

ALGEBRAIC DIFFERENTIAL CHARACTERS

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0. INTRODUCTION

In [5], Cheeger and Simons defined on a \mathcal{C}^∞ manifold X a group of differential characters $\hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$, which is an extension of the global \mathbb{R} valued closed forms of degree $2n$ having \mathbb{Z} periods, by the group $H^{2n-1}(X, \mathbb{R}/\mathbb{Z})$. (In fact, they write $\hat{H}^{2n-1}(X, \mathbb{R}/\mathbb{Z})$ but the notation $2n$ rather than $(2n - 1)$ fits better with weights in algebraic geometry). Similarly, there is a group of (complex) differential characters $\hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$, presented as an extension of the global \mathbb{C} valued closed forms of degree $2n$ having \mathbb{Z} periods, by the group $H^{2n-1}(X, \mathbb{C}/\mathbb{Z})$. The group $\hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$) is also presented as an extension of the Betti cohomology group $H^{2n}(X, \mathbb{Z})$ by global \mathbb{R} valued (resp. \mathbb{C} valued) differential forms of degree $2n - 1$, modulo the closed ones with \mathbb{Z} periods.

They define a ring structure on $\hat{H}^{2\bullet}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{H}^{2\bullet}(X, \mathbb{C}/\mathbb{Z})$) and show the existence of functorial and additive classes $\hat{c}_n(E, \nabla)_{\mathbb{R}} \in \hat{H}^{2n}(X, \mathbb{R}/\mathbb{Z})$ (resp. $c_n(E, \nabla) \in \hat{H}^{2n}(X, \mathbb{C}/\mathbb{Z})$) for a \mathcal{C}^∞ bundle with a connection ∇ , lifting the closed form $P_n(\nabla^2, \dots, \nabla^2)$ with \mathbb{Z} periods, where P_n is the homogeneous symmetric invariant polynomial of degree n which is the n -th symmetric function in the entries on diagonal matrices. When the connection ∇ is flat, that is when $\nabla^2 = 0$, then $\hat{c}_n(E, \nabla)_{\mathbb{R}} \in H^{2n-1}(X, \mathbb{R}/\mathbb{Z})$ (resp. $\hat{c}_n(E, \nabla) \in H^{2n-1}(X, \mathbb{C}/\mathbb{Z})$).

The aim of this article is to develop a similar construction when X is an algebraic smooth variety over a field k and (E, ∇) is an algebraic bundle with an algebraic connection. We define a group of *algebraic differential characters* $AD^n(X)$, which is an extension of global closed algebraic forms of degree $2n$, whose cohomology class in $\mathbb{H}^{2n}(X, \Omega_X^{\geq 2n})$ is algebraic, by the group $\mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d\log} \Omega_X^n \rightarrow \dots)$, introduced and studied in [7], [3]. When $k = \mathbb{C}$, $AD^n(X)$ maps to the group of *analytic differential characters* $D^n(X)$, defined as an extension of global analytic forms of degree $2n$, with $\mathbb{Z}(n)$ periods, by

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$H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$. The group $D^n(X)$ maps to the Deligne cohomology group $H_{\mathcal{D}}^{2n}(X, \mathbb{Z}(n))$. (The above description is valid when X is proper, otherwise one has to modify it a little bit). The group $AD^n(X)$ (resp. $D^n(X)$) is also presented as an extension of the subgroup $\text{Ker } CH^n(X) \rightarrow \mathbb{H}^n(X, \Omega_X^n \rightarrow \dots \rightarrow \Omega_X^{2n-1})$ of the Chow group $CH^n(X)$ (resp. of the subgroup $\text{Ker } H^{2n}(X_{\text{an}}, \mathbb{Z}(n)) \rightarrow H^n(X_{\text{an}}, \Omega_X^n \rightarrow \dots \rightarrow \Omega_X^{2n-1})$) of the Betti cohomology group $H^{2n}(X_{\text{an}}, \mathbb{Z}(n))$ by $\mathbb{H}^{n-1}(X, \Omega_X^n \rightarrow \dots \rightarrow \Omega_X^{2n-1})/H^{n-1}(X, \mathcal{K}_n)$ (resp. $\mathbb{H}^{n-1}(X_{\text{an}}, \Omega_X^n \rightarrow \dots \rightarrow \Omega_X^{2n-1})/H^{2n-1}(X_{\text{an}}, \mathbb{Z}(n))$). There is a ring structure on $AD^\bullet(X), D^\bullet(X)$. If (E, ∇) is an algebraic bundle with an algebraic connection, we define functorial and additive classes $c_n(E, \nabla) \in AD^n(X)$, lifting $P_n(\nabla^2, \dots, \nabla^2)$ and the algebraic Chern classes $c_n(E) \in CH^n(X)$. The group $AD^n(X)$ maps to $H^0(X, \Omega_X^{2n-1}/d\Omega_X^{2n-2})$. The class $c_n(E, \nabla)$ lifts the algebraic Chern-Simons class $w_n(E, \nabla)$, related to the algebraic equivalence relation on cycles, defined in [3]. When ∇ is flat, then $c_n(E, \nabla) \in \mathbb{H}^n(X, \mathcal{K}_n \xrightarrow{d \log} \Omega_X^n \rightarrow \dots)$ and is the class defined in [7].

We give two constructions of the classes. The first one is a *generalized splitting principle*. In [6] and [7], we had defined a modified splitting principle when ∇ is flat. However, our construction had the disadvantage to use the flag bundle of E , rather than simply its projective bundle $\mathbb{P}(E)$, and to introduce a “ τ cohomology” on the flag bundle which is not a free module over the corresponding “ τ cohomology” of the base X . We correct those two points here by introducing a slightly more complicated “ τ cohomology” on $\mathbb{P}(E)$, also defined when ∇ is not flat, which is free over the corresponding “ τ cohomology” on X . The product structure is then naturally defined following the recipe explained in [1] and [8]. The whole construction is now very closed to the construction of Hirzebruch-Grothendieck for bundles without supplementary structure.

The second construction relies on the generalized *Weil algebra* defined by Beilinson and Kazhdan in [2]. To a bundle E , they associate functorially a Weil algebra complex $\Omega_{X,E}^\bullet$ together with a group $H^{2n}(X, U_E(n))$, which is an extension of the homogeneous symmetric invariant polynomials P of degree n mapping to

$$\text{Im}(H^{2n}(X_{\text{an}}, \mathbb{Z}(n)) \rightarrow H^{2n}(X_{\text{an}}, \mathbb{C}))$$

via the Weil homomorphism, by $H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$. By evaluating their construction on the universal simplicial bundle, they defined functorial classes

$$c_n^{BK}(E) \in H^{2n}(X, U_E(n)).$$

A connection ∇ on E defines a map $\nabla : H^n(X, U_E(n)) \rightarrow D^n(X)$, and thereby classes in $D^n(X)$.

