

MINIMAL BUNDLES AND FINE MODULI SPACES

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ABSTRACT. We study sheaves E on a smooth projective curve X which are minimal with respect to the property that $h^0(E \otimes L) > 0$ for all line bundles L of degree zero. We show that these sheaves define ample divisors $D(E)$ on the Picard torus $\text{Pic}^0(X)$ (see Theorem 3.3). Next we classify all minimal sheaves of rank one (see Theorem 4.2) and two (see Theorem 4.3). As an application we show (see Proposition 5.6) that the moduli space parameterizing rank two bundles of odd degree can be obtained as a Quot scheme.

1. INTRODUCTION

In this article we resume the study of minimal bundles from [5]. We call a sheaf E on a smooth projective curve X minimal when $h^0(E \otimes L) > 0$ for all line bundles L of degree zero, but $H^0(E' \otimes L) = 0$ for all proper subsheaves $E' \subsetneq E$ and a general line bundle L of degree zero. For more equivalent characterizations of minimality see the definition after Theorem 2.1.

Using the Fourier-Mukai transform and the Poincaré line bundle \mathcal{P} on $X \times \text{Pic}^0(X)$ we find that for a minimal sheaf E on X the direct image sheaf $\text{pr}_{2*}(\mathcal{P} \otimes \text{pr}_1^*E)$ is a line bundle $\mathcal{O}_{\text{Pic}}(-D(E))$ where $D(E)$ is an effective ample divisor on the Picard torus $\text{Pic}^0(X)$. This is shown in Theorem 3.3 which is basically contained in [5, Theorem 4.6]. However, the proof given here (see Lemmas 3.1 and 3.2) is shorter than the old proof in [5]. In Proposition 4.1 we show that any sheaf E on X such that the direct image sheaf $\text{pr}_{2*}(\mathcal{P} \otimes \text{pr}_1^*E)$ is not zero contains a minimal bundle. In Theorems 4.2 and 4.3 we classify all minimal bundles of rank one and two. As an unexpected application we obtain that the fine moduli schemes $\text{Pic}^d(X)$, $\text{U}_X(2, 2k + 1)$ and $\text{SU}_X(2, 2k + 1)$ are special cases of Grothendieck's Quot schemes developed in [3]. See Propositions 5.4, 5.6, and 5.8 for the details.

The classical construction of geometric invariant theory ([7], see also [9]) uses the Quot schemes as a starting point and investigates a group action on this scheme. In the above cases we see that taking *the right Quot scheme* all stable objects appear as quotients, all quotients are stable, and for two quotients $[P \xrightarrow{\pi_1} E_1]$ and $[P \xrightarrow{\pi_2} E_2]$ we have an equivalence $E_1 \cong E_2 \iff \ker(\pi_1) = \ker(\pi_2)$.

It may be well known that fine moduli spaces of vector bundles on algebraic varieties which are projective appear as Quot schemes. For lack of reference we include this result in Proposition 5.2. In view of this result, the Propositions 5.6, and 5.8 show that the Poincaré bundle on $X \times \text{Pic}^0(X)$ and the Quot scheme construction are enough to construct $\text{U}_X(2, 2k + 1)$ and $\text{SU}_X(2, 2k + 1)$.

Since we exploit the properties of the Fourier-Mukai transform of the Fourier-Mukai pair of abelian varieties $(\text{Jac}(X), \text{Pic}^0(X))$, Raynaud's vector bundles P_m (see Theorem 2.1) play a crucial part. These bundles are a special case of GV-sheaves defined by Pareschi and Popa in [10].

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2. PRELIMINARIES

Notation. We fix an algebraically closed field k of arbitrary characteristic. In this article the word *scheme* means *scheme of finite type over k* . Let X be a smooth projective curve of genus g over k . We fix a point $P_0 \in X(k)$. We consider the following morphisms

$$\begin{array}{ccc} \text{Jac}(X) \times \text{Pic}^0(X) & \xrightarrow{p} & \text{Pic}^0(X) \\ & & \downarrow q \\ X & \xrightarrow{\iota} & \text{Jac}(X) \end{array}$$

The Poincaré bundle on $\text{Jac}(X) \times \text{Pic}^0(X)$ is denoted by \mathcal{P} . We normalize it by demanding that $\mathcal{P}|_{\{P_0\} \times \text{Pic}^0(X)} \cong \mathcal{O}_{\text{Pic}^0(X)}$. Mukai showed in [6] that the Fourier-Mukai transform

$$\text{FM}_{\mathcal{P}} : \text{D}^b(\text{Jac}(X)) \rightarrow \text{D}^b(\text{Pic}^0(X)) \quad e \mapsto p_*(\mathcal{P} \otimes q^*e)$$

is an equivalence of triangulated categories. Furthermore, he showed that the inverse $\text{FM}_{\mathcal{P}}^{-1} : \text{D}^b(\text{Pic}^0(X)) \rightarrow \text{D}^b(\text{Jac}(X))$ is given by $f \mapsto [-1]^*q_*(\mathcal{P} \otimes p^*f)[g]$ where $[-1]$ denotes the involution on $\text{Jac}(X)$ which assigns each element its inverse. For a coherent sheaf E on X we may regard ι_*E as a complex and obtain a complex $\text{FM}_{\mathcal{P}}(\iota_*E)$ in $\text{D}^b(\text{Pic}^0(X))$. For dimension reasons all the cohomology sheaves $\text{FM}^i(E) := \mathcal{H}^i(\text{FM}_{\mathcal{P}}(\iota_*E))$ vanish for $i \notin \{0, 1\}$.

