is empty or not might be a cumbersome question. However, if, for $v = 0, \dots, N$, the cones $\text{face}(\text{tail } \Delta, E_v)$ are mutually different facets of the cone $\text{tail}(\Delta)$, then one *always* has $\partial_v(\Delta_i) \neq \emptyset$.

Example 5.3. Let $N = 2$ and consider the action of $\tilde{T} = (\mathbb{K}^*)^2$ on $\mathbb{K}^3 = C(\mathbb{P}^2)$ that is given by the weights $\deg z_0 = [a, 1]$, $\deg z_1 = [b, 1]$, $\deg z_2 = [0, 1]$ with relatively prime integers $a \geq b \geq 1$. Choosing $A, B \in \mathbb{Z}$ with $0 < A \leq b$ and $0 \leq B < a$ and $Aa - Bb = 1$ yields $Y = \mathbb{P}^1$, $\tilde{\pi}'' = (b, -a, a - b)$, $s = (-B, -A, 0)^T$, and E_0, E_1, E_2 acting on $\tilde{N} = \mathbb{Z}^2$ as $[a, 1]$, $[b, 1]$, and $[0, 1]$, respectively. Hence, with $c^0 := 1$ and $c^\infty := -1$,

$$\mathcal{D} = (\text{conv}\{(\frac{B-A}{a-b}, \frac{1}{a-b}), (\frac{A}{b}, 0)\} + \sigma) \otimes \{0\} + ((\frac{-B}{a}, 0) + \sigma) \otimes \{\infty\}$$

with tail cone $\sigma := \langle (1, 0), (-1, a) \rangle \subseteq \mathbb{Q}^2$. We obtain the following picture of $p : \tilde{N} \rightarrow N$ which equals the projection $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ onto the first summand:



The action of the three linear forms E_0, E_1, E_2 is reflected by the three different slopes of $\partial\Delta_0$. However, it requires a closer look to realize that $\partial_1\Delta_\infty = \emptyset$ instead of $\partial_1\Delta_\infty = \{(\frac{-B}{a}, 0)\}$.

Theorem 5.4. *The projective T -variety $Z \subseteq \mathbb{P}_{\mathbb{K}}^n$ is given by the fancy divisor $p(\partial\mathcal{D}) := \{\sum_i \Delta_i^v \otimes D_i \mid v = 0, \dots, N\}$ with $\Delta_i^v := p(\partial_v \Delta_i)$.*

Proof. The inclusion of the two Chow quotients $Y \subseteq Y''$ may be obtained by converting s into a section $\tilde{M} \hookrightarrow \mathbb{Z}^{N+1}$ and using this to interpret the original equations of $Z \subseteq \mathbb{P}_{\mathbb{K}}^n$ or $C(Z) \subseteq \mathbb{K}^{n+1}$ as elements of $\mathbb{K}[M'']$. This leads to a closed subscheme of the dense torus $\text{Spec } \mathbb{K}[M''] \subseteq Y''$, and Y arises as the normalization of its closure in Y'' . Anyway, this procedure keeps the polytopal part unchanged, and we may assume, without loss of generality, that $Z = \mathbb{P}_{\mathbb{K}}^n$, $C(Z) = \mathbb{K}^{n+1}$, and $Y = Y''$.

Let $\mathcal{S} = \{\mathcal{D}^v \mid v = 0, \dots, N\}$ be the fancy divisor describing \mathbb{P}^N as a T -variety and denote by $E^0, \dots, E^N \in \mathbb{Z}^{N+1}$ the dual basis of E_\bullet . Then, up to the s shift, we have $\partial_v(\Delta_i) = \tilde{\pi}_{\mathbb{Q}}''^{-1}(c^i) \cap \langle E^0, \dots, \widehat{E^v}, \dots, E^N \rangle \subseteq \mathbb{Q}^{N+1}$. On the other hand, the polyhedral cone representing the v -th affine chart of $\mathbb{P}_{\mathbb{K}}^N$ equals $\sigma_v := \langle p(E^0), \dots, p(\widehat{E^v}), \dots, p(E^N) \rangle \subseteq \mathbb{Q}^{N+1}/\mathbb{1}$. Hence, up to the same s shift, the i -th

summand of the pp-divisor \mathcal{D}^v equals $\pi_{\mathbb{Q}}''^{-1}(c^i) \cap \sigma_v \subseteq \mathbb{Q}^{N+1}/\underline{1}$. Now, the claim follows from the fact that $p : \mathbb{Q}^{N+1} \rightarrow \mathbb{Q}^{N+1}/\underline{1}$ induces an isomorphism from $\partial_v(\Delta_i)$ to $\pi_{\mathbb{Q}}''^{-1}(c^i) \cap \sigma_v$. \square