We modify their construction to make it more algebraic. We obtain in this way $c_n^{ABK}(E, \nabla) \in AD^n(X)$. We show that $c_n(E, \nabla) = c_n^{ABK}(E, \nabla)$, reflecting the fact that the splitting principle and the universal bundle construction define the same Chern classes for bundles without supplementary structure.

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1. NOTATIONS

- 1) X is a smooth algebraic variety over a field k , D is a normal crossing divisor, $j : U = X - D \rightarrow X$ is the embedding.
- 2) $\lambda : X \rightarrow \bar{X}$ is a good compactification, such that $\Delta = \bar{X} - X$ and $\bar{X} - U = \bar{D}$ are normal crossing divisors, $\bar{j} = \lambda \circ j : U \rightarrow \bar{X}$ is the embedding.
- 3) $(\Omega_X^\bullet(\log D), \Omega_{\bar{X}}^{\geq n}(\log D))$ is the de Rham complex with logarithmic poles along D , filtered by its stupid filtration, and $\Omega_X^{< n}(\log D)$ is its quotient.
- 4) $\tau_0 : \Omega_X^\bullet(\log D) \rightarrow N^\bullet$ is a map of complexes, where N^\bullet is a differential graded algebra, such that the sheaves N^n are locally free, as well as $B^b = \text{Ker } \Omega_X^b(\log D) \rightarrow N^b$, such that $N^0 = \mathcal{O}_X$, and that if a is the smallest degree for which $B^a \neq 0$, then $B^b = B^a \wedge \Omega_X^{b-a}(\log D)$ for $b \geq a$ (see [3] (3.11) and [6] (2.1)).
- 5) \mathcal{K}_n is the image of the Zariski sheaf \mathcal{K}_n^M of Milnor K theory in $K_n^M(k(X))$, with its $d \log$ map $\mathcal{K}_n \xrightarrow{d \log} \Omega_X^{\geq n}[n]$, and its induced map $\tau_0 \circ d \log : \mathcal{K}_n \rightarrow N^{\geq n}[n]$. $N^\infty \mathcal{K}_n$ is the complex $(\mathcal{K}_n \rightarrow N^n \rightarrow N^{n+1} \rightarrow \dots)$.
- 6) (E, ∇) is an algebraic bundle of rank r with an algebraic τ_0 connection, also called a N^1 valued connection

$$\nabla : E \rightarrow N^1 \otimes_{\mathcal{O}_X} E,$$

that is a k linear map verifying the Leibniz rule

$$\nabla(\lambda e) = \tau_0 \circ d(\lambda) \otimes e + \lambda \nabla e.$$

∇^2 is the \mathcal{O}_X linear map $\nabla \circ \nabla : E \rightarrow N^2 \otimes E$, call the curvature of ∇ , with

$$\nabla : N^n \otimes E \rightarrow N^{n+1} \otimes E$$

defined by the sign convention

$$\nabla(\alpha \otimes e) = \tau_0 \circ d(\alpha) \otimes e + (-1)^n \alpha \wedge \nabla e.$$

If $\nabla^2 = 0$, one defines the τ_0 de Rham complex

$$(N^\bullet \otimes E, \nabla).$$

- 7) If $k = \mathbb{C}$, $a : X_{\text{an}} \rightarrow X_{\text{zar}}$ is the identity from X endowed with the analytic topology to X endowed with the Zariski topology. For a sheaf \mathcal{F} on X_{zar} , we denote by \mathcal{F}_{an} the sheaf $a^* \mathcal{F}$. When the context is clear, we still write \mathcal{F} for \mathcal{F}_{an} .
- 8) $H_{\mathcal{D}}^P(X, \mathbb{Z}(q))$ is the Deligne-Beilinson cohomology of X .
- 9) $\mathcal{F}^a \mathcal{H}^b$ is the Zariski sheaf associated to $F^a H_{DR}^b(U)$ (the Hodge filtration on the de Rham cohomology), $\mathcal{H}^n(\mathbb{C}/\mathbb{Z}(m))$ to $H^n(U, \mathbb{C}/\mathbb{Z}(m))$, $\mathcal{F}_{\mathbb{Z}(c)}^a \mathcal{H}^b = \text{Ker}(\mathcal{F}^a \mathcal{H}^b \rightarrow \mathcal{H}^b(\mathbb{C}/\mathbb{Z}(c)))$, $\mathcal{H}_{\mathcal{D}}^a(b)$ to $H_{\mathcal{D}}^a(U, \mathbb{Z}(b))$.
- 10) For $G = GL(r)$, of $\mathcal{G} = M(r)$, we denote by $P_n \in S^n(\mathcal{G}^*)^G$ the symmetric invariant polynomial of degree n which is the n -th symmetric function of the diagonal entries on diagonal matrices. We denote by $G_e \subset G$ the matrices fixing the subspace spanned by the r'' first canonical basis vectors, and by $\mathcal{G}_e \subset \mathcal{G}$ the corresponding algebra.
- 11) If E is a vector bundle, we denote by \mathcal{G}_E the endomorphisms of E , and by \mathcal{G}_E^* its dual (of course isomorph to \mathcal{G}_E). If

$$(e) : 0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$$

is an exact sequence of bundles with $r'' = \text{rank} E''$, $r' = \text{rank} E'$, we denote by $\mathcal{G}_{(E,e)} \subset \mathcal{G}_E$ the endomorphisms respecting (e) and by $\mathcal{G}_{(E,e)}^*$ its dual.

2. ALGEBRAIC DIFFERENTIAL CHARACTERS

Definition 2.0.1. *Let τ_0 be as in 1, 4). We denote by ι the natural embedding $N^{\geq 2n}[n] \rightarrow N^{\geq n}[n]$. We define on X the complex*

$$C(n)_{\tau_0} = \text{cone}(\mathcal{K}_n \oplus N^{\geq 2n}[n] \xrightarrow{d \log \oplus -\iota} N^{\geq n}[n])[-1].$$

If $k = \mathbb{C}$, we define on \bar{X}_{an}

$$\begin{aligned} DR(n)_{\tau_0} &= \text{cone} (\Omega_{\bar{X}}^{\leq n}(\log \bar{D}) \rightarrow R\lambda_*(N^{\geq n}[1])[-1]) \\ &= \mathcal{O}_X \rightarrow \cdots \rightarrow \Omega_{\bar{X}}^{n-1}(\log \bar{D}) \rightarrow \lambda_* N^n \rightarrow \lambda_* N^{n+1} \cdots \end{aligned}$$

$$C(n)_{\tau_0}^{\text{an}} = \text{cone} (R\lambda_* \mathbb{Z}(n) \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{\epsilon \oplus -\iota} DR(n)_{\tau_0})[-1]$$

where ϵ is induced by the map

$$\epsilon : R\lambda_* \mathbb{Z}(n) \rightarrow \Omega_{\bar{X}}^\bullet(\log \bar{D}).$$

We define

$$\begin{aligned} AD_{\tau_0}^n(X) &= \mathbb{H}^n(X, C(n)_{\tau_0}) \\ D_{\tau_0}^n(X) &= \mathbb{H}^{2n}(X, C(n)_{\tau_0}^{\text{an}}) \quad \text{if } k = \mathbb{C}. \end{aligned}$$

When $\tau_0 = \text{identity}$ and $D = \phi$, we simply write $AD^n(X)$ and $D^n(X)$.