The theta divisor $\Theta \subset \text{Pic}^0(X)$. We repeat some basic facts about the theta divisor on the Picard torus (cf. the first chapter of the book [1]). We start with fixing a theta characteristic ϑ , that is a line bundle such that $\vartheta^{\otimes 2} \cong \omega_X$. This allows a set-theoretical definition of the Θ -divisor

$$\Theta = \{[L] \in \text{Pic}^0(X) \mid h^1(L \otimes \vartheta) > 0\}.$$

For a line bundle M of degree $g-1$ we can likewise define the theta divisor Θ_M by $\Theta_M = \{[L] \in \text{Pic}^0(X) \mid h^1(L \otimes M) > 0\}$. Θ_M is the image of the summation morphism from the $(g-1)$ -fold symmetric product $X^{(g-1)}$ of X to $\text{Pic}^0(X)$ given by $(P_1, P_2, \dots, P_{g-1}) \mapsto \mathcal{O}_X(P_1 + P_2 + \dots + P_{g-1}) \otimes M^{-1}$. Θ defines a principal polarization on $\text{Pic}^0(X)$, that is Θ is ample, and $\int_{\text{Pic}^0(X)} \Theta^g = g!$. The divisors Θ and Θ_M are not rationally equivalent unless $M \cong \vartheta$. However, Θ_M is the translate of Θ by $[\vartheta \otimes M^{-1}] \in \text{Pic}^0(X)$. Such divisors are called geometrically equivalent, we write $\Theta \sim \Theta_M$.

Raynaud’s bundles P_m . For positive integers m Raynaud considered in [11] the vector bundles $P_m = \iota^*[-1]^*R^gq_*(\mathcal{P} \otimes p^*\mathcal{O}_{\text{Pic}}(-m\Theta))$. We repeat some properties of P_m next.

Theorem 2.1. *Let m be a positive integer, and E a sheaf of rank r on X .*

- (i) P_m is a vector bundle of rank m^g and degree gm^{g-1} .
- (ii) There exists a unique (up to scalar multiplication) surjection $P_{m+1} \rightarrow P_m$.
- (iii) We have equalities

$$\begin{aligned} \text{Hom}_X(P_m, E) &= \text{Hom}_{\text{Pic}(X)}(\mathcal{O}_{\text{Pic}}(-m\Theta), \text{FM}^0(E)) \\ &= \text{Hom}_{\text{D}^b(\text{Pic}(X))}(\mathcal{O}_{\text{Pic}}(-m\Theta), \text{FM}_{\mathcal{P}}(\iota_*E)). \end{aligned}$$

- (iv) The following equivalences hold

$$\begin{aligned} \text{FM}^0(E) \neq 0 &\iff \text{Hom}_{\text{Pic}(X)}(\mathcal{O}_{\text{Pic}}(-m\Theta), \text{FM}^0(E)) \neq 0 \text{ for } m \gg 0 \\ &\iff \text{Hom}_{\text{Pic}(X)}(\mathcal{O}_{\text{Pic}}(-m\Theta), \text{FM}^0(E)) \neq 0 \text{ for some } m > rg \\ &\iff h^0(E \otimes L) > 0 \text{ for all line bundles } L \text{ of degree zero} \end{aligned}$$

Proof. (i) is 3.1 in Raynaud's article [11], (ii) is 2.4 in [5], (iii) is 2.5 in [5], and (iv) is the content of 2.5 and 3.7 in [5]. Partially, this is contained in Corollary 7.3 in [10]. \square

Minimal bundles. We call a vector bundle E on X minimal, when $\mathrm{FM}^0(E) \neq 0$, and for all proper subsheaves $E' \subsetneq E$ we have $\mathrm{FM}^0(E') = 0$. In view of Theorem 2.1 minimality can be expressed by the following equivalent conditions

$$\begin{aligned} E \text{ is minimal} &\iff \mathrm{FM}^0(E) \neq 0, \text{ and } \mathrm{FM}^0(E') = 0 \text{ for all proper subsheaves } E' \subset E. \\ &\iff \mathrm{Hom}(P_m, E) \neq 0, \text{ and } \mathrm{Hom}(P_m, E') = 0 \text{ for all proper subsheaves } E' \subset E, \text{ and } m \gg 0. \\ &\iff \mathrm{Hom}(P_m, E) \neq 0, \text{ and } \mathrm{Hom}(P_m, E') = 0 \text{ for all proper subsheaves } E' \subset E, \text{ and } m > rg. \\ &\iff \mathrm{Hom}(P_m, E) \neq 0 \text{ for some } m > rg \text{ and every } \pi \in \mathrm{Hom}(P_m, E) \text{ is either zero or surjective.} \\ &\iff h^0(E \otimes L) \geq 1, \text{ and } h^0(E' \otimes L) = 0 \text{ for all proper subsheaves } E' \subset E, \text{ and a general line bundle } L \text{ of degree zero.} \end{aligned}$$

Note that the definition of *minimal bundle* differs slightly from the definition given in [5].

3. $\mathrm{FM}^0(E)$ OF MINIMAL BUNDLES

Taking an elementary transformation $0 \rightarrow E' \rightarrow E \rightarrow k(P) \rightarrow 0$, we obtain an exact sequence

$$0 \rightarrow \mathrm{FM}^0(E') \rightarrow \mathrm{FM}^0(E) \rightarrow L_P$$

where $L_P = \mathrm{FM}^0(k(P))$ is the numerical trivial line bundle on $\mathrm{Pic}^0(X)$ parameterized by the point P . The minimality of E implies that $\mathrm{FM}^0(E') = 0$, thus

$$\mathrm{FM}^0(E) \subset L_P \text{ for all points } P \in X(k).$$

In particular we have $\mathrm{FM}^0(E) \subset L_{P_0} = \mathcal{O}_{\mathrm{Pic}^0(X)}$. We fix an embedding $\mathrm{FM}^0(E) \rightarrow \mathcal{O}_{\mathrm{Pic}}$, and we write $\mathrm{FM}^0(E) = \mathcal{J}_Z$ for a closed subscheme $Z \subset \mathrm{Pic}^0(X)$. We decompose $Z = D \cup Z'$, where \mathcal{O}_D is the quotient of \mathcal{O}_Z modulo the maximal subsheaf of codimension two. This way we can write $\mathcal{J}_Z = \mathcal{O}_{\mathrm{Pic}}(-D) \otimes \mathcal{J}_{Z'}$ with Z' of codimension at least two.

