FINITENESS RESULTS FOR TEICHMÜLLER CURVES

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Abstract. We show that for each genus there are only finitely many algebraically primitive Teichmüller curves C, such that i) C lies in the hyperelliptic locus and ii) C is generated by an abelian differential with two zeros of order $g-1$.
We prove moreover that for these Teichmüller curves the trace field of the affine group is not only totally real but cyclotomic.

Introduction

A Teichmüller curve is an algebraic curve in the moduli space of curves of genus g, denoted by M_g, whose preimage in Teichmüller space is a complex geodesic for the Teichmüller metric. Teichmüller geodesics are obtained as the orbit of a pair (X^0, q^0) of a Riemann surface X^0 plus a quadratic differential q^0 on X^0 under the action of $\mathrm{SL}_2(\mathbb{R})$. Those (few) pairs (X^0, q^0) that give Teichmüller curves are called Veech surfaces. We restrict ourselves to the case when $q^0 = (\omega^0)^2$ is a square of a holomorphic differential. The case of proper squares might be analysed using the canonical double covering of X^0, that makes the pullback into a square. We remark that Teichmüller curves naturally lift to the bundle ΩM_g over M_g of holomorphic one-forms. This bundle is stratified according to the multiplicity of the zeros of the one-form. See Section 1 for more details.

The first examples of Teichmüller curves were obtained as coverings of the torus ramified over one point. There are infinitely many of them in each stratum. First examples not of this type were discovered by Veech ([Ve89]). In particular the trace field of the affine group (see Section 1) is not $\mathbb{Q}$ in Veech's examples.
In genus 2 there are infinitely many non-torus coverings in the stratum with one double zero but only a single one in the stratum with a two zeros. If one fixes one additional discrete parameter (the discriminant of the order all curves parametrized by such a Teichmüller curve have real multiplication with), the number becomes finite also for Teichmüller curves in the stratum with one double zero. In fact there are one or two of them according to the congruence class of the discriminant mod 8. This classification is contained in [McM04a], [McM04b] and [McM04c].

To go beyond genus 2 we recall from [Mö04a] that the family of Jacobians over a Teichmüller curve splits into an r-dimensional part with real multiplication and some rest, where r is the field extension degree of the trace field over $\mathbb{Q}$.
A Teichmüller curve in M_g is called *algebraically primitive* if the trace field has degree g over $\mathbb{Q}$. This implies that the curve is *geometrically primitive*, i.e. that the pair (X^0, ω^0) does not arise from a surface of lower genus plus a differential via a covering construction. Both notions coincide in genus two, but in general the converse implication is not true.

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At the time of writing the following is known about primitive Teichmüller curves:
Among Veech's examples there are infinitely many algebraically primitive ones, but at most one for each genus. Besides this there are series of examples and sporadic ones in [Vo96], [Wa98], [KeSm00]. Only finitely many of them are algebraically primitive and for each genus there are only finitely many examples. The recent work of McMullen ([McM05]) contains infinitely many geometrically primitve (though not algebraically primitive examples) for the genera 3, 4 and 5.

The purpose of the present work is to obtain some finiteness results valid in all genera. We cannot hope for such results for imprimitive curves without fixing additional discrete parameters. For geometrically primitive but algebraically imprimitive Teichmüller curves it seems unclear what to expect. If we restrict to algebraically primitive Teichmüller curves we show, generalizing [McM04c]:

Theorem 3.1: For fixed genus g there are only finitely many algebraically primitive Teichmüller curves in the connected component of the stratum $\Omega M_g(g-1, g-1)$, that parametrizes hyperelliptic curves.

We will consider the family of curves $f : X \to C$ over a Teichmüller curve C or over a suitable cover of C. Recall from [Mö04b] that the zeros of the generating differential ω^0 determine sections of f. In the algebraically primitive case the difference of any two of those sections is a torsion element of the relative Jacobian. The theorem is an instance of the philosophy that torsion points on families of curves are rare. It might be possible to show the same type of result for differentials with more zeros instead of hyperelliptic ones. But using the same methods the combinatorics become quite complicated then.

We briefly outline the strategy of our proof:

i) From an argument in [Mö04b] we deduce that an algebraically primitive Teichmüller curve in $\Omega M_g(g-1, g-1)^{\rm hyp}$ has a reducible and an irreducible degeneration (Thm. 2.3) in say the vertical and horizontal direction.
ii) The irreducible degeneration is used to bound the torsion order (Prop. 3.5 and Section 4). This limits the suitably normalized *widths* of the cylinders in the horizontal direction to a finite set. It generalizes the discussion of sine ratios in Section 2 of [McM04c]. Prop. 3.5 has the flavour of the toric case of the Mordell-Lang conjecture. Yet none of the versions in the literature seems strong enough to cover what we need.
iii) The reducible degneration is used to relate the torsion order and the *moduli* of the cylinders in the vertical direction (Thm. 2.4).
iv) The combination of these informations limits the possibilties for the flat geometry of a Veech surface to a finite number (see the prototype in Figure 2 and the end of Section 3).

As a byproduct of the proof we obtain:

Corollary 3.8: The trace field of an algebraically primitive Teichmüller curve in the stratum $\Omega M_g(g-1, g-1)^{\rm hyp}$ is cyclotomic.

The cyclotomic fields appear roughly as follows: The normalisations of some degenerate fibres in the family over the Teichmüller curve are isomorphic to $\mathbb{P}^1$. Arranging the position of the zeros of the generating differential suitably, the preimages of the nodes are forced by the torsion condition to lie at roots of unities in $\mathbb{P}^1$. We deduce that enough periods of ω^0 lie in this cyclotomic field to conclude that the trace field is cyclotomic.

We remark that the trace fields of all presently known Teichmüller curves are cyclotomic. Based on the above Corollary one might conjecture that this holds in general, at least for Teichmüller curves with more than one zero.

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1. Notation

Strata of ΩM_g

We denote the tautological bundle over M_g by ΩM_g. Its points are pairs (X^0, ω^0) of a Riemann surface X^0 of genus g and a holomorphic differential (or equivalently: a one-form) $\omega^0 \in \Gamma(X^0, \Omega^1_{X^0})$. This space is naturally stratified by the type of multiplicities of the zeros of ω. Kontsevich and Zorich have determined the connected components of the strata ([KoZo03]).