Proposition 2.0.2.

- 1) The group $D_{\tau_0}^n(X)$ does not depend on λ (this justifies the notation).
- 2) There are exact sequences
 - i)

$$\begin{aligned} 0 \rightarrow \mathbb{H}^n(X, N^\infty \mathcal{K}_n) \rightarrow AD_{\tau_0}^n(X) \\ \rightarrow \text{Ker} (H^0(X, N_{cl}^{2n}) \rightarrow \frac{\mathbb{H}^{2n}(X, N^{\geq n})}{CH^n(X)}) \rightarrow 0 \end{aligned}$$

ii)

$$\begin{aligned} 0 \rightarrow \mathbb{H}^{2n}(X_{\text{an}}, \text{cone} (R\lambda_* \mathbb{Z}(n) \rightarrow DR(n)_{\tau_0})[-1]) \\ \rightarrow D_{\tau_0}^n(X) \rightarrow \text{Ker} (H^0(X_{\text{an}}, N_{cl}^{2n}) \\ \rightarrow \frac{\mathbb{H}^{2n}(\bar{X}_{\text{an}}, DR(n)_{\tau_0})}{H^{2n}(X_{\text{an}}, \mathbb{Z}(n))}) \rightarrow 0 \end{aligned}$$

For $\tau_0 = \text{identity}$, $D = \phi$ and X proper, this reads

$$\begin{aligned} 0 \rightarrow H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \rightarrow D^n(X) \\ \rightarrow \text{Ker} (H^0(X_{\text{an}}, \Omega_{cl}^{2n}) \rightarrow H^{2n}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))) \rightarrow 0 \end{aligned}$$

For $\tau_0 = \text{identity}$, $D = \phi$ and X non-proper, then one has a splitting $\mathbb{H}^m(X_{\text{an}}, DR(n)) = H_{DR}^m(X) \oplus R(m, n)$ and

$$\text{Ker } D^n(X) \xrightarrow{\text{restriction}} \mathbb{H}^{2n}(X_{\text{an}}, C(n)^{\text{an}}|_{X_{\text{an}}}) = R(2n-1, n).$$

3) If $k = \mathbb{C}$, there is a commutative diagram

$$\begin{array}{ccc} AD_{\tau_0}^n(X) & \xrightarrow{\alpha} & CH^n(X) \\ \psi_{\tau_0} \downarrow & & \downarrow \psi \\ D_{\tau_0}^n(X) & \xrightarrow{\beta} & H_{\mathcal{D}}^{2n}(X, \mathbb{Z}(n)) \end{array}$$

Proof.

1) As usual, if λ and λ' are two good compactifications, one constructs a third one λ_1 dominating λ and λ' , with $\sigma : \bar{X}_1 \rightarrow \bar{X}$. One just has to compare the λ and the λ_1 constructions. Then σ induces a map

$$\sigma^* : \mathbb{H}^m(\bar{X}, C(n)_{\tau_0, \lambda}^{\text{an}}) \rightarrow \mathbb{H}^m(\bar{X}_1, C(n)_{\tau_0, \lambda_1}^{\text{an}})$$

(where we put a λ index to underline the dependance), and from the cone definition of $C(n)_{\tau_0, \lambda}^{\text{an}}$, one just has to see that

$$\sigma^* : \mathbb{H}^m(\bar{X}, DR(n)_{\tau_0, \lambda}) \rightarrow \mathbb{H}^m(\bar{X}_1, DR(n)_{\tau_0, \lambda_1})$$

is an isomorphism. But one has a long exact sequence

$$\begin{aligned} \rightarrow \mathbb{H}^m(X_{\text{an}}, N^{\geq n}) &\rightarrow \mathbb{H}^m(\bar{X}, DR(n)_{\tau_0, \lambda}) \\ &\rightarrow H_{DR}^m(X)/F^n H_{DR}^m(X) \rightarrow \dots \end{aligned}$$

and σ^* induces an isomorphism on the two terms $\mathbb{H}^m(X_{\text{an}}, N^{\geq n})$ and $H_{DR}^m(X)/F^n H_{DR}^m(X)$.

2) i) We just regard the exact triangle

$$\rightarrow N^\infty \mathcal{K}_n \rightarrow C(n)_{\tau_0} \rightarrow N^{\geq 2n}[n] \xrightarrow{[1]}$$

ii) Similarly we regard the exact triangle

$$\begin{aligned} \rightarrow \text{cone}(R\lambda_* \mathbb{Z}(n) \rightarrow DR(n)_{\tau_0})[-1] \\ \rightarrow C(n)_{\tau_0}^{\text{an}} \rightarrow R\lambda_* N_{\text{an}}^{\geq 2n}[n] \xrightarrow{[1]} . \end{aligned}$$

When $\tau_0 = \text{identity}$ and $D = \phi$, the maps

$$\rightarrow \Omega_{\bar{X}_{\text{an}}}^\bullet(\log \bar{D}) \rightarrow DR(n) \rightarrow R\lambda_*(\Omega_{\bar{X}_{\text{an}}}^\bullet)$$

define a splitting

$$\mathbb{H}^m(\bar{X}_{\text{an}}, DR(n)) = H_{DR}^m(X) \oplus R(m, n),$$

with $H^m(X_{\text{an}}, \mathbb{Z}(n))$ mapping to $H_{DR}^m(X)$. This shows that the restriction map is injective on the right hand side of the exact sequence ii) and that the Kernel of the left hand side is $R(2n - 1, n)$.