Lemma 3.1. [5, Lemma 4.10] *The divisor D is ample.*

Proof. We consider the map of abelian varieties $\mathrm{Pic}^0(X) \xrightarrow{\phi(D)} \mathrm{Jac}(X)$ given by $\phi(D) : P \mapsto T_P^* \mathcal{O}(D) \otimes \mathcal{O}(-D)$. Its kernel $K(D)$ is the biggest subscheme where D is trivial (see [8, Corollary 5 on page 131].) If the kernel is discrete, then D is ample, and we are done. Let $K_0(D)$ be the connected component of zero in $K(D)$. The injection $K_0(D) \rightarrow \mathrm{Pic}^0(X)$ defines a surjection $\mathrm{Jac}(X) \xrightarrow{\psi(D)} \mathrm{Pic}^0(K_0(D))$ of the dual abelian varieties. If L_P is the line bundle parameterized by the point $P \in X(k)$, then we have $H^0(L_P \otimes \mathcal{O}_{\mathrm{Pic}}(D)) = \mathrm{Hom}(\mathcal{O}_{\mathrm{Pic}}(-D), L_P) = \mathrm{Hom}(\mathcal{J}_Z, L_P)$ because the codimension of Z' is at least two. As we remarked above there exists an embedding $\mathcal{J}_Z = \mathrm{FM}^0(E) \subset L_P$ for all points $P \in X(k)$. The global section of $H^0(L_P \otimes \mathcal{O}_{\mathrm{Pic}}(D))$ shows that $L_P \otimes \mathcal{O}_{\mathrm{Pic}}(D)$ is trivial on all fibers of $\phi(D)$. So it is trivial on $K_0(D)$. We conclude that the kernel of $\psi(D)$ contains all points $P \in X(k) \subset \mathrm{Jac}(X)(k)$. Since there exists no proper abelian subvariety in $\mathrm{Jac}(X)$ which contains X , we conclude that the kernel of $\psi(D)$ is $\mathrm{Jac}(X)$. Thus, $K_0(D)$ is of dimension zero which implies the ampleness of D . \square

Lemma 3.2. [5, Lemma 4.11] *$Z' = \emptyset$ which implies $\mathrm{FM}^0(E) = \mathcal{O}(-D)$.*

Proof. We start with the short exact sequence $0 \rightarrow \mathcal{J}_Z \rightarrow \mathcal{O}_{\text{Pic}}(-D) \rightarrow \mathcal{O}_{Z'}(-D) \rightarrow 0$. Since D is ample we have $R^i q_*(\mathcal{P} \otimes p^* \mathcal{O}_{\text{Pic}}(-D)) = 0$ for all $i \neq g$. Since the dimension of Z' is at most $g - 2$ we have $R^i q_*(\mathcal{P} \otimes p^* \mathcal{O}_{Z'}(-D)) = 0$ for $i \geq g - 1$. It follows that $\text{FM}_{\mathcal{P}}^{-1}(\mathcal{J}_Z)$ has cohomology only in nonpositive degrees, and the zero cohomology $\mathcal{H}^0(\text{FM}_{\mathcal{P}}^{-1}(\mathcal{J}_Z)) = \mathcal{H}^0(\text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D)))$. Since $\mathcal{J}_Z = \text{FM}^0(E)$ we have a canonical morphism $\mathcal{J}_Z \rightarrow \text{FM}_{\mathcal{P}}(E)$. We obtain a morphism $\rho : \text{FM}_{\mathcal{P}}^{-1} \mathcal{J}_Z \rightarrow \iota_* E$. The only non vanishing term in the Eilenberg-Moore spectral sequence (see [2, page 263])

$$\text{Ext}^q(\mathcal{H}^q(\text{FM}_{\mathcal{P}}^{-1}(\mathcal{J}_Z)), \iota_* E) \implies \text{Hom}(\text{FM}_{\mathcal{P}}^{-1}(\mathcal{J}_Z), \iota_* E) = \text{Hom}(\mathcal{J}_Z, \text{FM}_{\mathcal{P}}(E))$$

is $\text{Hom}(\mathcal{H}^0(\text{FM}_{\mathcal{P}}^{-1} \mathcal{J}_Z), \iota_* E)$. So we conclude that ρ factors through $\text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D))$. So $\mathcal{J}_Z \rightarrow \text{FM}^0(E)$ factors through $\mathcal{O}_{\text{Pic}}(-D)$ which gives $\mathcal{J}_Z = \mathcal{O}_{\text{Pic}}(-D)$. \square

Theorem 3.3. *Let E be a minimal bundle on the smooth projective curve X .*

- (i) $\text{FM}^0(E) = \mathcal{O}_{\text{Pic}}(-D)$ where $D = D(E)$ is an ample divisor.
- (ii) $\text{FM}^1(E)$ has projective dimension at most two.
- (iii) $\text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E)))$ is a vector bundle of rank $h^0(\mathcal{O}_{\text{Pic}}(D(E)))$, and the natural map $\iota^* \text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E))) \rightarrow E$ is surjective.
- (iv) E is simple.

Proof. (i) follows from Lemma 3.1 and 3.2.