A pair (X, ω) belongs to a *hyperelliptic stratum* if X is a hyperelliptic curve with involution σ and quotient map $\pi : X \to X/\langle\sigma\rangle \cong \mathbb{P}^1$, such that $\omega^2 = \pi^* q$ for a quadratic differential q on $\mathbb{P}^1$ with (a) $2g+1$ simple poles and a zero of order $2g-3$ or (b) $2g+2$ simple poles and a zero of order $2g-2$. In case (a) the pair belongs to $\Omega M_g(2g-2)$ while in case (b) the pair belongs to $\Omega M_g(g-1, g-1)$. The hyperellipic strata form connected components ([KoZo03] Thm. 1). They will be denoted by a superscript *hyp*. For $(X, \omega) \in \Omega M_g(g-1, g-1)^{\mathrm{hyp}}$ the involution σ interchanges the two zeroes of ω.

Note that there are other types of zeros of a pair (X, ω) such that X is hyperelliptic and such that ω^2 is the pullback of a quadratic differential on the quotient. But the above two cases are the only ones, where a connected component consists entirely of such hyperelliptic pairs.

The two hyperelliptic strata are the natural generalisation of the only two strata that exist for $g = 2$.

$\mathrm{SL}_2(\mathbb{R})$-action

There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on ΩM_g minus the zero section: Apply the $\mathbb{R}$-linear transformation to the local complex charts of X given by integrating ω to obtain a new complex structure and apply the $\mathbb{R}$-linear transformation to the real and imaginary parts of ω to obtain a new one-form, which is holomorphic for the new complex structure. For more details see e.g. [MaTa02] or [McM03]. This action obviously preserves the stratification of ΩM_g.

Teichmüller curves

Teichmüller curves are algebraic curves $C \to M_g$ in the moduli space of curves of that are geodesic for the Teichmüller metric. We deal here exclusively with Teichmüller curves generated by a pair (X^0, ω^0) i.e. whose natural lift to the bundle of quadratic differentials over M_g lies in the image of ΩM_g. Here $C = \mathbb{H}/\Gamma$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is the image of the affine group of (X^0, ω^0) (see e.g. [Mö04a]). Let $K = \mathbb{Q}(\mathrm{tr}(\gamma), \gamma \in \Gamma)$ be the trace field of Γ and $r := [K : \mathbb{Q}]$. Let $f : X \to C$ denote the universal family over some finite unramified cover of C, abusively denoted by the same letter. Let $\mathrm{Jac}\,(f) : \mathrm{Jac}\, X/C \to C$ denote the family of Jacobians. Recall from [Mö04a] that $\mathrm{Jac}\, X/C$ splits up to isogeny into a product of a family $g : A \to C$ of abelian varieties of dimension r with real multiplication by K and a family of abelian varieties of dimension $g - r$. Since the splitting up to isogeny is not unique we take $g : A \to C$ to be the maximal quotient in its isogeny class. This letter should cause no confusion with the genus of X^0.

We extend all the above families to families over $\overline{C}$, i.e. let $\overline{f} : \overline{X} \to \overline{C}$ be the stable model and $\widetilde{f} : \widetilde{X} \to \overline{C}$ the minimal semistable model with smooth total space $\tilde{X}$. Also let $\overline{g} : \overline{A} \to \overline{C}$ be the corresponding family of semiabelian varieties.

Néron Models for families of Jacobians
Let F denote the function field of the curve C. The Néron Model $\tilde{g} : Q \to \overline{C}$ of a family $\overline{g} : \overline{A} \to \overline{C}$ of semiabelian varieties is a group scheme, whose fibre over the generic point of $\overline{C}$ coincides with $\overline{A}_F$ and such that for any given smooth group scheme $\overline{Y} \to \overline{C}$ any map $\overline{Y}_F \to Q_F$ over F extends uniquely to a map $\overline{Y} \to Q$ over all $\overline{C}$. In particular sections of g extend to sections of $\tilde{g}$. In all the cases we consider Néron Models exist, see [BLR90].
In case of algebraically primitive Teichmüller curves, i.e. for $g = r$, the family $\overline{g}$ is just $\mathrm{Pic}^0(\overline{X}/\overline{C})$, i.e. line bundles on $\overline{X}$ that are of degree zero on each component of each fibre. The connected component of 1 of Q, denoted by Q^0, coincides with $\mathrm{Pic}^0(\overline{X}/\overline{C})$ in this case ([BLR90] Thm. 9.5.4 b)).

Torsion
Let $f : X \to C$ be the universal family over a Teichmüller curve generated by (X^0, ω^0) in the stratum $\Omega M_g(k_1, \ldots, k_r)$. Recall from [Mö04b] that, maybe after passing to a finite unramified cover of C, the zeros of ω^0 define sections $s_1, \ldots, s_r$ of f. For any pair (i, j) the difference $s_i - s_j$ is a torsion section of g. It extends to a section of $\tilde{g}$. Since (in characteristic zero) the kernel of multiplication by some integer is étale on any group scheme, in particular on the Néron Model ([BLR90] Lemma 7.3.2), the order of $(s_i - s_j)$ restricted to any fibre of $\tilde{g}$ equals the same number $N(i, j)$. In particular this holds for the fibres over the cusps.

2. Degenerations

We study the degenerate fibres and give a relation between the geometry of a degenerate fibre and the torsion order of the difference of the two zeroes, if $(X^0, \omega^0) \in \Omega M_g(g-1, g-1)$ generates an algebraically primitive Teichmüller curve.