3) α is induced by

$$C(n)_{\tau_0} \rightarrow \mathcal{K}_n$$

and β by

$$C(n)_{\tau_0}^{\text{an}} \rightarrow \text{cone}(R\lambda_*\mathbb{Z}(n) \rightarrow \Omega_{\bar{X}}^{\leq n}(\log \bar{D}))[-1].$$

The cycle map is compatible with restriction to open sets. From the commutative diagrams of exact sequences

$$\begin{array}{ccccccc} & & \mathbb{H}^{2n-1}(X, N^{\geq n}/N^{\geq 2n}) & \xlongequal{\quad} & \mathbb{H}^{2n-1}(X, N^{\geq n}/N^{\geq 2n}) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker} & \longrightarrow & H & \longrightarrow & AD_{\tau_0}^n(X) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker} & \longrightarrow & CH^n(\bar{X}) & \longrightarrow & CH^n(X) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \mathbb{H}^n(X, N^{\geq n}/N^{\geq 2n}) & \xlongequal{\quad} & \mathbb{H}^n(X, N^{\geq n}/N^{\geq 2n}) \end{array}$$

$$\begin{array}{ccccccc} & & \mathbb{H}^{2n-1}(X_{\text{an}}, N^{\geq n}/N^{\geq 2n}) & \xlongequal{\quad} & \mathbb{H}^{2n-1}(X_{\text{an}}, N^{\geq n}/N^{\geq 2n}) & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \text{Ker}^{\text{an}} & \longrightarrow & H^{\text{an}} & \longrightarrow & D_{\tau_0}^n(X) \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ker}^{\text{an}} & \longrightarrow & H_{\mathcal{D}}^{2n}(\bar{X}, \mathbb{Z}(n)) & \longrightarrow & H_{\mathcal{D}}^{2n}(X, \mathbb{Z}(n)) \\ & & & & \downarrow & & \downarrow \\ & & & & \mathbb{H}^{2n}(X_{\text{an}}, N^{\geq n}/N^{\geq 2n}) & \xlongequal{\quad} & \mathbb{H}^{2n}(X_{\text{an}}, N^{\geq n}/N^{\geq 2n}) \end{array}$$

with

$$H = \mathbb{H}^n(\bar{X}, \bar{C}(n)_{\tau_0})$$

$$H^{\text{an}} = \mathbb{H}^{2n}(\bar{X}, \bar{C}(n)_{\tau_0}^{\text{an}})$$

$$\bar{C}(n)_{\tau_0} = \text{cone}(\mathcal{K}_n \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{d \log \oplus -\iota} R\lambda_*(N^{\geq n}))[-1]$$

$$\bar{C}(n)_{\tau_0}^{\text{an}} = \text{cone}(\mathbb{Z}(n) \oplus R\lambda_*(N^{\geq 2n})[n] \xrightarrow{\epsilon \oplus -\iota} \bar{D}R(n)_{\tau_0})[-1]$$

$$\bar{D}R(n)_{\tau_0} = \text{cone}(\Omega_{\bar{X}}^{\leq n} \rightarrow R\lambda_*N^{\geq n}[1])[-1].$$

One sees that it is enough to lift $\psi_{\bar{X}} : CH^n(\bar{X}) \rightarrow H_{\mathcal{D}}^{2n}(\bar{X}, \mathbb{Z}(n))$ to $\psi_{\bar{X}, \tau_0} : H \rightarrow H^{\text{an}}$. One has

$$R^i a_* \bar{C}(n)_{\tau_0}^{\text{an}} = \mathcal{H}^{i-1}(\mathbb{C}/\mathbb{Z}(n)) \quad \text{for } i \leq n-1.$$

By the Bloch-Ogus vanishing theorem ([4], (6.2)), this implies

$$\begin{aligned} \mathbb{H}^{2n}(\bar{X}, \bar{C}(n)_{\tau_0}^{\text{an}}) = \\ \mathbb{H}^{2n}(\bar{X}, \text{cone}(\tau_{\leq n-1} Ra_* \bar{C}(n)_{\tau_0}^{\text{an}} \rightarrow Ra_* \bar{C}(n)_{\tau_0}^{\text{an}})) \end{aligned}$$

One has an exact sequence

$$0 \rightarrow \mathcal{H}^{n-1}(\mathbb{C}/\mathbb{Z}(n)) \rightarrow \mathcal{H}_{\mathcal{D}}^n(\mathbb{Z}(n)) \rightarrow \mathcal{F}_{\mathbb{Z}(n)}^n \mathcal{H}^n \rightarrow 0$$

and the map of complexes

$$\begin{aligned} (\mathcal{H}_{\mathcal{D}}^n(\mathbb{Z}(n)) \rightarrow (a_* \lambda_* N)^{\geq n} / (a_* \lambda_* N)^{\geq 2n} [n]) \\ \rightarrow \text{cone}(\tau_{\leq n-1} Ra_* \bar{C}(n)_{\tau_0}^{\text{an}} \rightarrow Ra_* \bar{C}(n)_{\tau_0}^{\text{an}}) \end{aligned}$$

factors through

$$\begin{aligned} (\mathcal{F}_{\mathbb{Z}(n)}^n \mathcal{H}^n \rightarrow (a_* \lambda_* N)^{\geq n} / (a_* \lambda_* N)^{\geq 2n} [n]) \\ \xrightarrow{\sigma} \text{cone}(\tau_{\leq n-1} Ra_* \bar{C}(n)_{\tau_0}^{\text{an}} \rightarrow Ra_* \bar{C}(n)_{\tau_0}^{\text{an}}). \end{aligned}$$

To obtain $\psi_{\bar{X}, \tau_0} : H \rightarrow H^{\text{an}}$, we first map $\mathcal{K}_n \rightarrow (\lambda_* N)^{\geq n} / (\lambda_* N)^{\geq 2n} [n]$ to $\mathcal{F}_{\mathbb{Z}(n)}^n \mathcal{H}^n \rightarrow (a_* \lambda_* N)^{\geq n} / (a_* \lambda_* N)^{\geq 2n} [n]$ via the $d \log$ map on \mathcal{K}_n and the change of topology map on $\lambda_* N^i$, and then we apply σ .

□

2.1. Products. In this section, we want to define products on $AD_{\tau_0}^n(X)$ and $D_{\tau_0}^n(X)$, compatibly with the products in the Chow groups and in the Deligne cohomology. To this aim we follow the pattern explained in [1] and [8].