In order to see (ii) consider a surjection $M^{\oplus N} \xrightarrow{\pi} E$ with M a line bundle of negative degree. By base change we obtain that $\text{FM}^1(M^{\oplus N})$ and $\text{FM}^1(\ker \pi)$ are vector bundles, and $\text{FM}^0(M^{\oplus N}) = 0$. Hence, we obtain a resolution of $\text{FM}^1(E)$ by vector bundles

$$0 \rightarrow \text{FM}^0(E) \rightarrow \text{FM}^1(\ker \pi) \rightarrow \text{FM}^1(M^{\oplus N}) \rightarrow \text{FM}^1(E) \rightarrow 0$$

(iii) By Serre duality $\text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E))) = [-1]^* R^g q_*(\mathcal{P} \otimes p^* \mathcal{O}_{\text{Pic}}(-D(E)))$ is a vector bundle of rank $h^0(\mathcal{O}_{\text{Pic}}(D(E)))$. Since $\text{Hom}(P_m, E) = \text{Hom}(\mathcal{O}_{\text{Pic}}(-m\Theta), \text{FM}^0(E))$, every morphism $P_m \rightarrow E$ factors through the image of $\iota^* \text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E))) \rightarrow E$.

(iv) From the surjection $\iota^* \text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E))) \rightarrow E$ we deduce that $\text{Hom}(E, E) \subset \text{Hom}(\iota^* \text{FM}_{\mathcal{P}}^{-1}(\mathcal{O}_{\text{Pic}}(-D(E))), E)$. Since $\text{FM}_{\mathcal{P}}$ is an equivalence of categories the last vector space is isomorphic to $\text{Hom}(\mathcal{O}_{\text{Pic}}(-D(E)), \text{FM}_{\mathcal{P}}(\iota_* E)) = \text{Hom}(\mathcal{O}_{\text{Pic}}(-D(E)), \text{FM}^0(E))$ which is one dimensional by definition of $D(E)$. \square

4. EXAMPLES OF MINIMAL BUNDLES

Proposition 4.1. *Any coherent sheaf E on X with $\text{FM}^0(E) \neq 0$ contains a minimal sheaf.*

Proof. Set $m = \text{rk}(E)g + 1$. The non trivial homomorphisms $\phi : P_m \rightarrow E$ form a bounded family. So there exists a ϕ_{\min} such that the image $E' = \text{im}(\phi_{\min})$ has minimal Hilbert polynomial. By construction any morphism $P_m \rightarrow E'$ is either zero or surjective. In consequence, E' is minimal. \square

Remark. So any sheaf E of positive Euler characteristic $\chi(E)$ contains a minimal sheaf. This minimal sheaf is by no means uniquely defined. As an example take a line bundle M of degree $2g$. This line bundle contains all line bundles L of degree g which are all minimal as we show next.

Theorem 4.2. *A line bundle L is minimal if and only if its degree $\text{deg}(L) = g$. For a line bundle L of degree g the divisor $D = D(L)$ is geometrically equivalent to the Θ -divisor on $\text{Pic}^0(X)$.*

Proof. If $\chi(L) \leq 0$, then a general line bundle L' of degree $\deg(L)$ has no sections. Thus, we have $h^0(L \otimes (L' \otimes L^{-1})) = 0$ which implies $\mathrm{FM}^0(L) = 0$. If $\chi(L) > 1$, then there are proper subsheaves $L' \subset L$ of degree g for which $\mathrm{FM}^0(L') \neq 0$. For $\deg(L) = g$, we have seen that $\mathrm{FM}^0(L) \neq 0$, and all proper subsheaves $L' \subset L$ satisfy $\mathrm{FM}^0(L') = 0$. So the minimality is shown.

Let now be L a line bundle of degree g . From $0 \rightarrow L(-P_0) \rightarrow L \rightarrow k(P_0) \rightarrow 0$ we deduce that

$$(\mathrm{FM}^0(L(-P_0)) = 0) \rightarrow (\mathrm{FM}^0(L) = \mathcal{O}_{\mathrm{Pic}}(-D(L))) \rightarrow \mathcal{O}_{\mathrm{Pic}} \rightarrow \mathrm{FM}^1(L(-P_0)).$$

$\mathrm{FM}^1(L(-P_0))$ is a sheaf of projective dimension one concentrated on the Θ divisor $\Theta_{L(-P_0)}$ which is geometrically equivalent to Θ . The linear system $|\Theta_{L(-P_0)}|$ consists only of the divisor $\Theta_{L(-P_0)}$ which is a translate of the irreducible Θ divisor. Since $\Theta_{L(-P_0)}$ is irreducible and $D(L)$ is ample, we have $D(L) = \Theta_{L(-P_0)}$. \square

Theorem 4.3. *A rank two vector bundle E is minimal if and only if it is stable and of degree $2g - 1$. For a minimal rank two bundle E the divisor $D(E)$ is geometrically equivalent to 2Θ .*

The statement of the proposition is a consequence of the three successive Lemmas 4.4–4.6.

Lemma 4.4. *Any stable rank two vector bundle E of degree $2g - 1$ is minimal.*

Proof. Suppose E is a stable vector bundle of rank two with $\deg(E) = 2g - 1$. Let $\pi : E \rightarrow k(P)$ be a surjection and let E' be its kernel. The stability of E implies the semistability of E' . Therefore (see [11, Proposition 1.6.2]) the sheaf $\mathrm{FM}^0(E')$ is zero. Since $\chi(E) = 1$ we have $\mathrm{FM}^0(E) \neq 0$. However any proper subsheaf $E'' \subset E$ is contained in a sheaf E' as above which implies $\mathrm{FM}(E'') = 0$. This gives the minimality of E . \square

Lemma 4.5. *Let E be a minimal vector bundle of rank two. Then E is stable and of degree $\deg(E) = 2g - 1$.*

Proof. First let us assume that E were not stable. Then we would have a short exact sequence $0 \rightarrow L_1 \rightarrow E \rightarrow L_2 \rightarrow 0$ with $\deg(L_1) \geq \deg(L_2)$. Thus, the vanishing of $\mathrm{FM}^0(L_1)$ implies that $\mathrm{FM}^0(L_2) = 0$. Since $\mathrm{FM}^0(E) \neq 0$, we conclude that $\mathrm{FM}^0(L_1) \neq 0$ which contradicts the minimality of E . We deduce that E is stable.