Theorem 2.1. *Let $\overline{f} : \overline{X} \to \overline{C}$ be the universal family over a Teichmüller curve. The sum of the genera of the components of a singular fibre of $\overline{f}$ is at most $g - r$. In particular the degenerate fibres of an algebraically primitive Teichmüller curve have only rational components.*

Proof: A family of abelian varieties with real multiplication degenerates to a semi-abelian variety whose abelian part is trivial (see e.g. [Go02] Lemma 2.23). Hence the abelian part of the fibre of $\overline{g}$ over any cusp has dimension at most $g - r$.
Alternatively this can be deduced from the Clemens-Schmid exact sequence for a degeneration of Hodge structures and the explicit description of the local system in [Mö04a]. □

We recall how the degeneration of a Teichmüller curve is described via the euclidian geometry defined by (X^0, ω^0): A geodesic on X^0 has a well-defined slope and all geodesics with this slope form a *direction*. Veech dichotomy ([Ve89]) states that each direction that contains a geodesic joining two zeros or one zero to itself (a *saddle connection*) is *periodic* i.e. each geodesic in this direction is periodic or closed.
The closed geodesics of a periodic direction (say the horizontal one) sweep out cylinders C_i and we denote their core curves by γ_i. Consider the degenerate fibre obtained by applying $\mathrm{diag}(e^t, e^{-t})$ to (X^0, ω^0) for $t \to \infty$. Say this point corresponds to the cusp $c \in \overline{C} \setminus C$.

By [Ma75] the stable model of the singular (or 'degenerate') fibre X_c of f is obtained by squeezing the core curves of the C_i to points. Topologically the irreducible components of X_c are obtained by cutting along the γ_i.

Corollary 2.2. *Each direction of a Teichmüller curve in $\Omega M_g(k_1, \dots, k_s)$ has at least r and at most $r+s-1$ cylinders.*

Proof: Each component of the degeneration in the given direction contains at least one zero. □

For the rest of this section we suppose that (X^0, ω^0) generates *an algebraically primitive Teichmüller curve.*

Theorem 2.3. *For any two zeros Z_1 and Z_2 of ω^0 with $Z_1 \neq Z_2$ there is a direction with the following property:*
Let X_c denote the singular fibre corresponding to the degeneration in this direction and s_i the sections defined by the Z_i. Then s_1 and s_2 intersect X_c in different irreducible components.

Proof: We know that s_i does not intersect the degenerate fibre in a node. Suppose the statement was wrong. Then $s_1 - s_2$ defines a non-zero section of $\mathrm{Pic}^0(\overline{X}/\overline{C})$ over the completed Teichmüller curve $\overline{C}$. This is not possible by [Mö04b] Thm. 3.1: Its proof shows not only that there are only finitely many sections of g, but also that there are none of $\overline{g}$. □

Suppose from now on that the Teichmüller curve C is generated by a differential with two zeros of order $g-1$. By Thm. 2.3 there is a direction, say the vertical one, such that the corresponding singular fibre X_v has two components. The vertical direction has hence $g+1$ cylinders. Let γ_i denote the core curves of the cylinders. We number them in such a way that for $i = 1, \dots, a$ the curve γ_i degenerates to a node on the first component of X_v, while for $i = a+1, \dots, a+b$ the curve γ_i degenerates to a node on the second component. We enumerate the components of X_v such that $a \leq b$. Note that $a + b \leq g - 1$ since the two components of X_v intersect in at least two points: a core curve of a cylinder is not separating.

We denote by h_i^v the height and by b_i^v the width of the i-th vertical cylinder, i.e. the length of γ_i. Moreover let $m_i^v = h_i^v / b_i^v$ be the modulus of the i-th vertical cylinder.
It is remarked in Veech [Ve89] that the moduli m_i^v for $i = 1, \dots, g$ are commensurable. It is no loss of generality for the purposes below to rescale the generating differential of the Teichmüller curve such that $m_i^v \in \mathbb{N}$ and $\gcd(m_i^v, i = 1, \dots, g) = 1$.
A small simple loop in C around the cusp c obtained by degenerating in the vertical direction corresponds (compare [Ve89] Prop. 2.4) to the product

$$(\prod_{i=1}^{g+1} D_{\gamma_i}^{m_i^v})^k, \quad \text{where } D_{\gamma_i} \text{ is a Dehn twist along } \gamma_i.$$

Here k is some positive integer, which appears since we have taken (with abuse of notation) coverings of the Teichmüller curve C that may ramify at the cusps. Hence the loop is not necessarily a generator of the corresponding parabolic subgroup of the affine group. This means that in the stable model the node in X_v corresponding to γ_i is given by $xy = t^{m_i^v k}$, where t is a local coordinate of C at the cusp c and x, y are local coordinates of an embedding of a neighborhood of the node in the stable fibre into $\mathbb{C}^3$. In fact, the statement is local in the base C and in the total space and it reduces for $m_i^v k = 1$ to the easiest case of the Picard-Lefschetz transformation. The general case is obtained via base change. After resolving this singularity the fibre $\widetilde{X_v}$ of the semistable model with smooth total space of $\overline{f}$ contains a chain of $m_i^v k - 1$ rational (-2)-curves in the preimage of the node.

Theorem 2.4. *Let N denote the order of $s_2 - s_1$. Suppose that the moduli m_i^v are integers and let $m^v = \mathrm{lcm}\{m_i^v, i = a+b+1, \ldots, g+1\}$. Then the torsion order and the moduli of the cylinders are related by*

$$\sum_{i=a+b+1}^{g+1} \frac{m^v}{m_i^v} \text{ divides } N.$$

Proof: By the preceding discussion the fibre $\widetilde{X_v}$ looks as in the following figure. Lines correspond to components of the semistable fibre, intersection points are nodes and Z_1, Z_2 are the intersection points of the section s_i with the stable fibre. Section 9.6 in [BLR90],