Definition 2.1.1. *Let $\alpha \in \mathbb{R}$. We define*

$$C(m)_{\tau_0} \times C(n)_{\tau_0} \xrightarrow{\cup_{\alpha}} C(m+n)_{\tau_0}$$

by

$$\begin{aligned}
x \cup_{\alpha} y &= \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
&= 0 & x \in \mathcal{K}_m, y \in N^{\geq 2n}[n] \\
&= (1 - \alpha)d \log x \wedge y & x \in \mathcal{K}_m, y \in N^{\geq n}[n] \\
&= 0 & x \in N^{\geq 2m}[m], y \in \mathcal{K}_n \\
&= x \wedge y & x \in N^{\geq 2m}[m], y \in N^{\geq 2n}[n] \\
&= (-1)^{\deg x} \alpha x \wedge y & x \in N^{\geq 2m}[m], y \in N^{\geq n}[n] \\
&= \alpha x \wedge d \log y & x \in N^{\geq m}[m], y \in \mathcal{K}_n \\
&= (1 - \alpha)x \wedge y & x \in N^{\geq m}[m], y \in N^{\geq 2n}[n] \\
&= 0 & x \in N^{\geq m}[m], y \in N^{\geq n}[n]
\end{aligned}$$

Proposition 2.1.2. 1) *These formulae define for each $\alpha \in \mathbb{R}$ a product, compatibly with the product on $\mathcal{K}_n, N^{\geq 2n}$, and with the product*

$$\begin{aligned}
\mathcal{K}_m \times N^{\geq 2n} &\xrightarrow{d \log \times 1} \Omega_X^{\geq m}[m] \times N^{\geq 2n} \xrightarrow{\tau_0} N^{\geq m+2n}[m] \\
N^{\geq 2m} \times \mathcal{K}_n &\xrightarrow{1 \times d \log} N^{\geq 2m} \times \Omega^{\geq n}[n] \xrightarrow{\tau_0} N^{2m+n}[n]
\end{aligned}$$

- 2) *One has $x \cup_{\alpha} y = (-1)^{\deg x \deg y} y \cup_{(1-\alpha)} x$.*
3) *For α and β , the products \cup_{α} and \cup_{β} are homotopic. We denote by \cup the induced product in cohomology. In particular*
a) *\cup is commutative on $AD_{\tau_0}^n(X)$.*
b) *The restriction of \cup to $\mathbb{H}^n(X, N^{\infty} \mathcal{K}_n)$ (2.0.2, 2 ii)) is given by*

$$\begin{aligned}
\{x, y\} & \quad x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
d \log x \wedge y & \quad x \in \mathcal{K}_m, y \in N^{\geq n}[n] \\
0 & \quad \text{otherwise}
\end{aligned}$$

Proof. The verification is exactly as in [8], (3.2), (3.3), (3.5), where one replaces

$$\mathbb{Z}(n) \quad \text{by} \quad \mathcal{K}_n$$

$$F^n \quad \text{by} \quad N^{\geq 2n}[n]$$

$$\Omega_X^\bullet \quad \text{by} \quad N^{\geq n}[n].$$

In particular, the homotopy between \cup_α and \cup_β is given by

$$\begin{aligned} h(x \otimes y) &= (-1)^\mu (\alpha - \beta)x \wedge y \quad \text{if } x \in (N^{\geq m}[m])^{\mu-1} y \in (N^{\geq n}[n])^{\mu'-1} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

for

$$h : (C(m)_{\tau_0} \otimes_{\mathbb{Z}} C(n)_{\tau_0})^\ell \rightarrow C(m+n)_{\tau_0}^\ell.$$

□

3. THE SPLITTING PRINCIPLE FOR BUNDLES WITH CONNECTION

3.1. The τ complex. Let $\tau_0 : \Omega_X^\bullet(\log D) \rightarrow N^\bullet$ and (E, ∇) be as in 1.4), 6), with $B^b = B^a \wedge \Omega_X^{b-a}(\log D)$. Let $\pi : P := \mathbb{P}(E) \rightarrow X$ be the projective bundle of E , and $D' := \pi^{-1}(D)$.

Definition 3.1.1.

$$\Omega_{\mathbb{P}, \tau_0}^n(\log D') = \frac{\pi^* N^n \oplus \Omega_{\mathbb{P}}^n(\log D') / \pi^* B^a \wedge \Omega_{\mathbb{P}}^{n-a}(\log D')}{\pi^*(\Omega_X^n(\log D) / B^a \wedge \Omega_X^{n-a}(\log D))}.$$

For example, if $\Omega_X^\bullet(\log D)$ surjects onto N^\bullet , then this is

$$\mathcal{F}^n = \Omega_{\mathbb{P}}^n(\log D') / \pi^* B^a \wedge \Omega_{\mathbb{P}}^{n-a}(\log D'),$$

and in general it is the push-down of this sheaf via

$$\begin{array}{ccc} \pi^*(\Omega_X^n(\log D) / B^a \wedge \Omega_X^{n-a}(\log D)) & \xrightarrow{\subset} & \mathcal{F}^n \\ \subset \downarrow & & \\ N^n & & \end{array}$$

The Leibniz rule implies that $\pi^* B^a \wedge \Omega_{\mathbb{P}}^{n-a}(\log D')$ is a subcomplex of $\Omega_{\mathbb{P}}^\bullet(\log D')$, and therefore τ_0 induces a map of differential graded algebras, still denoted by τ_0 :

Definition 3.1.2.

$$\tau_0 : (\Omega_{\mathbb{P}}^\bullet(\log D'), d) \rightarrow (\Omega_{\mathbb{P}, \tau_0}^\bullet(\log D'), \tau_0 \circ d).$$

Then ∇ induces a τ_0 connection, still denoted by ∇ :

$$\nabla : \pi^* E \rightarrow \Omega_{\mathbb{P}, \tau_0}^1(\log D') \otimes \pi^* E.$$

Recall from [6], § 2, that the connection

$$\nabla : E \rightarrow N^1 \otimes E$$

induces a splitting

$$\tau : \Omega_{\mathbb{P}, \tau_0}^1(\log D') \rightarrow \pi^* N^1$$

of the exact sequence

$$0 \rightarrow \pi^* N^1 \rightarrow \Omega_{\mathbb{P}, \tau_0}^1(\log D') \rightarrow \Omega_{\mathbb{P}/X}^1 \rightarrow 0,$$

such that the $\pi^* N^1$ valued connection $\tau \circ \nabla$ on $\pi^* E$ respects the canonical filtration

$$0 \rightarrow \Omega_{\mathbb{P}/X}^1(1) \rightarrow \pi^* E \rightarrow \mathcal{O}(1) \rightarrow 0,$$

i.e.:

$$\tau \circ \nabla(\Omega_{\mathbb{P}/X}^1(1)) \subset \pi^* N^1 \otimes \Omega_{\mathbb{P}/X}^1(1).$$

The induced connection on $\Omega_{\mathbb{P}/X}^1 = \Omega_{\mathbb{P}/X}^1(1) \otimes \mathcal{O}(1)^*$ is then given by applying first the splitting into $\Omega_{\mathbb{P}, \tau_0}^1(\log D')$, then applying the differential d of $\Omega_{\mathbb{P}, \tau_0}^\bullet(\log D')$, then projecting onto the factor $\pi^* N^1 \otimes \Omega_{\mathbb{P}/X}^1$ of $\Omega_{\mathbb{P}, \tau_0}^1(\log D')$. We write for short