Raynaud showed in Proposition 1.6.2 in [11] that for a semistable E we have an equivalence $\mathrm{FM}^0(E) \neq 0 \iff \chi(E) > 0$. So we deduce by minimality that $\chi(E) = 1$. Riemann-Roch gives the stated degree. \square

Lemma 4.6. *Let E be a minimal bundle of rank two. The divisor $D = D(E)$ is geometrically equivalent to the the double of the Θ -divisor on $\mathrm{Pic}^0(X)$.*

Proof. By Lemma 4.5 the vector bundle E is stable of degree $\deg(E) = 2g - 1$. We consider a surjection $\pi : E \rightarrow k(P_0)$. The bundle $E' = \ker(\pi)$ is semistable of Euler characteristic $\chi(E') = 0$. Thus, we have that $\mathrm{FM}^0(E') = 0$ and $c_1(\mathrm{FM}^1(E')) \sim 2\Theta$. Once we show that the support of $\mathrm{FM}^1(E)$ is at most $g - 2$ dimensional we conclude, as in the proof of Theorem 4.2 that $D(E) \sim 2\Theta$.

Case 1: E contains a line bundle L_1 of degree $g - 1$.

We choose the surjection π in such a way that $\pi|_{L_1} = 0$. This way $L_1 \subset E'$. Therefore E' is properly semistable, and the Θ -divisor of E' is the union of the irreducible divisors Θ_{L_1} and Θ_{E'/L_1} . From the short exact sequence $0 \rightarrow E' \rightarrow E \rightarrow k(P_0) \rightarrow 0$ we deduce the long exact sequence

$$0 \rightarrow (\mathrm{FM}^0(E) = \mathcal{O}_{\mathrm{Pic}}(-D(E))) \rightarrow \mathcal{O}_{\mathrm{Pic}} \rightarrow \mathrm{FM}^1(E') \rightarrow \mathrm{FM}^1(E).$$

Since the support of $\text{FM}^1(E')$ is the union of Θ_{L_1} and Θ_{E/L_1} we conclude that $D(E)$ is one of the four divisors in the set $\{\emptyset, \Theta_{L_1}, \Theta_{E/L_1}, \Theta_{L_1} + \Theta_{E/L_1}\}$. Since $D(E)$ is ample (see Theorem 3.3 (i)) we can exclude \emptyset . We can exclude Θ_{L_1} because $h^0(\mathcal{O}_{\text{Pic}}(\Theta_{L_1})) = 1$. So there can not be a surjection from $\iota^*\text{FM}_P^{-1}(\mathcal{O}_{\text{Pic}}(\Theta_{L_1}))$ to E (see Theorem 3.3 (iii)). For the same reason we can exclude Θ_{E/L_1} . So we have $D(E) = \Theta_{L_1} + \Theta_{E/L_1}$.

Case 2: E contains no line bundle of degree $g - 1$.

A line bundle $[L] \in \text{Pic}^0(X)(k)$ is in the support of $\text{FM}^1(E) \iff H^1(E \otimes L) \neq 0$. By Serre duality this is equivalent to $\text{Hom}(E, \omega_X \otimes L^{-1}) \neq 0$. So it is enough to show that the line bundles M of degree $2g - 2$ which admit a morphism $E \xrightarrow{\neq 0} M$ form a family of dimension at most $g - 2$. Now any morphism $E \xrightarrow{\alpha} M$ factors through its image $E \rightarrow \text{im}(\alpha) \rightarrow M$. Let $\text{deg}(\text{im}(\alpha)) = 2g - 2 - d$. The semistability of E implies $d \leq g - 2$. Since for a given line bundle $\text{im}(\alpha)$ the extensions $\text{im}(\alpha) \rightarrow M$ are parameterized by effective divisors of degree d we have to show that the image of the Quot scheme $\text{Quot}_E^{1, 2g-2-d}$ of line bundle quotients of E of degree $2g - 2 - d$ is of dimension at most $g - 2 - d$. We consider the following morphisms

$$\text{Quot}_E^{1, 2g-2-d} \xrightarrow[\substack{\beta \\ [E \rightarrow L] \mapsto L}]{} \text{Pic}^{2g-2-d}(X) \xrightarrow[\substack{\gamma \\ L \mapsto L^{\otimes 2} \otimes \det(E)^{-1}}]{} \text{Pic}^{2g-3-2d}(X)$$

Since γ is a finite morphism we have to show that the image of $\gamma \circ \beta$ is of dimension at most $g - 2 - d$.

The Quot scheme parameterizing quotients $[E \xrightarrow{\pi} L]$ has tangent space $\text{Hom}(\ker(\pi), L) = H^0(L \otimes \ker(\pi)^{-1})$. The line bundle $L \otimes \ker(\pi)^{-1} \cong L^{\otimes 2} \otimes \det(E)^{-1}$ is of degree $2g - 3 - 2d$. From Clifford's theorem [4, Theorem IV.5.5] we see that the $h^0(L^{\otimes 2} \otimes \det(E)^{-1}) \leq g - 1 - d$. We assume now that there exists a irreducible component $Q \subset \text{Quot}_E^{1, 2g-2-d}$ of dimension $g - 1 - d$ such that its image $\bar{Q} = \gamma \circ \beta(Q)$ is also of dimension $g - 1 - d$. We consider the cartesian diagram

$$\begin{array}{ccc} Z \cong \bar{Q} \times_{\text{Pic}} X^{(2g-3-2d)} & \longrightarrow & X^{(2g-3-2d)} \\ \xi \downarrow & & \downarrow \\ \bar{Q} & \longrightarrow & \text{Pic}^{2g-3-2d} \end{array}$$