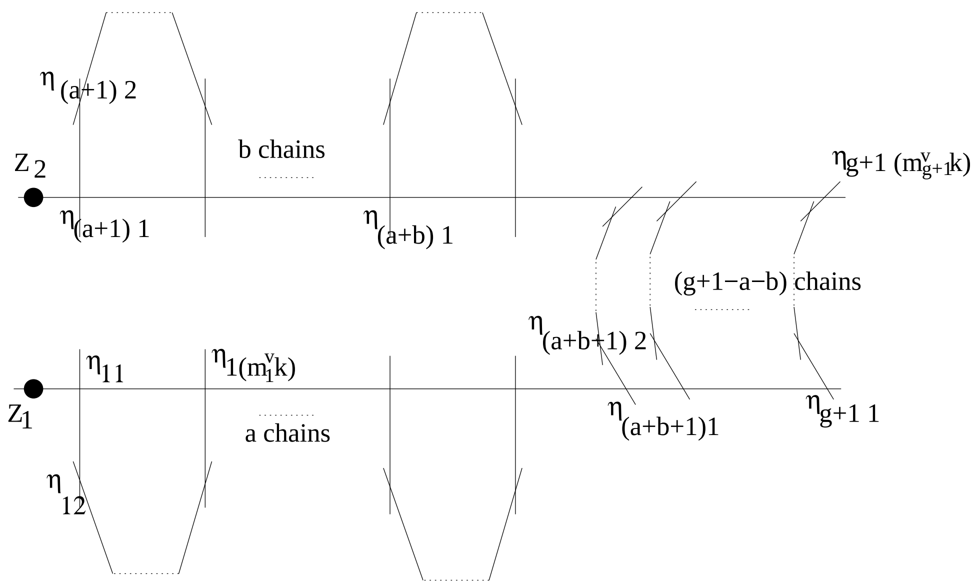

FIGURE 1. Semistable model of $\widetilde{X_v}$

in particular p. 283, gives a presentation of the component group Q/Q^0 of the Néron model of a flat family of curves with smooth total space over a discrete valuation ring. This can be applied to the localization of $\tilde{f}$ at the 'vertical' cusp: Let η_{ij} denote the nodes of the singular fibre, i.e. the edges of the dual (intersection) graph used in loc. cit. The component group is generated by the η_{i1} and $\eta_{ij} - \eta_{i\,j-1}$ for $i = 1, \ldots, g+1$ and

$j = 2, \dots, m_i^v k$ with the following relations: The components of $\widetilde{X_v}$ contribute

$$\begin{array}{rcll} \eta_{ij} - \eta_{i\,j-1} & = & 0 & i = 1, \dots, g+1,\ j = 2, \dots, m_i^v k \\ \sum_{i=a+b+1}^{g+1} \eta_{i1} + \sum_{i=1}^{a} (\eta_{i1} - \eta_{i\,m_i^v k}) & = & 0 & \\ -\sum_{i=a+b+1}^{g+1} \eta_{i\,m_i^v k} + \sum_{i=a+1}^{a+b} (\eta_{i1} - \eta_{i\,m_i^v k}) & = & 0 & \end{array}$$

and the fundamental group of the intersection graph contributes the relations

$$\begin{array}{rcll} \sum_{j=1}^{m_i^v k} \eta_{ij} & = & 0 & i = 1, \dots, a+b \\ \sum_{j=1}^{m_i^v k} \eta_{ij} - \sum_{j=1}^{m_{g+1}^v k} \eta_{g+1\,j} & = & 0 & i = a+b+1, \dots, g \end{array}$$

The difference $s_2 - s_1$ defines a section of Q, hence of $G := Q/Q^0$, which is given in this presentation e.g. by

$$[s_2 - s_1] = \sum_{j=1}^{m_{g+1}^v k} \eta_{g+1\,j}$$

We shall show that the order of $[s_2 - s_1]$ in G equals $\sum_{i=a+b+1}^{g+1} \frac{m^v}{m_i^v}$.
We may simplify the presentation of G using only the generators $\eta_{i,1}$ for $1, \dots, g+1$ and relations

$$\begin{array}{lcll} (m_i^v k)\,\eta_{i,1} & = & 0 & i = 1, \dots, a+b \\ (m_i^v k)\,\eta_{i,1} - (m_g^v k)\eta_{g+1,1} & = & 0 & i = a+b+1, \dots, g+1 \\ \sum_{i=a+b+1}^{g+1} \eta_{i,1} & = & 0 & \end{array}$$

In this presentation $[s_2 - s_1] = (m_i^v k)\,\eta_{i,1}$ for any $i = a+b+1, \dots, g+1$. We have

$$(\sum_{i=a+b+1}^{g+1} \frac{m^v}{m_i^v})[s_2 - s_1] = (m^v k) \sum_{i=a+b+1}^{g+1} \eta_{i,1} = 0.$$

To see that the order is not smaller, we consider $[s_2 - s_1]$ in the group H with the same generators and all but the last relation. If $n \cdot [s_2 - s_1] = 0$ in G then there is $n' \in \mathbb{Z}$ such that $n \cdot [s_2 - s_1] = n' \sum_{i=a+b+1}^{g+1} \eta_{i,1}$ in H. Listing the equivalence class of the right hand side in H this means that we can write $n = \sum_{i=a+b+1}^{g+1} n_i$ such that

$$n \cdot [s_2 - s_1] = \sum_{i=a+b+1}^{g+1} n_i (m_i^v k)\,\eta_{i,1} = \sum_{i=a+b+1}^{g+1} n' \eta_{i,1}.$$

Hence n' is a common multiple of all $m_i^v k$ and hence $m^v k$, which appeared above, is minimal.

Since the order of $s_2 - s_1$ in the component group divides N we are done. $\square$

Example 2.5. In case of the decagon, the unique primitive Teichmüller curve in $\Omega M_2(1,1)$ one has $N = 5$ and the moduli are $(1,2,1)$ (see [McM04c]). This is confirmed by

$$(2/1 + 2/2 + 2/1) \mid 5.$$

3. Algebraically primitive Teichmüller curves in $\Omega M_g(g-1, g-1)^{\mathrm{hyp}}$

In this section we prove the following theorem:

Theorem 3.1. *There are only finitely many algebraically primitive Teichmüller curves in the component* $\Omega M_g(g-1, g-1)^{\mathrm{hyp}}$ *for each* $g \geq 2$.

We specialize the results of Section 2 to the algebraically primitive case and the hyperelliptic stratum. We start with a direction that contains a saddle connection joining the two zeros, say the horizontal one. By Thm. 2.1 and Cor. 2.2 this direction contains precisely g cylinders. Similarly as for the vertical cylinders we denote by h_i^h, b_i^h and m_i^h the (respective) heights, widths and moduli of the horizontal cylinders.