Definition 3.1.3.

$$\tau d : \Omega_{\mathbb{P}/X}^1 \rightarrow \pi^* N^1 \otimes \Omega_{\mathbb{P}/X}^1.$$

Lemma 3.1.4. *In the splitting*

$$\Omega_{\mathbb{P}, \tau_0}^n(\log D') = \bigoplus_{a=0}^n \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{n-a}$$

one has

$$\begin{aligned} & \tau \circ d(\Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{n-a}) \\ & \subset \Omega_{\mathbb{P}/X}^{a-1} \otimes \pi^* N^{n-a+2} \oplus \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{n-a+1} \\ & \qquad \qquad \qquad \oplus \Omega_{\mathbb{P}/X}^{a+1} \otimes \pi^* N^{n-a} \end{aligned}$$

Proof. The map $\tau \circ d$ has 3 components:

$$\begin{aligned} d_{\text{rel}} : \Omega_{\mathbb{P}/X}^1 & \rightarrow \Omega_{\mathbb{P}/X}^2 \\ & \tau d \\ \mu : \Omega_{\mathbb{P}/X}^1 & \rightarrow \pi^* N^2, \end{aligned}$$

where μ is $\mathcal{O}_{\mathbb{P}}$ linear (see [3], (4.3.2) for the study of μ). One just applies the Leibniz rule. \square

Definition 3.1.5. 1) We define a decreasing filtration Φ on $\Omega_{\mathbb{P},\tau_0}^n(\log D')$ by

$$\Phi^\ell \Omega_{\mathbb{P},\tau_0}^n(\log D') = \bigoplus_{a=\ell}^n \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{n-a}$$

for $\ell \leq n$

$$\Phi^\ell \Omega_{\mathbb{P},\tau_0}^n(\log D') = 0 \text{ for } \ell > n,$$

$$\Phi^0 \Omega_{\mathbb{P},\tau_0}^n(\log D') = \Omega_{\mathbb{P},\tau_0}^n(\log D')$$

fulfilling

$$d\Phi^\ell \Omega_{\mathbb{P},\tau_0}^n(\log D') \subset \Phi^{\ell-1} \Omega_{\mathbb{P},\tau_0}^{n+1}(\log D')$$

2) We define the complex

$$\begin{aligned} (M^{\geq n})^\ell &= 0 && \ell < n \\ &= \Omega_{\mathbb{P},\tau_0}^\ell(\log D') / \Phi^{2n-\ell} \Omega_{\mathbb{P},\tau_0}^\ell(\log D') \\ &= \bigoplus_{a < 2n-\ell} \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{\ell-a} \end{aligned} .$$

In particular, $(M^{\geq n})^\ell = 0$ for $\ell \geq 2n$.

3) We define the complex

$$\begin{aligned} \Gamma(n)_\tau &= \text{cone} \left(\mathcal{K}_n \xrightarrow{\tau \circ d \log} M^{\geq n}[n] \right)[-1] \\ &= \text{cone} \left(\mathcal{K}_n \oplus (M^{\geq n})^{\geq 2n}[n] \xrightarrow{\tau \circ d \log \oplus 0} M^{\geq n}[n] \right)[-1] \end{aligned}$$

endowed with the map

$$\pi^{-1} : C(n)_{\tau_0} \rightarrow R\pi_* \Gamma(n)_\tau$$

induced by

$$\begin{aligned} \pi^{-1} \mathcal{K}_n &\rightarrow \mathcal{K}_n \\ \pi^{-1} N^{\geq 2n} &\rightarrow 0 = (M^{\geq n})^{\geq 2n}, \\ \pi^{-1} N^{\geq n} &\rightarrow M^{\geq n}. \end{aligned}$$

4) We define the multiplication

$$\Gamma(m)_\tau \times \Gamma(m)_\tau \rightarrow \Gamma(m+n)_\tau$$

as in 2.1.1, replacing $N^{\geq n}$ by $M^{\geq n}$, and $N^{\geq 2n}$ by 0, and observing that for $x \in \Omega_{\mathbb{P},\tau_0}^m(\log D')$, $y \in \bigoplus_{a < 2n-\ell} \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{\ell-a}$, then

$x \wedge y \in$

$$\begin{aligned} &\bigoplus_{a < 2n+m-\ell} \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{m+\ell-a} \\ &= \bigoplus_{a < 2(n+m)-(\ell+m)} \Omega_{\mathbb{P}/X}^a \otimes \pi^* N^{m+\ell-a}. \end{aligned}$$

Thus concretely

$$\begin{aligned}
x \cup_{\alpha} y &= \{x, y\} \text{ for } x \in \mathcal{K}_m, y \in \mathcal{K}_n \\
&= (1 - \alpha)d \log x \wedge y \in (M^{\geq m+n})^{m+\ell} \\
&\quad \text{for } x \in \mathcal{K}_m, y \in (M^{\geq n})^{\ell} \\
&= \alpha x \wedge d \log y \in (M^{\geq m+n})^{n+\ell} \\
&\quad \text{for } x \in (M^{\geq m})^{\ell}, y \in \mathcal{K}_n \\
&= 0 \text{ for } x \in M^{\geq m}[m], y \in M^{\geq n}[n].
\end{aligned}$$

Again \cup_{α} does not depend on α on cohomology. We denote it by \cup . In particular, the product induces an action

$$\mathbb{H}^a(\mathbb{P}, M^{\geq b}[b]) \times \mathbb{H}^{a'}(\mathbb{P}, \Gamma(b')_{\tau}) \xrightarrow{\cup} \mathbb{H}^{a+a'}(\mathbb{P}, M^{\geq (b+b')}[b+b'])$$

making Image $\mathbb{H}^{\bullet-1}(\mathbb{P}, N^{\geq \bullet\bullet}[\bullet\bullet])$ in $\mathbb{H}^{\bullet}(\mathbb{P}, \Gamma(\bullet\bullet)_{\tau})$ into an ideal of square 0.

- 5) The group $\mathbb{H}^1(\mathbb{P}, \Gamma(1)_{\tau}) = \mathbb{H}^1(\mathbb{P}, \mathcal{K}_1 \rightarrow \pi^*N^1)$ is the group of isomorphism classes of rank 1 bundles with a π^*N^1 valued connection. We denote by ξ the class of $(\mathcal{O}(1), \tau \circ \nabla)$ in it (3.1)

Theorem 3.1.6. *The maps*

$$\pi^{-1} : AD_{\tau_0}^a(X) \rightarrow \mathbb{H}^a(\mathbb{P}, \Gamma(a)_{\tau})$$

are injective for all $a \geq 0$. One has a splitting

$$\mathbb{H}^m(\mathbb{P}, \Gamma(n)_{\tau}) = \bigoplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, C(n-i)_{\tau_0}) \cup \xi^i$$

(where r is the rank of E (1, 6)).