Since for all line bundles \bar{L} parameterized by \bar{Q} we have $h^0(L) = g - 1 - d$ we follow that ξ is a \mathbb{P}^{g-2-d} bundle. So Z is of dimension $2g - 3 - d$. So we must have $Z = X^{(2g-3-2d)}$. However by Poincaré's formula (cf I.§5 in [1]) the image of $X^{(2g-3-2d)}$ is of dimension $2g - 3 - 2d$. So we conclude that $g - 2 - d = 0$ which implies that $d = g - 2$. This means that E has a quotient line bundle $[E \xrightarrow{\pi} L]$ of degree $\text{deg}(L) = g$. The kernel of π is a line bundle of degree $g - 1$ which we excluded in case 2. \square

5. QUOT SCHEMES AS FINE MODULI SPACES

5.1. The general case. Let $(X, \mathcal{O}_X(1))$ be a polarized projective variety, and $\chi : \mathbb{Z} \rightarrow \mathbb{Z}$ a polynomial. We say that a scheme M is up to first order a fine

moduli space of simple χ -bundles on X when the following conditions are satisfied:

- (i) There exists a vector bundle \mathcal{E} on $X \times M$.
- (ii) For any point $P \in M(k)$ the vector bundle $\mathcal{E}_P := \text{pr}_{1*}(\mathcal{E} \otimes \text{pr}_2^* k(P))$ has Hilbert polynomial χ with respect to $\mathcal{O}_X(1)$.
- (iii) For $P, Q \in M(k)$ we have $\text{hom}(\mathcal{E}_P, \mathcal{E}_Q) = \begin{cases} 1 & \text{when } P = Q \\ 0 & \text{otherwise.} \end{cases}$
- (iv) For any $P \in M(k)$ the Kodaira-Spencer map $T_{M,P} \rightarrow \text{Ext}^1(\mathcal{E}_P, \mathcal{E}_P)$ is an isomorphism.

Note that any open subset of a fine moduli space is a up to first order a fine moduli space. The following notion is very convenient: For a point $P \in M(k)$ we have the vector bundle $E = \mathcal{E}_P$ parameterized by P on X . Therefore we denote the point P by $[E]$.

Proposition 5.2. *Let $(X, \mathcal{O}_X(1))$ be a polarized projective Gorenstein variety, and $\chi : \mathbb{Z} \rightarrow \mathbb{Z}$ a polynomial. Assume there exists a up to first order fine moduli space $M = M_X^\chi$ of simple χ -bundles on X . Furthermore, we assume that M is smooth, connected, and projective over k . Under these assumptions there exists a vector bundle P on X such that a connected component Q_1 of the Quot scheme $\text{Quot}_{P/X}^\chi$ of quotients of P with Hilbert polynomial χ gives an isomorphism*

$$Q_1 \xrightarrow[\text{[}Q \rightarrow E \text{]} \mapsto \text{[}E \text{]}]{\sim} M$$

which identifies the universal quotient \mathcal{E} of $\text{pr}_1^* P$ on $X \times Q_1$ with the universal bundle on $X \times M$.

Proof. First, we fix an ample polarization $\mathcal{O}_M(1)$ on M . We consider the morphisms $X \xleftarrow{p} X \times M \xrightarrow{q} M$. Any twist $\mathcal{E}(N) := \mathcal{E} \otimes q^* \mathcal{O}_M(N)$ is also a universal bundle on $X \times M$. For a closed point $[E] \in M(k)$ with ideal sheaf $\mathcal{J}_E \subset \mathcal{O}_M$ we obtain a long exact sequence on X :

$$\cdots \rightarrow p_* \mathcal{E}(N) \rightarrow p_*(\mathcal{E}(N) \otimes q^* k([E])) \rightarrow R^1 p_*(\mathcal{E}(N) \otimes q^* \mathcal{J}_E) \rightarrow \cdots$$

We may choose $N \gg 0$ such that the higher direct image sheaves $R^i p_*(\mathcal{E}(N) \otimes q^* \mathcal{J}_E)$ all vanish for all points $[E] \in M(k)$, and all $i \geq 1$. This forces $p_* \mathcal{E}(N)$ to be a vector bundle. By definition the vector bundle E is isomorphic to $p_*(\mathcal{E}(N) \otimes q^* k([E]))$. Summing up, we obtain a surjection $p_* \mathcal{E}(N) \twoheadrightarrow E$ for all $[E] \in M(k)$.

Next we want to show that $\text{Hom}(p_* \mathcal{E}(N), E)$ is one dimensional given that N is sufficiently big. By Serre duality $\text{Hom}(p_* \mathcal{E}(N), E)$ is dual to $\text{Ext}^{\dim X}(E \otimes \omega_X^\vee, p_* \mathcal{E}(N))$. Since p^* is the left adjoint of p_* we find $\text{Ext}^{\dim X}(E \otimes \omega_X^\vee, p_* \mathcal{E}(N)) = \text{Ext}^{\dim X}(p^*(E \otimes \omega_X^\vee), \mathcal{E}(N))$. This vector space is isomorphic to $H^{\dim X}(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N))$. To compute this cohomology group we use the Leray spectral sequence

$$H^k(R^l q_*(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N))) \implies H^{k+l}(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N)).$$

Choosing N sufficiently high, all the higher cohomology groups $H^k(\dots)$ on the left vanish and we obtain that $H^{\dim X}(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N)) = H^0(R^{\dim X} q_*(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N)))$. By Serre duality, base change, and our condition (iii) we see that $R^{\dim X} q_*(p^* E^\vee \otimes p^* \omega_X \otimes \mathcal{E}(N))$ is the skyscraper sheaf $k([E])$. This proves that $\text{hom}(p_*(\mathcal{E}(N)), E) = 1$.

