$\mathrm{SU}_X(r, L)$ IS SEPARABLY UNIRATIONAL

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Abstract. We show that the moduli space of $\mathrm{SU}_X(r, L)$ of rank r bundles of fixed determinant L on a smooth projective curve X is separably unirational.

1. Introduction

In a discussion V. B. Mehta pointed out to me that for certain applications about the cohomology of moduli of vector bundles on smooth projective curves over algebraically closed fields in characteristic p it is necessary to have that $\mathrm{SU}_X(r, L)$ is separably unirational. This short note provides us with a proof of this statement.

Theorem 1. *Let X be a projective curve of genus $g \geq 2$ over an algebraically closed field k of arbitrary characteristic. We fix a line bundle L on X. The moduli space $\mathrm{SU}_X(r, L)$ of S-equivalence classes of semistable vector bundles of rank r with determinant isomorphic to L is separably unirational, that means there exists an open subset $U \subset \mathbb{P}^{(r^2-1)(g-1)}$, and an étale morphism $U \to \mathrm{SU}_X(r, L)$.*

For a discussion of the notion *separable unirationality* and typical applications see the lecture notes [2, 1.10 and 1.11]. If the characteristic of k is zero, then separably unirational and unirational coincide. Thus, in this case the result is well known (see for example page 53 in Seshadri's lecture notes [1]).

2. Proof of theorem 1

Let $d = \deg(L)$ be the degree of L. Fix a line bundle M such that $\deg(M) < \frac{d}{r} - 2g$. Let E be any semistable vector bundle on X with $\mathrm{rk}(E) = r$, and $\det(E) \cong L$. Set $M_0 := M^{\oplus(r+1)}$, and $M_1 := M^{\otimes(r+1)} \otimes L^{-1}$.

If $\mathrm{Ext}^1(M, E) \neq 0$, then by Serre duality $\mathrm{Hom}(E, M \otimes \omega_X) \neq 0$. $M \otimes \omega_X$ is a stable bundle of slope $\mu(M \otimes \omega_X) = \deg(M) + 2g - 2 < \mu(E)$. Thus, there can be only the zero morphism in $\mathrm{Hom}(E, M \otimes \omega_X)$. So we have $\mathrm{Ext}^1(M, E) = 0$.

By the same argument we conclude that $\mathrm{Ext}^1(M, E(-P)) = 0$ for every point $P \in X(k)$. Therefore $\mathrm{Hom}(M, E) \to \mathrm{Hom}(M, E \otimes k(P))$ is a surjection. We conclude that $\mathrm{Hom}(M, E) \otimes M \to E$ is surjective. Since X is of dimension one, for a general subspace $V \subset \mathrm{Hom}(M, E)$ of dimension $r + 1$ the restriction $V \otimes M \to E$ is surjective. We

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obtain a surjection $M_0 \xrightarrow{\pi} E$. The kernel of π is a line bundle and its determinant is $\det(\ker(\pi)) \cong \det(M_0) \otimes \det(E)^{-1} = M^{\otimes(r+1)} \otimes L^{-1} = M_1$. Taking one isomorphism $M_1 \xrightarrow{\sim} \ker(\pi)$ we obtain for any E as before the existence of a short exact sequence

$$0 \to M_1 \xrightarrow{\iota} M_0 \xrightarrow{\pi} E \to 0\,. \tag{1}$$

We use this to parameterize all bundles in $\mathrm{SU}_X(r, L)$ as cokernels of morphisms $M_1 \to M_0$. To do so, we define $V := \mathrm{Hom}(M_1, M_0)^\vee$, consider $\mathbb{P}(V) \xleftarrow{p} \mathbb{P}(V) \times X \xrightarrow{q} X$, and the natural morphism $p^*\mathcal{O}_{\mathbb{P}(V)}(-1) \otimes q^*M_1 \xrightarrow{\alpha} q^*M_0$. We denote the cokernel of α by $\mathcal{E}$. Let U_1 be the open subset of points $u \in \mathbb{P}(V)$ such that $\mathcal{E}_u := q_*(\mathcal{E} \otimes p^*k(u))$ is a semistable bundle on X. We see that the resulting morphism gives a surjection $U_1 \xrightarrow{\rho} \mathrm{SU}_X(r, L)$.

Next we show that ρ is infinitesimal surjective. We take a point $[\iota] \in \mathbb{P}(V)$ corresponding to a short exact sequence (1). Let $D = k[\varepsilon]/\varepsilon^2$ be the ring of dual numbers. To give an infinitesimal deformation of E corresponds to give a flat family E_D on $X_D = \mathrm{Spec}(D) \times X$ which specializes to E when restricting to the reduced fiber $X_0 \cong X$. Since we want to consider deformations with fixed determinant we have an isomorphism $\det(E_D) \cong q_D^*L$ where q_D is the projection $X_D \to X$. The flat deformation E_D yields a short exact sequence $0 \to E \to E_D \to E \to 0$ on $\mathrm{Spec}(D) \times_k X$ which gives the exact sequence

$$\begin{array}{ccccc} \mathrm{Hom}_{X_D}(q_D^*M_0, E_D) & \longrightarrow & \mathrm{Hom}_{X_D}(q_D^*M_0, E) & \longrightarrow & \mathrm{Ext}^1_{X_D}(q_D^*M_0, E) \\ & & \| & & \| \\ & & \pi \in \mathrm{Hom}_X(M_0, E) & & \mathrm{Ext}^1_X(M_0, E) \end{array}$$

Since $M_0 = M^{\oplus(r+1)}$, we conclude that $\mathrm{Ext}^1(M_0, E) = 0$ from the fact that $\mathrm{Ext}^1(M, E) = 0$. So π is the restriction of some $\pi_D \in \mathrm{Hom}_{X_D}(q_D^*M_0, E_D)$ to the reduced fiber. The morphism $\pi_D : q_D^*M_0 \to E_D$ is surjective. Again $\ker(\pi_D)$ is isomorphic to the line bundle $\det(q_D^*M_0) \otimes \det(E_D)^{-1} = q_D^*(\det(M_0) \otimes L^{-1}) = q_D^*M_1$. Fixing such an isomorphism we obtain a short exact sequence of $\mathcal{O}_{X_D}$ bundles

$$0 \to q_D^*M_1 \xrightarrow{\iota_D} q_D^*M_0 \xrightarrow{\pi_D} E_D \to 0\,.$$

We conclude that any deformation E_D of E is induced by a deformation ι_D of ι. Now for a general linear subspace $L \subset \mathbb{P}(V)$ of dimension $\dim \mathrm{SU}_X(r, L) = (r^2 - 1)(g - 1)$ passing through a stable $[E] \in \mathbb{P}(V)$ the composition of tangent maps is an isomorphism to $T_{\mathrm{SU}_X(r,L),E}$. Thus, on some Zariski open subset $U \subset (L \cap U_1)$ containing $[E]$ the morphism $U \to \mathrm{SU}_X(r, L)$ is étale. $\square$

References

[1] C. S. Seshadri, *Fibrés vectoriels sur les courbes algébriques*, Astérisque **96**, Société Mathématique de France, Paris, 1982.

[2] K. Smith with an appendix by J. Rosenberg, *Rational and Non-Rational Algebraic Varieties: Lectures of János Kollár*, `alg-geom/9707013`.

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