Problem 5.5. If, as in Example 5.3, the assumption of the second remark after Definition 5.2 fails, then one can, nevertheless, still consider the polyhedral subdivisions $\mathcal{S}_i := p(\partial\Delta_i)$ of $N_{\mathbb{Q}}$. To turn the formal sum $\mathcal{S} = \sum_i \mathcal{S}_i \otimes D_i$ into a decent fancy divisor \mathcal{S} , we just need to know about the labels as explained at the end of Section 2. Is there an easy, direct way to find out about them, i.e. to decide about the emptiness of the $\partial_v(\Delta_i)$ or their images Δ_i^v ?

6. THE AFFINE CONE OVER $\text{Grass}(2, n)$

This section consists of the proof of Theorem 4.2. Since we are going to use the affine cone over the Plücker embedding $\text{Grass}(2, n) \subseteq \mathbb{P}^{\binom{n}{2}-1}$, we will specify the situation of the previous section into

$$0 \longrightarrow M'' \longrightarrow \mathbb{Z}^{\binom{n}{2}} \xrightarrow{\text{deg}} \widetilde{M} \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \widetilde{N} \xrightarrow{\text{deg}^*} \mathbb{Z}^{\binom{n}{2}} \xrightarrow{\pi := \widetilde{\pi}''} N'' \longrightarrow 0,$$

$$E_{ij} \longmapsto e_i + e_j \qquad \qquad \qquad E^{ij} \longmapsto c^{ij}$$

where $\widetilde{M} := \{u \in \mathbb{Z}^n \mid 2 \mid \sum_i u_i\}$. Note that we wrote $\mathbb{Z}^{\binom{n}{2}}$ instead of $\Lambda^2 \mathbb{Z}^n$ – the reason being that its basis elements represent the exponents of the Plücker coordinates z_{ij} , i.e. we have $z_{ji} = -z_{ij}$, but $E_{ji} = E_{ij}$. Moreover, $\widetilde{N} = \mathbb{Z}^n + (\frac{1}{2} + \mathbb{Z})^n$, and there is a canonical surjection $\widetilde{N} \rightarrow \Lambda_R^*$ with $e^i \mapsto \ell_i$, and $e := \frac{1}{2}\underline{1}$ generating the kernel.

In Section 4, we considered partitions $B = [B' \sqcup B'' = \{1, \dots, n\}]$. Extending the set of basis vectors $E^{ij} \in \mathbb{Z}^{\binom{n}{2}}$, we define for a subset $B' \subseteq \{1, \dots, n\}$

$$E^{B'} := \sum_{i < j, \text{both} \in B'} E^{ij} = \frac{1}{\#B' - 2} \sum_{b \in B'} E^{B' \setminus b},$$

where the second equality requires that $\#B' \neq 2$. Thus, the fact $\text{deg}^*(e^i) = \sum_{j \neq i} E^{ij}$ does upgrade to the formula

$$\text{deg}^* : e^{B'} := \sum_{b \in B'} e^b \mapsto 2E^{B'} + \sum_{i \in B', j \in B''} E^{ij}.$$

In particular, $\text{deg}^*(e^{B'} - e^{B''}) = 2(E^{B'} - E^{B''})$. Defining $c^{B'} := \pi(E^{B'}) \in N''$, this implies that $c^{B'} = \sum_{i \in B', j \in B''} c^{ij} = c^{B''}$, and this element will be just called c^B .

Lemma 6.1. *With $\sigma := \widetilde{N}_{\mathbb{Q}} \cap \mathbb{Q}_{\geq 0}^{\binom{n}{2}} = \text{cone}(\Delta(2, n))^{\vee}$, the “positive fiber” of c^B is*

$$\pi^{-1}(c^B) \cap \mathbb{Q}_{\geq 0}^{\binom{n}{2}} = \overline{E^{B'} E^{B''}} + \sigma.$$

In particular, c^B induces the subdivision $\Delta(2, n) = \Delta(2, n)'_B \cup \Delta(2, n)''_B$ with the big cells $\Delta(2, n)'_B = \text{conv}\{e_i + e_j \mid i \in B'\}$ and $\Delta(2, n)''_B = \text{conv}\{e_i + e_j \mid j \in B''\}$.