Since the vector bundles parameterized by M form a bounded family, we may choose N such that $\text{hom}(p_*(\mathcal{E}(N)), E) = 1$ for all $[E] \in M(k)$.

Next we construct a morphism from M to the Quot scheme $\text{Quot}_{p_* \mathcal{E}(N)/X}^\chi$ as follows: On $X \times M \times M$ we consider the surjection $\text{pr}_{13}^* \mathcal{E}(N) \rightarrow \text{pr}_{13}^* \mathcal{E}(N) \otimes \text{pr}_{23}^* \mathcal{O}_{\Delta(M)}$. Applying pr_{12*} to this surjection we obtain a surjection $\text{pr}_{12*} \text{pr}_{13}^* \mathcal{E}(N) \twoheadrightarrow \text{pr}_{12*}(\text{pr}_{13}^* \mathcal{E}(N) \otimes \text{pr}_{23}^* \mathcal{O}_{\Delta(M)})$.

The direct image sheaf $\mathrm{pr}_{12*}\mathrm{pr}_{13}^*\mathcal{E}(N)$ is isomorphic to $p^*p_*\mathcal{E}(N)$, and the quotient $\mathrm{pr}_{12*}(\mathrm{pr}_{13}^*\mathcal{E}(N)\otimes\mathrm{pr}_{23}\mathcal{O}_{\Delta(M)})$ becomes isomorphic to E when specialized to $X\times\{[E]\}$. This way we obtain a morphism from $M\rightarrow\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ which sends $[E]$ to $[p_*\mathcal{E}(N)\xrightarrow{\pi}E]$ where π is the unique (up to scalar) surjective morphism in $\mathrm{Hom}(p_*\mathcal{E}(N),E)$.

To finish the proof we show that the constructed morphism $M\rightarrow\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ identifies M with one connected component of $\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$. By replacing the Quot scheme by the connected component containing the image of M , we may assume $\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ is connected. Since all the sheaves parameterized by M are different the morphism $M\rightarrow\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ is injective on closed points. Thus, $\dim\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X\geq\dim M$. For any $[E]\in M(k)$ we consider the surjection $p_*\mathcal{E}(N)\xrightarrow{\pi}E$. This gives a long exact sequence

$$0\rightarrow\mathrm{Hom}(E,E)\xrightarrow{\alpha}\mathrm{Hom}(p_*\mathcal{E}(N),E)\xrightarrow{\beta}\mathrm{Hom}(\ker\pi,E)\xrightarrow{d_E}\mathrm{Ext}^1(E,E)\rightarrow\dots$$

The morphism α is a morphism between two vector spaces of dimension one. We deduce that β is zero, and d_E is injective. d_E is the tangent map for the forgetful morphism $\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X\rightarrow M$ which assigns $[p_*\mathcal{E}(N)\rightarrow E]\mapsto[E]$ in the point $[p_*\mathcal{E}(N)\rightarrow E]$. We obtain

$$\dim\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X\leq\mathrm{hom}(\ker\pi,E)\leq\mathrm{ext}^1(E,E)=\dim(M).$$

Putting together this inequality with the one obtained before ($\dim\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X\geq\dim M$) we see that all \leq become equalities and d_E is an isomorphism. Since M is projective its image on $\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ is projective. We conclude that $M\rightarrow\mathrm{Quot}_{p_*\mathcal{E}(N)/X}^X$ is an isomorphism. \square

Remark. The assumptions of Proposition 5.2 are fulfilled when X is a smooth projective curve and $M=U_X(r,d)$ is the moduli space of rank r stable vector bundles of degree d on X provided that r and d are coprime integers. Therefore, we find (under the assumption $(r,d)=1$) that all the moduli spaces $U_X(r,d)$ are quot schemes $\mathrm{Quot}_{P/X}^{r,d}$ for certain sheaves P on X .

5.3. The case of algebraic curves. From now on X is a smooth projective curve over k of genus g . We show, how the minimality of certain bundles allows the construction of the moduli spaces of these bundles. Whereas in Proposition 5.2 we can not say much about the vector bundle P , we see that Raynaud's bundles P_m can be taken in the situations of 5.4–5.8.

Proposition 5.4. *The Quot scheme $\mathrm{Quot}=\mathrm{Quot}_{P_2/X}^{1,g}$ of quotients of the Raynaud bundle P_2 of rank one and degree d is isomorphic to $\mathrm{Pic}^g(X)$. The universal quotient of $\mathrm{pr}_1^*P_2$ on $X\times\mathrm{Quot}$ is a Poincaré bundle.*

Proof. Let $P_2\rightarrow\rightarrow Q$ be surjection to a sheaf Q on X with $\mathrm{rk}(Q)=1$, and $\mathrm{deg}(Q)=g$. If Q has torsion subsheaf T different from zero, then there exists a surjection $P_2\rightarrow\rightarrow L=Q/T$ to a line bundle of degree $\mathrm{deg}(L)\leq g-1$. However, L is properly contained in a line bundle M of degree g which contradicts Theorem 4.2. We conclude that Q is torsion free, hence a line bundle.

Let now Q be a line bundle of degree g on X . We have seen that $\mathrm{FM}^0(Q)=\mathcal{O}_{\mathrm{Pic}}(-D)$ for some divisor $D\sim\Theta$. We conclude that $\mathrm{Hom}(\mathcal{O}_{\mathrm{Pic}}(-2\Theta),\mathrm{FM}^0(E))$ is one dimensional. This implies (see (iii) of Theorem 2.1) that $\mathrm{Hom}(P_2,E)$ is one dimensional. Thus, we have a nontrivial morphism $\pi:P_2\rightarrow Q$. The minimality of Q (Theorem 4.2) implies that the image of π cannot be contained in a proper subsheaf. We eventually conclude

that $\text{Hom}(P_2, Q)$ is a one dimensional vector space, generated by an element π which is surjective.