Proof. We denote by $[\xi]$ the class of $\mathcal{O}(1)$ in $H^1(\mathbb{P}, \mathcal{K}_1)$. One has the projective bundle formula for the cohomology of the sheaves \mathcal{K}

$$\mathbb{H}^m(\mathbb{P}, \mathcal{K}_n) = \bigoplus_{i=0}^{r-1} H^{m-i}(X, \mathcal{K}_{n-i}) \cup [\xi]^i.$$

One also has obviously

$$\mathbb{H}^m(\mathbb{P}, M^{\geq n}[n]) = \bigoplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, N^{\geq n-i}[n-i]) \cup \xi^i.$$

One regards the exact sequences

$$\begin{aligned}
\rightarrow H^{m-1}(\mathbb{P}, \mathcal{K}_n) &\rightarrow \mathbb{H}^{m-1}(\mathbb{P}, M^{\geq n}[n]) \rightarrow \\
&\mathbb{H}^m(\mathbb{P}, \Gamma(n)_{\tau}) \rightarrow H^m(\mathbb{P}, \mathcal{K}_n) \rightarrow \mathbb{H}^m(\mathbb{P}, M^{\geq n}[n]) \rightarrow
\end{aligned}$$

and

$$\begin{aligned} \rightarrow \bigoplus_{i=0}^{r-1} H^{m-1-i}(X, \mathcal{K}_{n-i}) &\rightarrow \bigoplus_{i=0}^{r-1} \mathbb{H}^{m-1-i}(X, N^{\geq r-1-i}[r-1-i]) \\ &\rightarrow \bigoplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, C(n-i)_{\tau_0}) \rightarrow \bigoplus_{i=0}^{r-1} H^{m-i}(X, \mathcal{K}_{n-i}) \\ &\rightarrow \bigoplus_{i=0}^{r-1} \mathbb{H}^{m-i}(X, M^{\geq(n-i)}[n-i]) \rightarrow . \end{aligned}$$

The map π^{-1} induces an isomorphism on the \mathcal{K} and M cohomology, thus on the Γ cohomology as well. \square

Definition 3.1.7. *Let $(E, \nabla), \tau_0, \xi$ be as in 3.1.5. We define*

$$c_0(E, \nabla) = 1 \in AD_{\tau_0}^0(X) = \mathbb{Z}$$

and

$$c_n(E, \nabla) \in AD_{\tau_0}^n(X), n = 1, \dots, r$$

by the formula

$$\sum_{n=0}^r (-1)^n c_n(E, \nabla) \cup \xi^{r-n} = 0$$

via 3.1.6. We define $c_n(E, \nabla) = 0$ for $n > m$.

Theorem 3.1.8.

- 1) **Functoriality:** *Let $\pi : Y \rightarrow X$ be a morphism of smooth varieties, such that $D' = \pi^{-1}D$ is a normal crossing divisor. Let $\tau_0 : \Omega_Y^\bullet(\log D') \rightarrow \Omega_{Y, \tau_0}^\bullet(\log D')$ be the map of differential graded algebras with $\Omega_{Y, \tau_0}^n(\log D')$ defined as in 3.1.1 by*

$$\Omega_{Y, \tau_0}^n(\log D') = \frac{\pi^* N^n \oplus \Omega_Y^n(\log D') / \pi^* B^a \wedge \Omega_Y^{n-a}(\log D')}{\pi^*(\Omega_X^n(\log D) / B^a \wedge \Omega_X^{n-a}(\log D))}.$$

*Then ∇ induces on π^*E a $\Omega_{Y, \tau_0}^1(\log D')$ valued connection $\pi^*\nabla$, and π^{-1} induces a map*

$$\pi^{-1} : AD_{\tau_0}^n(X) \rightarrow AD_{\tau_0}^r(Y).$$

*Then $c_n(\pi^*E, \pi^*\nabla) = \pi^{-1}c_n(E, \nabla)$. For all further maps*

$$\tau : \Omega_{Y, \tau_0}^\bullet(\log D') \rightarrow M^\bullet$$

of differential graded algebras, one has

$$c_n(\pi^*E, \pi^*\nabla) = \text{Im } c_n(\pi^*E, \pi^*\nabla)$$

in $AD_{\tau \circ \tau_0}^n(X)$.

- 2) **First class:** If E is of rank 1, and τ_0, ∇ are as in 1, 4), 6), then $c_1(E, \nabla)$ is the isomorphism class of (E, ∇) in the group of isomorphism classes of rank 1 bundles with a τ_0 connection $AD_{\tau_0}^1(X) = \mathbb{H}^1(X, \mathcal{K}_1 \rightarrow \pi^* N^1)$.
- 3) **Whitney product formula:** Let

$$0 \rightarrow (E'', \nabla'') \rightarrow (E, \nabla) \rightarrow (E', \nabla') \rightarrow 0$$

be an exact sequence of bundles with τ_0 connection, that is the bundle sequence is exact and

$$\nabla E'' \subset N^1 \otimes E'' \quad , \quad \nabla'' = \nabla|_{E''}, \nabla'$$

is the quotient connection. Then one has

$$c_n(E, \nabla) = \bigoplus_{a=0}^n c_a(E', \nabla') \cup c_{n-a}(E'', \nabla'')$$

in $AD_{\tau_0}^n(X)$ for all $n \geq 0$.

- 4) **Characterization of the classes:** The classes $c_n(E, \nabla) \in AD_{\tau_0}^n(X)$ are uniquely determined by the functoriality property, the definition of the first class and the Whitney product formula.