Proof. Let $P \in \pi^{-1}(c^B) \cap \mathbb{Q}_{\geq 0}^{\binom{n}{2}}$, i.e. $P = E^{B'} + \deg^*(v)$ with $v = \sum_{i \in B'} \lambda_i e^i + \sum_{j \in B''} \mu_j e^j$ and $\lambda_\bullet + \lambda_\bullet + 1 \geq 0$, $\lambda_\bullet + \mu_\bullet \geq 0$, and $\mu_\bullet + \mu_\bullet \geq 0$. If the minimal value of $\lambda_\bullet + \lambda_\bullet$ is negative, then we denote it by $-c \geq -1$. Then, $E := (1-c)E^{B'} + cE^{B''}$ sits on the line segment, and $P = E + \deg^*(\tilde{v})$ with \tilde{v} having the coordinates $\tilde{\lambda}_i = \lambda_i + c/2$ and $\tilde{\mu}_j = \mu_j - c/2$, hence satisfying $\tilde{\lambda}_\bullet + \tilde{\lambda}_\bullet \geq 0$, $\tilde{\lambda}_\bullet + \tilde{\mu}_\bullet \geq 0$, and $\tilde{\mu}_\bullet + \tilde{\mu}_\bullet \geq 0$. \square

Since the decompositions of $\Delta(2, n)$ induced from partitions with $\#B', \#B'' \geq 2$ are proper and the coarsest possible, it follows that the associated elements c^B are rays in the fan Σ'' , hence, they provide divisors $\overline{\text{orb}(c^B)} \subseteq \mathbb{T}\mathbb{V}(\Sigma'', N'') = \mathbb{P}^{\binom{n}{2}-1} //^{\text{ch}} T$. Moreover, these decompositions are ‘‘matroid decompositions’’ in the sense of [Kap, Definition 1.2.17]. Thus, they correspond to 1-codimensional so-called Chow strata in $\text{Grass}(2, n) //^{\text{ch}} T$ which are, by Lemma 5.1, the restrictions of $\overline{\text{orb}(c^B)}$ via the Plücker embedding.

Eventually, [Kap, Theorem 4.1.8] tells us that $\text{Grass}(2, n) //^{\text{ch}} T \cong \overline{M}_{0, n}$. Under this isomorphism, by [Kap, Corollary 4.1.12], the Chow stratum $\overline{\text{orb}(c^B)}|_{\text{Grass}(2, n) //^{\text{ch}} T}$ corresponds to the divisor $D_B \subseteq \overline{M}_{0, n}$ introduced right before Theorem 4.2.

Proposition 6.2. *The affine \tilde{T} -variety $\text{cone}(\text{Grass}(2, n))$ corresponds to the pp-divisor $\mathcal{D} = \sum_B \Delta_B \otimes D_B$ on $(\overline{M}_{0, n}, \tilde{N})$ with $b := \#B'$ and*

$$\Delta_B = \frac{b-1}{n-2} e^{B'} - \frac{(b-1)b}{2(n-2)(n-1)} \cdot \mathbb{1} + \frac{1}{2} \overline{0(e^{B''} - e^{B'})} + \sigma.$$

Proof. The pp-divisor in question is the restriction of the pp-divisor describing the affine \tilde{T} -space $\mathbb{C}^{\binom{n}{2}}$. Hence, for every ray $c \in \Sigma''^{(1)}$, we have to determine the shifted ‘‘positive fiber’’ via π – yielding the coefficient of $\text{orb}(c)$ restricted to $\text{Grass}(2, n) //^{\text{ch}} T \cong \overline{M}_{0, n}$.