Thus, every point $[P_2 \xrightarrow{\pi} Q]$ of $\text{Quot}(k)$ corresponds to exactly one line bundle $[Q] \in \text{Pic}^d(X)(k)$. Let $\text{Quot} = \text{Quot}_1 \cup \dots \cup \text{Quot}_M$ be the decomposition into irreducible components. Since the forgetful morphism $[P_2 \xrightarrow{\pi} Q] \mapsto [Q]$ is surjective, we may assume that $\dim \text{Quot}_1 \geq g = \dim \text{Pic}^d(X)$. Take a closed point $[P_2 \xrightarrow{\pi} Q] \in \text{Quot}_1(k)$. We have a long exact sequence

$$0 \rightarrow \text{Hom}(Q, Q) \xrightarrow{\alpha} \text{Hom}(P_2, Q) \xrightarrow{\beta} \text{Hom}(\ker(\pi), Q) \xrightarrow{\gamma} \text{Ext}^1(Q, Q).$$

We have seen that $\text{Hom}(P_2, Q)$ is one dimensional, so it follows that α is an isomorphism, $\beta = 0$, and γ is injective. $\text{Hom}(\ker(\pi), Q)$ is the tangent space of Quot_1 at $[P_2 \xrightarrow{\pi} Q]$, so it is of dimension at least g . $\text{Ext}^1(Q, Q)$ is the tangent space of $\text{Pic}^d(X)$ at the point $[Q]$, a g -dimensional vector space. Thus, we see that $\text{hom}(\ker \pi, Q) = g$ for all points in Quot_1 , and γ is an isomorphism. This implies that Quot_1 is regular. We see that $\text{Quot}_1 \rightarrow \text{Pic}^d(X)$ is injective on closed points and an isomorphism on the tangent spaces. Since Quot_1 is projective, we deduce that $\text{Quot} = \text{Quot}_1 \xrightarrow{\sim} \text{Pic}^d(X)$. \square

Corollary 5.5. *For any integer $d \in \mathbb{Z}$ there exists a sheaf P_2^d such that the Quot scheme $\text{Quot}_d = \text{Quot}_{P_2^d/X}^{1,d}$ of rank one quotients of P_2^d is isomorphic to $\text{Pic}^d(X)$ and the universal quotient of $\text{pr}_1^* P_2^d$ on $X \times \text{Quot}_d$ is a Poincaré bundle.*

Proof. Indeed let L be any line bundle of degree one, set $P_2^d = P_2 \otimes L^{\otimes d-g}$, then the result follows from Proposition 5.4. \square

Proposition 5.6. *The Quot scheme $\text{Quot} = \text{Quot}_{P_3/X}^{2,2g-1}$ of rank two quotients of degree $2g-1$ of the Raynaud bundle P_3 is isomorphic to the moduli space $U_X(2, 2g-1)$ of stable rank two bundles of degree $2g-1$. The universal quotient of $\text{pr}_1^* P_3$ on $X \times \text{Quot}$ is a Poincaré bundle.*

Proof. Let $P_3 \xrightarrow{\pi} F$ be a surjection to a sheaf of rank two of degree $2g-1$. Let T be the torsion part of F , and set $F'' = F/T$.

Step 1: F'' does not surject to a line bundle M of degree smaller than g .

If there were such a surjection $F'' \rightarrow M$, then we could embed M as a proper subsheaf of a line bundle M' of degree g , and would obtain from the composition

$P_3 \twoheadrightarrow F \twoheadrightarrow F'' \twoheadrightarrow M \xrightarrow{\subsetneq} M'$ a nonzero morphism $P_3 \rightarrow M'$ which is not surjective, which contradicts Theorem 4.2.

Step 2: We take now an elementary transformation $F'' \rightarrow F'$ such that $\deg(F') = 2g-1$.

Step 3: F' is stable.

Indeed, suppose that $F' \rightarrow M$ would be a destabilizing quotient. Then $\deg(M) = \mu(M) \leq g-1$. The image of F'' in M were a line bundle of degree at most $g-1$ which is impossible by step 1.

Step 4: $T = 0$

Indeed if $T \neq 0$, then $P_3 \rightarrow F'$ would be a nontrivial morphism which is not surjective which contradicts the minimality of F' by Theorem 4.3.

We eventually conclude that $F = F'$ is stable. Thus, we have seen that any quotient $[P_3 \rightarrow F]$ yields a stable vector bundle F . Stable vector bundles F (with $\text{rk}(F) = 2$, and $\deg(F) = 2g-1$) are by Theorem 4.3 quotients of P_3 . The rest follows along the lines of the proof of Proposition 5.4. \square

Corollary 5.7. *For any odd integer d there exists a vector bundle P_3^d on X such that the Quot scheme $\text{Quot} = \text{Quot}_{P_3^d/X}^{2,d}$ of rank two quotients of degree d of the vector bundle P_3^d is isomorphic to the moduli space $U_X(2, d)$ of stable rank two bundles of degree d . The universal quotient of $\text{pr}_1^*P_3^d$ on $X \times \text{Quot}$ is a Poincaré bundle. \square*

Proposition 5.8. *The Quot scheme $\text{Quot} = \text{Quot}_{P_2/X}^{2,2g-1}$ of rank two quotients of degree $2g - 1$ of the Raynaud bundle P_2 is isomorphic to the moduli space $\text{SU}_X(2, \omega_X(P_0))$ of stable rank two bundles of determinant isomorphic to $\omega_X(P_0)$. As before, the universal quotient of $\text{pr}_1^*P_2$ on $X \times \text{Quot}$ is a Poincaré bundle.*

Proof. The birational divisor class $D(E)$ does not vary when $\det(E)$ stays constant. Thus, all vector bundles E of determinant $\omega_X(P_0)$ have the same divisor $D(E)$. \square

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