Suppose that the vertical direction is chosen as in the paragraph preceding Thm. 2.4. Since the hyperelliptic involution interchanges the zeros it also interchanges the components of such a degenerate fibre and hence $a = b$. Moreover since the hyperelliptic involution on a smooth hyperelliptic curve has $2g+2$ fixed points, these fixed points have to degenerate to $g+1$ nodes joining the two components. We have shown:

Lemma 3.2. *For a degenerate fibre of an algebraically primitive Teichmüller curve in the component* $\Omega M_g(g-1, g-1)^{\mathrm{hyp}}$ *we have* $a = b = 0$.

We describe a prototype for a Veech surface in $\Omega M_g(g-1, g-1)^{\mathrm{hyp}}$: Suppose the hyperelliptic involution fixes n of the g horizontal cylinders and interchanges the remaining $g-n$ cylinders in pairs. This implies that we have precisely $2n$ Weierstraß points contained in the interiors of the horizontal cylinders. The Weierstraß points on the boundary define sections of the family f (again maybe after passing to an unramified cover of C) that do not pass through the nodes of the degenerate fibre. Hence they are fixed points of the 'hyperelliptic' involution that acts on the normalization of the degenerate fibre. Since this normalization is isomorphic to $\mathbb{P}^1$ there are precisely two fixed points and hence precisely two Weierstraß points on the boundary. To obtain $2g+2$ Weierstraß points alltogether we must have $n = g$, i.e. all the horizontal cylinders are fixed by the involution. We conclude that such a Veech surface looks as in Figure 2. The dots correspond to the Weierstraß points, the square and the cross denote the zeroes of ω^0. Vertical edges are glued by horizontal translations. The horizontal edges containing the Weierstraß points are glued on the same horizontal cylinder. In the other cases the 'free' top horizontal saddle connection of the i-th cylinder is glued to the 'free' bottom saddle connection of the $i+1$-st cylinder. For g even the square and the star have to be switched in the lowest parallelogram.

We normalize the prototype (by $\mathrm{GL}_2^+(\mathbb{R})$-action) by imposing that

$$h_1^h = 1 \quad \text{and} \quad b_1^h = 1 + w_1, \quad \text{i.e. that } m_{g+1}^v = 1, \tag{1}$$

where the the $g+1$-st vertical cylinder sits in the upper left corner of the picture. We suppose no longer (as we did in Section 2) that the vertical moduli are all integers but only that they are rational.

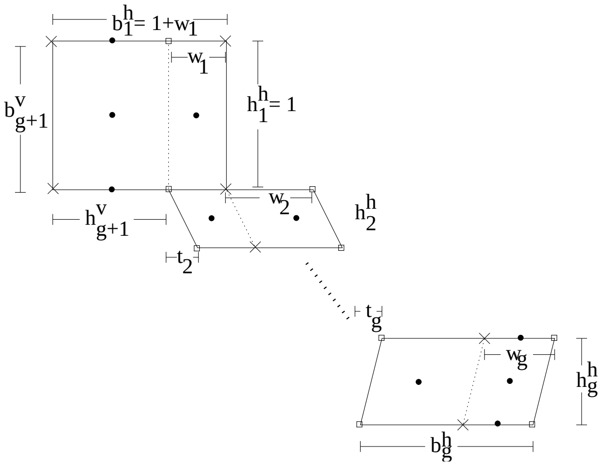

FIGURE 2. Prototype of a Veech surface in $\Omega M_g(g-1,g-1)^{\rm hyp}$

Remark 3.3. By [Mö04a] Thm. 2.6 the Jacobian of X^0 has real multiplication and the prototype above looks similar to the ones in [McM04a], Section 3. Nevertheless we do not claim that the elliptic curves obtained by glueing the horizontal slits are isogenous (what is known to be true in genus 2).

Horizontal degeneration.
Let X_h be the singular fibre obtained by degeneration in the horizontal direction. By Thm. 2.1 the degenerate fibre X_h is a singular rational curve with $g+1$ nodes. We denote by $X_h^{\rm norm}$ its normalization. We suppose that the intersection of X_h with s_2 and s_1 lift to the points 0 and ∞ on $X_h^{\rm norm}$, respectively. Since the hyperelliptic involution interchanges the two sections, we may suppose that one of its fixed points is 1, i.e. that it acts by $z \mapsto 1/z$. The Weierstraß points degenerate to ± 1 and g pairs $(x_i, 1/x_i)$ on $X_h^{\rm norm}$ that are glued together on X_h.
Let $\mathcal{L} \subset f_*\omega_{\overline{X}/\overline{C}}$ be the distinguished subbundle, whose restriction to X^0 is $\mathbb{C}\cdot\omega^0$. We choose a generator ω_h of $\mathcal{L}|_{X_h}$ and denote by $\omega_h^{\rm norm}$ its pullback to $X_h^{\rm norm}$. The differential $\omega_h^{\rm norm}$ has zeros of order $g-1$ at 0 and infinity and simple poles at $x_i^{\pm 1}$ such that the residues differ by a factor -1.

Since $\mathrm{Jac}\,(f) = \overline{g}$ for an algebraically primitive curve, the torsion condition implies that there is a map $t : X \to \mathbb{P}^1$ whose fibre over 0 (resp. ∞) equals s_1 (resp. s_2) with multiplicity N. The map extends to $t : X_{\rm bl} \to \mathbb{P}^1$ for some suitable blowup of $\overline{X}$. Since X_h is irreducible and contains both a point that maps to 0 and to ∞, the map t has to be non-constant on X_h. Hence t has the form $z \mapsto z^N$ on $X_h^{\rm norm}$ and factors through X_h. This implies that x_i are $2N$-th roots of unity.

Since the degenerate fibre can be obtained from (X^0, ω^0) by applying $\mathrm{diag}(e^t, e^{-t}) \subset \mathrm{SL}_2(\mathbb{R})$ the residues of $\omega_h^{\rm norm}$ around the g poles coincide up to a common scalar multiple with the integrals of ω^0 along the core curves of the horizontal cylinders, i.e. with the b_i^h.