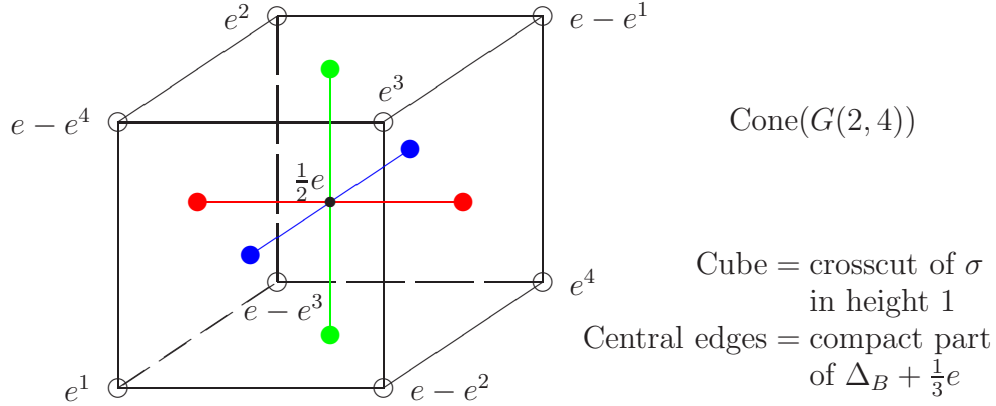
First, the choice of a section $s : N'' \rightarrow \mathbb{Z}^{\binom{n}{2}}$ is equivalent to the choice of a retraction $t : \mathbb{Z}^{\binom{n}{2}} \rightarrow \tilde{N}$ of \deg^* . Moreover, according to a remark right after Theorem 4.2, we prefer to preserve symmetries, hence we would rather choose a *rational* retraction $t : \mathbb{Q}_{\geq 0}^{\binom{n}{2}} \rightarrow \tilde{N}_{\mathbb{Q}}$. Defining $t(E^{ij}) := \frac{1}{n-2} e^i + \frac{1}{n-2} e^j - \frac{1}{(n-2)(n-1)} \mathbb{1}$, one obtains

$$t(E^{B'}) = \frac{\#B' - 1}{n-2} e^{B'} - \frac{(\#B' - 1) \cdot \#B'}{2(n-2)(n-1)} \cdot \mathbb{1},$$

which does nicely fit with the previous formula $e^{B'} - e^{B''} = 2(E^{B'} - E^{B''})$. Now, we use t to shift the ‘‘positive fiber’’ of $c = c^B$ from Lemma 6.1 into $\tilde{N}_{\mathbb{Q}}$. Since $t|_{\sigma} = \text{id}$, hence $t(\sigma) = \sigma$, and the result is

$$\begin{aligned} \Delta_B &= t(\pi^{-1}(c^B) \cap \mathbb{Q}_{\geq 0}^{\binom{n}{2}}) = \overline{t(E^{B'})t(E^{B''})} + \sigma \\ &= t(E^{B'}) + \overline{0(E^{B''} - E^{B'})} + \sigma. \end{aligned}$$

On the other hand, if c is not of the form c^B , then Lemma 5.1 implies that, inside $\mathbb{T}\mathbb{V}(\Sigma'', N'')$, the divisor $\overline{\text{orb}(c)}$ is disjoint to the closed subvariety $\text{Grass}(2, n) //^{\text{ch}}$. \square



Now, we can finish the proof of Theorem 4.2. According to Theorem 5.4, it remains to project the facets of the polyhedra Δ_B along the map $\tilde{N}_{\mathbb{Q}} \rightarrow \Lambda_R^*$ with $e^i \mapsto \ell_i$ and $e^{B'} \mapsto \ell_{B'} = \sum_{i \in B'} \ell_i$. The relation $\ell_1 + \dots + \ell_n = 0$ translates into $\ell_{B''} = -\ell_{B'}$ in Λ_R^* . Hence, the vertices of the compact edge of Δ_B turn into $\frac{b-1}{n-2} \ell_{B'}$ and $\frac{n-b-1}{n-2} \ell_{B''} = \frac{b+1-n}{n-2} \ell_{B'}$. Its center is $\frac{2b-n}{2(n-2)} \ell_{B'}$, and, as a vector, it equals $\ell_B := \pm \ell_{B'} = \mp \ell_{B''}$. The tail cones of the Δ_B -facets yield the tail fan which was already described in Proposition 4.1. \square

7. APPENDIX: THE GLOBAL/LOCAL COMPARISON

As additional information, a comparison of the global situation with that of the affine chart $\mathbb{A} = \text{Grass}(2, n) \cap [z_{12} \neq 0]$ might provide some further insight. While the Plücker embedding does not go along with accompanying maps between the lattices Λ_R^* and \tilde{N} (it is not equivariant with respect to some map $T \hookrightarrow (\mathbb{C}^*)^{\binom{n}{2}}$), the retraction behaves well. The expression of the local coordinates in terms of the quotients z_{ij}/z_{12} leads to the following commutative diagram which connects the exact sequences of the beginning of Section 6 (first line) and those of Recipe 3.3 at the end of Section 3 (second line with $\ell = 2(n-2)$).