Lemma 3.4. *The residues b_i^h $(i = 1, \ldots, g)$ of $\omega_h^{\rm norm}$, normalized such that $b_1^h = 1$, form a basis of $K/\mathbb{Q}$.*

Proof: Let λ denote a primitive element of $K/\mathbb{Q}$ and consider it ([Mö04a] Thm. 2.6) as an endomorphism T_λ of the family of semiabelian varieties $\overline{g}$. In particular T_λ acts on

X_h. The differential ω_h is an eigenform for the action of T_λ. Hence T_λ acts $\mathbb{Q}$-linearly on the periods b_i^h of ω_h and $(b_1^h, \ldots, b_g^h)$ form an eigenvector for the eigenvalue $\lambda \in K$. Since $b_1^h = 1$ we can express all powers of λ as $\mathbb{Q}$-linear combinations of the b_i^h. Since λ is primitive and $[K : \mathbb{Q}] = g$ the b_i^h form a basis. □

We have shown that we can write ω_h^{norm} in two ways

$$\omega_h^{\text{norm}} = \sum_{i=1}^{g} \left(\frac{b_i^h}{z - x_i} - \frac{b_i^h}{z - x_i^{-1}} \right) dz = C \frac{z^{g-1}}{\prod_{i=1}^{g} \left((z - x_i)(z - x_i^{-1}) \right)} dz,$$

where x_i are roots of unity, b_i^h form a basis of a real subfield $K \subset \mathbb{Q}(x_1, \ldots, x_g)$ and C is some real number.

Proposition 3.5. *For fixed g, there are only finitely many g-tupels $(x_1, \ldots, x_g)$ of roots of unity such that there exist real numbers $b_1^h, \ldots, b_g^h$ which form a $\mathbb{Q}$-basis of some real number field $K \subset \mathbb{Q}(x_1, \ldots, x_g)$ and a real constant C, such that we have the following identity of rational functions*

$$\sum_{i=1}^{g} \left(\frac{b_i^h}{z - x_i} - \frac{b_i^h}{z - x_i^{-1}} \right) = C \frac{z^{g-1}}{\prod_{i=1}^{g} \left((z - x_i)(z - x_i^{-1}) \right)}$$

In particular the least common multiple of the orders of the x_i satisfying the above condition ist bounded by a function depending only on g.

The proof does not require any properties of Teichmüller curves and will be given in the next section.

Corollary 3.6. *There is only a finite number of period tupels $(b_2^h, \ldots, b_g^h, w_1)$ that can occur for a Veech curve normalized as in (1).*

Proof: The finiteness of possibilities for the b_i^h is an immediate consequence. The period w_1 is the integral of ω_h along a path from 0 to ∞ that crosses the unit circle once (in a point different from ± 1 and $x_i^{\pm 1}$). Since the x_i are fixed up to a finite number of choices this determines w_1 up to a finite number of choices. □

Vertical direction

The work has been done in the previous section. We record that Thm. 2.4 implies:

Lemma 3.7. *For fixed N there is only a finite number of possibilities for the moduli m_i^v.*

Proof of Thm. 3.1: Fix one of the finitely many possibilities for the b_i^h, w_1 and hence w_i $(i = 1, \ldots, g)$ and for the moduli m_j^v $(j = 1, \ldots, g)$. For all j the heights h_j^v are bounded above by $\max\{w_i; i = 1, \ldots, g-1\}$. Hence all the b_j^v are bounded above.

Let $J_1 \subset \{1, \ldots, g\}$ be the indices of vertical cylinders intersecting w_1. For $j \in J_1$ the heights h_j^v are bounded away from zero since b_j^v is bounded away from zero by $h_1^h = 1$ and the m_j^v are fixed. Since the b_i^h are fixed and h_i^v are bounded away from zero, there is a only finite number of possibilities for the intersection numbers $e_{ij} := \gamma_i^h \cdot \gamma_j^v$ for $i = 1, \ldots, g$ and $j \in J_1$. We fix one possibility.

Claim: For at least one (say the i_0-th) of the horizontal cylinders intersected by some $j \in J_1$ the height $h_{i_0}^h$ is bounded away from zero by a constant depending only on w_1, the

moduli m_j^v and the intersection numbers fixed so far.
In fact, we know that

$$w_1 = \sum_{j \in J_1} e_{1j} h_j^v$$

and by definition

$$h_j^v = m_j^v \sum_{i=1}^{g} e_{ij} h_j^h.$$

Putting these equations together we obtain using $h_1^h = 1$

$$w_1 - \sum_{j \in J_1} m_j^v e_{1j}^2 = \sum_{i=2}^{g} h_i^h \cdot (\sum_{j \in J_1} m_i^v e_{1j} e_{ij}).$$

The left hand side of this equation is non-negative and if it were zero this would imply that the vertical cylinders crossing w_1 do not intersect any other horizontal cylinder but the first. This is absurd. Hence it is positive and depends only on quantities fixed so far. This implies that not all the h_i^h for $i = 2, \dots, g$ can be simultaneously arbitrarily small.

Using the claim we denote by J_2 the set of cylinders that intersect γ_1 or γ_{i_0}. As above this limits the e_{ij} for $j \in J_2$ to a finite number. We now proceed inductively analysing w_j in place of w_1 to conclude that all intersection numbers e_{ij} vary through a finite list.

Fix one of the finitely many possibilities for the intersection numbers. We know that

$$b_i^h = \sum_{j=1}^{g+1} e_{ij} h_j^v = \sum_{j=1}^{g+1} e_{ij} m_j^v b_j^v$$

for $i = 1, \dots, g$ and for $j = 1, \dots, g$ we know by definition

$$b_j^v = \sum_{i=1}^{g} e_{ij} h_i^h.$$

Let E denote the $g \times (g+1)$-matrix with entries e_{ij}. From [HuLa05] we deduce that $E\text{diag}(m_i^v)E^t$ is regular. In fact they show that the eigenvalues of $E\text{diag}(m_i^v)E^t\text{diag}(m_i^h)$ form a basis of $K/\mathbb{Q}$. Hence we may plug the second equation above in the first and solve uniquely for the h_i^h, since we know the b_i^h. This also determines the b_j^v and consequently the h_j^v.

We know all heights and widths of the cylinders and it remains to limit the possible twists t_i for $i = 2, \dots, g$ to a finite number. The absolute values of the twists are bounded by the intersection numbers times b_i^h and they can only vary in positive integral linear combinations of the h_i^v. Hence there is only a finite number of possibilities for the twists. □

Corollary 3.8. *The trace field K of an algebraically primitive Teichmüller curve in $\Omega M_g(g-1, g-1)^{\text{hyp}}$ is abelian.*

Proof: In the above proof of Thm.3.1 we have seen that the periods b_i^h lie in the field $\mathbb{Q}(x_i)$, where x_i are roots of unity. The field generated by the b_i^h coincides with the trace field of Γ by Lemma 3.4. □

Since it fits into this context, we end our finiteness discussion by the following complement:

Theorem 3.9. *Fix a genus g and consider all Teichmüller curves $C \to M_g$. If we fix moreover the Euler characteristic $\chi(C) = 2g-2+n$ of C then there is only a finite number of possibilities for the monodromy, in particular for the trace field of such a Teichmüller curve.*

Proof: The Euler characteristic of the corresponding curve in the moduli space $M_g^{[3]}$ of curves with level-3-structure is also bounded. Hence we can apply Prop. 3.10 in [De87]. □

4. Proof of Prop. 3.5

Suppose we are given a rational function as in the statement of Prop. 3.5. Choosing $\prod_{i=1}^g (z - x_i)(z - x_i^{-1})$ as common denominator and comparing coefficients of z^0 to z^{g-2} translates into the following system of equations for $e = 1, \ldots, g-1$ (the coefficients of z^g to z^{2g-2} provide the same system):

$$\sum_{i=1}^{g} \left(b_i^h (x_i - x_i^{-1}) \sum_{\substack{j_1 < \ldots < j_e \\ \text{all } j_k \neq i}} \prod_{k=1}^{e-1} (x_{j_k} + x_{j_k}^{-1}) \right) = 0 \tag{2}$$

We subtract in the first step $\sum_{j=1}^g (x_j + x_j^{-1})$ times the equation with $e = 1$ from $e = 2$ to obtain an equation denoted by $(Eq : 2')$. Then subtracting $\sum_{j_1 < j_2} (x_j + x_j^{-1})$ times the equation with $e = 1$ from $e = 3$ and adding $\sum_{j=1}^g (x_j + x_j^{-1})$ times the equation $(Eq : 2')$ we obtain an equation denoted by $(Eq : 3')$. Proceeding in this way we obtain the simplified system

$$\sum_{i=1}^{g} \left(b_i^h (x_i - x_i^{-1})(x_i + x_i^{-1})^{e-1} \right) = 0. \tag{$Eq : e'$}$$

This system of equations is equivalent to the system

$$\sum_{i=1}^{g} \left(b_i^h (x_i^e - x_i^{-e}) \right) = 0. \tag{$Eq : e''$}$$

for $e = 1, \ldots, g-1$, which will be used in the sequel.

We say that an equation

$$\sum_{i=1}^{k} a_i \zeta_i = 0 \tag{3}$$

where the ζ_i are pairwise different roots of unity and where the a_i lie in the number field K form a *K-relation of length k*. The relation is called irreducible, if $\sum_{i=1}^k b_i \zeta_i = 0$ and $b_i(a_i - b_i) = 0$ for all i implies that $b_i = 0$ for all i or $a_i - b_i = 0$ for all i. Each relation is a sum of irreducible relations, but there may be several ways of writing a relation as sum of irreducible relations.

Lemma 4.1. *Let $\sum_{i=1}^k a_i \zeta_i = 0$ be an irreducible K-relation with $K \subset \mathbb{Q}^{\mathrm{ab}}$ and $[K : \mathbb{Q}] = g$. Then multiplying the relation by a suitable root of unity we can achieve that*

$$\zeta_i \in \mathbb{Q}(e^{2\pi i/N}) \quad \textit{where} \quad N = \prod_{p \leq 2kg \textit{ prime}} p^{\nu_0(p)} \quad \textit{and} \quad \nu_0(p) = \lceil \log_p(1 + \frac{g}{p-1})) \rceil.$$

In particular the ζ_i appearing in such a relation with the normalization $\zeta_1 = 1$ belong to a finite set.

Proof: The following argument extends a theorem of Mann ([Mn65]) from the case of rational coefficients to the case of coefficients in a field of bounded degree over $\mathbb{Q}$.
Suppose the irreducible relation has $\zeta_i \in \mathbb{Q}(e^{2\pi i/N})$ and $N = p^\nu N'$ for some $\nu \geq \nu_0 + 1$ and $\gcd(p, N') = 1$. Let ζ be a primitive p^ν-th root of unity and let ρ be a primitive N/p-th root of unity. Resorting the relation according to powers of ζ we obtain

$$\sum_{j=0}^{p-1} b_j \zeta^j = 0 \quad \text{where} \quad b_j = \sum_{i \in \Lambda_j} a_i \rho^{\alpha_i} \tag{4}$$

where $\Lambda_j = \{i : \exists \alpha_i \in \mathbb{N} \,\text{such that}\, \rho^{\alpha_i}\zeta^j = \zeta_i\}$. The coefficients b_j belong to $L = K(\rho)$. Since $K \subset \mathbb{Q}^{\text{ab}}$ and $[K : \mathbb{Q}] \leq g$ we know that

$$K \subset \mathbb{Q}(e^{2\pi i/p^{j_0(p)}}, p \text{ prime}).$$

Since cyclotomic fields for powers of different primes are linearly disjoint over $\mathbb{Q}$ we deduce that

$$[L(\zeta) : L] = [\mathbb{Q}(\rho, \zeta), \mathbb{Q}(\rho)] = p.$$

Hence $b_j = 0$ for $j = 0, \ldots, p-1$. Since the original relation was irreducible, this is only possible if $\Lambda_{j_0} = \{1, \ldots, k\}$ for some j_0 (and the other Λ_j are empty). This means that we can reduce N by multiplying the original relation with a suitable power of ζ.