We have used the symbols f^i ($i = 1, 2$) and g^j ($j = 3, \dots, n$) to denote a basis for \mathbb{Z}^2 and \mathbb{Z}^{n-2} , respectively. The latter elements satisfy $\sum_j g^j = 0$ in N' . Moreover, for a subset $J \subseteq \{3, \dots, n\}$, we define $g^J := \sum_{j \in J} g^j$.

In the local chart \mathbb{A} , only special partitions B are “visible” – the elements 1 and 2 are not allowed to sit in a common set B' or B'' . This is reflected in the rightmost vertical map of the above diagram: If B is “invisible” in \mathbb{A} , then c^B maps to 0. If B does separate 1 and 2, then we mean with “ $B \setminus 2$ ” the set among B' and B'' originally containing 2, but removing it then.

Applying the functor $\mathbb{T}\mathbb{V}(\cdot)$ of toric varieties to the map $(N'', \Sigma'') \rightarrow (N', \Sigma')$ yields $Y'' \rightarrow Y'$. Composing this with the Plücker embedding $Y \hookrightarrow Y''$ yields the birational modification $\overline{M}_{0,n} = Y \rightarrow Y'$ mentioned at the end of Section 3.

$$\begin{array}{ccccccc}
\mathbb{Z} \hookrightarrow \frac{1}{2} \cdot \mathbb{1} & & e^i \longmapsto & & \sum_{\bullet \neq i} E^{i \bullet} & & \\
0 \longrightarrow & (\tilde{N} = \mathbb{Z}^n + (\frac{1}{2} + \mathbb{Z})^n) & \xrightarrow{\text{deg}^*} & \mathbb{Z}^{\binom{n}{2}} & \xrightarrow{\pi := \tilde{\pi}''} & N'' & \longrightarrow 0 \\
& \downarrow e^i \mapsto \ell_i & & \downarrow \begin{array}{l} e^{1j} \mapsto f^2 \otimes g^j \\ e^{2j} \mapsto f^1 \otimes g^j \\ e^{12} \mapsto -\sum f^\bullet \otimes g^\bullet \end{array} & & \downarrow c^B \mapsto g^{B \setminus 2} & \\
0 \longrightarrow & (\Lambda_R^* = \langle \ell_1, \dots, \ell_n \rangle) & \xrightarrow{\text{deg}_{\text{loc}}^*} & \mathbb{Z}^2 \otimes \mathbb{Z}^{n-2} & \xrightarrow{\pi_{\text{loc}} := \pi'} & (\mathbb{Z}^{n-2} / \mathbb{1} = N') & \longrightarrow 0 \\
& \swarrow & & \downarrow \begin{array}{l} \ell_{i=1,2} \mapsto -f^i \otimes \sum_j g^j \\ \ell_{j \geq 3} \mapsto f^1 \otimes g^j + f^2 \otimes g^j \end{array} & & & \\
0 & & & & & & \\
& & & & & f^1 \otimes g^j \longmapsto g^j & \\
& & & & & f^2 \otimes g^j \longmapsto -g^j. &
\end{array}$$

REFERENCES

- [AlHa] Altmann, K.; Hausen, J.: Polyhedral Divisors and Algebraic Torus Actions. *Math. Ann.* **334**, 557-607 (2006).
- [AHS] Altmann, K.; Hausen, J.; Süß: Algebraic Torus Actions. *ArXiv:math.AG/0606772*.
- [FuHa] Fulton, W.; Harris, Joe: Representation Theory – A First Course. *Graduate Texts in Mathematics* **129**, Springer-Verlag 1991.
- [Kap] Kapranov, M.M.: Chow Quotients of Grassmannians I. *Advances in Soviet Mathematics*, **16/2**, 29-110 (1993).
- [KSZ] Kapranov, M.M.; Sturmfels, B.; Zelevinsky, A.V.: Quotients of toric varieties. *Math. Ann.* **290** (1991), 643-655. Torische Chow-Quotienten.
- [Spr] Springer, T.A.: Linear Algebraic Groups. *Progress in Mathematics* **9**, Birkhäuser-Verlag Boston.

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