We have bounded the exponents that occur in the factorization of N. It remains to bound the size of primes dividing N. Suppose that p is prime and divides N to the order $\nu \leq \nu_0(p)$. As above let ζ be a primitive p^ν-th root of unity, but we let now ρ be a primitive N/p^ν-th root of unity. Resorting the relation according to powers of ζ we obtain

$$f(\zeta) := \sum_{j=0}^{p^\nu - 1} b_j \zeta^j = 0 \quad \text{where} \quad b_j = \sum_{i \in \Lambda_j} a_i \rho^{\alpha_i} \tag{5}$$

The coefficients b_j of f lie in $K(\rho)$. Since $\mathbb{Q}(\rho) \cap \mathbb{Q}(\zeta) = \mathbb{Q}$ the polynomial f is a multiple of the minimal polynomial $f_{\zeta/K}$ of ζ over K, which has degree at least $\phi(p^\nu) - g$. Here ϕ denotes Euler's ϕ-function. On the other hand by construction at most k of the coefficients b_j are non-zero. Hence there is somewhere a gap of size p^ν/k between non-zero b_j. Multiplying the relation by a suitable power ζ we may suppose from the beginning that

$$\deg f \leq p^\nu(1 - \frac{1}{k})$$

If $p^\nu/k - 1 \geq g + p^{\nu-1}$ this leads to a contradiction to the degree of $f_{\zeta/K}$. This condition is fulfilled if the rough bound $p \leq 2kg$ is violated. □

Proof of Prop. 3.5: Suppose the finiteness statement was wrong. Then there exists a sequence $(b^h_{i,n}, x_{i,n})$ for $n \in \mathbb{N}$ satisfying $(Eq : e'')$ for all $e = 1, \ldots, g-1$ and such that least common multiple $N(n)$ of the orders of the $x_{i,n}$ is unbounded. We interpret the (solutions of the) equations as relations between roots of unity

$$\sum_{i=1}^{g} b^h_{i,n} x^e_{i,n} + \sum_{i=-g}^{-1} b^h_{i,n} x^e_{i,n} = 0$$

with the convention that $b^h_{i,n} = b^h_{-i,n}$ and $x_{i,n} = x^{-1}_{-i,n}$.
For each n and each e we may write the relation in a (non-unique) way as a sum of irreducible relations. The summands occuring in such an irreducible relation form a partition of $I := \{-g, \ldots, -1, 1, \ldots, g\}$. Since this set admits only finitely many partitions we pass to a subsequence of $(b^h_{i,n}, x_{i,n})$ and suppose without loss of generality that there are partitions P_e consisting of subsets $P_{e,j}$ of I such that for $e = 1, \ldots, g-1$, for all j and for all $n \in \mathbb{N}$

$$\sum_{i \in P_{e,j}} b^h_{i,n} x^e_{i,n} = 0$$

is an irreducible relation. We apply Lemma 4.1 to these relations and write

$$x^e_{i,n} = \zeta_{i,e,n}\sigma_{i,e,n} \tag{6}$$

with the following two properties: First, the $\zeta_{i,e,n}$ are roots of unity of order bounded by a function depending only on g since the relations are of length $\leq 2g$. Second if i and i' are both in $P_{e,j}$ then $\sigma_{i,j,n} = \sigma_{i',j,n}$. Passing to a subsequence again we may suppose

$$\zeta_{i,e,n} = \zeta_{i,e} \quad \text{for all} \quad n \in \mathbb{N}.$$

We want to limit the possible choices for the $\sigma_{i,e,n}$ to a finite set in order to obtain a contradiction. From (6) we deduce that

$$\sigma_{i,e,n} = \sigma^e_{i,1,n}\frac{\zeta^e_{i,1}}{\zeta_{i,e}}.$$

This means that the $\sigma_{i,e,n}$ for different second arguments are closely related. Since they coincide when the first argument varies in a fixed partition set $P_{e,j}$ there is, roughly speaking, a partition of I that controls the $\sigma_{i,e,n}$ for all e simultaneously. More precisely, consider the following equivalence relation: $i \sim i'$ if there exists (e, j) such that $P_{e,j} \supset \{i, i'\}$. Denote the corresponding partition by $P_0 = \cup_j P_{0,j}$. Then $\sigma_{i,e,n}$ and $\sigma_{i',e,n}$ differ for $P_{0,j} \supset \{i, i'\}$ at worst by a product of $\zeta_{i,e}$ and a $(g-1)!$-th root of unity. We may suppose that they actually coincide by recording the discrepancy in modified $\zeta_{i,e}$. I.e. we write

$$x^e_{i,n} = \widetilde{\zeta_{i,e}}\sigma^e_{i,n} \tag{7}$$

with the following two properties: First, $\widetilde{\zeta_{i,e}}$ is root of unity of order bounded by a function depending only on g and second if i and i' are both in $P_{0,j}$ then $\sigma_{i,n} = \sigma_{i',n}$.
Suppose $P_{0,j}$ does not contain a pair $\{i, -i\}$. Then the cardinality k of $P_{0,j}$ is at most g. We fix $n \in \mathbb{N}$. The $b^h_{i,n}$ are solutions of the system of equations

$$\sum_{i \in P_{o,j}} b^h_{i,n} x^e_{i,n} = 0, \quad e \in \{1, \ldots, k-1\}. \tag{8}$$

Since the $b^h_{i,n}$ are real we may take complex conjugates to see that they also solve the system of equations for $e \in \{-k+1, \ldots, -1\}$. Since $x_{i,n} \neq x_{j,n}$ for $i \neq j$ the only solution is $\sum_{i \in P_{o,j}} x^e_{i,n} = 0$, i.e. all $b^h_{i,n}$ for $i \in P_{0,j}$ are equal. This contradicts that the b^h_i form a $\mathbb{Q}$-basis of K.

On the other hand if $P_{0,j}$ contains $\{i_0, -i_0\}$ we deduce from

$$\zeta_{-i_0,1}\sigma_{i_0,n} = \zeta_{-i_0,1}\sigma_{-i_0,n} = x_{-i_0,n} = x^{-1}_{i_0,n} = (\zeta_{i_0,1}\sigma_{i_0,n})^{-1}$$

that $\sigma_{i_0,n}$ runs for $n \in \mathbb{N}$ through a finite set. By construction the same holds for $\sigma_{i,n}$ for all $i \in P_{0,j}$. Applying this to all partition sets of P_0 the orders of the $\sigma_{i,n}$ and hence the orders of the $x_{i,n}$ are bounded. This contradicts the choice of the sequence $x_{i,n}$. □

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