Proof. 1) and 2) are clear, and 4) is clear once one knows 3). For 3) , we mimic the classical proof [9], 6.10. One has

$$\begin{array}{ccc} \mathbb{P}' := \mathbb{P}(E') & \xrightarrow{i} & P \xleftarrow{j} & U : \mathbb{P} - \mathbb{P}' \\ & & & \downarrow p \\ & & & \mathbb{P}'' := \mathbb{P}(E'') \end{array}$$

where p is an affine filtration with the obvious notations $\Gamma(m)_{\tau'}, r'', \dots$ one defines

$$\begin{aligned} \alpha &= \sum_{n=0}^{r'} (-1)^n c_n(E', \nabla') \cup \xi^{r'-n} \in \mathbb{H}^{r'}(\mathbb{P}, \Gamma(r')_{\tau}) \\ \beta &= \sum_{n=0}^{r''} (-1)^n c_n(E'', \nabla'') \cup \xi^{r''-n} \in \mathbb{H}^{r''}(\mathbb{P}, \Gamma(r'')_{\tau}) \end{aligned}$$

Then $\gamma = \alpha \cup \beta \in \mathbb{H}^r(\mathbb{P}, \Gamma(r)_{\tau})$ fulfills

$$j^* \beta = p^{-1} \left(\sum_{n=0}^{r''} (-1)^n c_n(E'', \nabla'') \cup \xi^{r''-n} \right) = 0$$

and $\alpha|_{\mathbb{P}'} = 0$. Let $\beta' \in \mathbb{H}_{\mathbb{P}'}^r(\mathbb{P}, \Gamma(r)_{\tau})$, with $\rho(\beta') = \beta$ in the localization sequence

$$\mathbb{H}_{\mathbb{P}'}^r(\mathbb{P}, \Gamma(r)_{\tau}) \xrightarrow{\rho} \mathbb{H}^r(\mathbb{P}, \Gamma(r)_{\tau}) \xrightarrow{j^*} \mathbb{H}^r(U, \Gamma(r)_{\tau}).$$

Since the product

$$\mathbb{H}^a(\mathbb{P}, \Gamma(b)_\tau) \times \mathbb{H}^{a'}(\mathbb{P}, \Gamma(b')_\tau) \rightarrow \mathbb{H}^{a+a'}(\mathbb{P}, \Gamma(b+b')_\tau)$$

induces a compatible product

$$\mathbb{H}^a(\mathbb{P}, \Gamma(b)_\tau) \times \mathbb{H}_{\mathbb{P}'}^{a'}(\mathbb{P}, \Gamma(b')_\tau) \rightarrow \mathbb{H}_{\mathbb{P}'}^{a+a'}(\mathbb{P}, \Gamma(b+b')_\tau)$$

one has $\gamma = \rho(\alpha \cup \beta')$. On the other hand, one has a Gysin isomorphism

$$\mathbb{H}^{a-r''}(\mathbb{P}', \Gamma(b-r'')_{\tau''}) \xrightarrow{i_*} \mathbb{H}_{\mathbb{P}'}^a(\mathbb{P}, \Gamma(b)_\tau).$$

To see this, one observes that i_* induces an isomorphism on the \mathcal{K} cohomology. For the $M^{\geq b}$ cohomology, one makes a dévissage with the stupid filtration. This then reduces to the obvious isomorphism

$$\mathbb{H}^a(\mathbb{P}', \Omega_{\mathbb{P}'/X}^b \otimes \pi'^* N^c) \xrightarrow{i_*} \mathbb{H}_{\mathbb{P}'}^{a+r''}(\mathbb{P}, \Omega_{\mathbb{P}/X}^{b+r''} \otimes \pi^* N^c).$$

We write $\beta' = i_*(\beta'')$ with $\beta'' \in \mathbb{H}^{r''}(\mathbb{P}', \Gamma(r'')_{\tau'})$, and $\gamma = \rho(\alpha \cup i_*\beta'')$. It remains to see that

$$x \cup i_*y = i_*(x|_{\mathbb{P}'} \cup y)$$

for $x \in \mathbb{H}^a(\mathbb{P}, \Gamma(b)_\tau)$, $y \in \mathbb{H}^{a'-r''}(\mathbb{P}', \Gamma(b'-r'')_{\tau'})$, as $\alpha|_{\mathbb{P}'} = 0$. One checks that the diagram

$$\begin{array}{ccc} \Gamma(b)_\tau \times \text{cone}(\Gamma(b')_\tau \rightarrow Rj_*\Gamma(b')_\tau)[-1] & \xrightarrow{\cup} & \text{cone}(\Gamma(b+b') \rightarrow Rj_*\Gamma(b+b')_\tau)[-1] \\ \text{restriction} \times \text{residue} \downarrow & & \downarrow \text{residue} \\ \Gamma(b)_{\tau'}|_{\mathbb{P}'} \times \Gamma(b'-r'')_{\tau'}|_{\mathbb{P}'} & \xrightarrow{\cup} & \Gamma(b+b'-r'')_{\tau'}|_{\mathbb{P}'} \end{array}$$

is commutative. \square

3.2. Comparison with the algebraic Chern-Simons invariants of [3].

Theorem 3.2.1. 1) Let (τ_0, E, ∇) be as in 3.1.6, with τ_0 surjective. Then the image of $c_n(E, \nabla)$ in $H^0(X, N_{cl}^{2n})$ (2.0.2 2,i) is $P_n(\nabla^2, \dots, \nabla^2)$ (1, 10).

2) Assume $\tau_0 = \text{identity}$. Then the image of $c_n(E, \nabla)$ under

$$AD_{\tau_0}^n(X) \rightarrow H^0(X, \Omega_X^{2n-1}(\log D)/d\Omega_X^{2n-2}(\log D))$$

coincides with the algebraic Chern-Simons class $w_n(E, \nabla)$ defined in [3], § 2.

Proof. 1) Let $p : G \rightarrow X$ be the flag bundle of E , $\Delta = p^{-1}(D)$. Define as in [3], (4.3.3) the sheaf

$$\bar{M}^{2n-\ell} = p^* N^{2n+\ell} / \mu(\Omega_{G/X}^1 \otimes p^* N^{2n+\ell-2})$$

where μ is as in the proof of 3.1.4, for $\ell \geq 0$. One defines $\Omega_{G,\tau_0}^\bullet(\log D)$ as in 3.1.8, 1), and $\Gamma(n)_\tau$ similarly. Let

$$\bar{\Gamma}(n)_\tau = \text{cone} (\mathcal{K}_n \rightarrow \bar{M}^{\geq n}[n])[-1]$$

with

$$\begin{aligned} (\bar{M}^{\geq n})^\ell &= (M^{\geq n})^\ell & \ell \leq 2n-1 \\ &= \bar{M}^\ell & \ell \geq 2n. \end{aligned}$$

Then $\bar{\Gamma}(n)_\tau$ maps to $\Gamma(n)_\tau$. We lift the product in $\Gamma(n)_\tau$ to $\bar{\Gamma}(n)_\tau$ defined in 3.1.5 by setting

$$\begin{aligned} x \cup_\alpha y &= \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= (1-\alpha)d \log x \wedge y & x \in \mathcal{K}_m, y \in (\bar{M}^{\geq n}[n])^\ell \\ &= \alpha x \wedge d \log y & x \in (\bar{M}^{\geq m}[m])^\ell, y \in \mathcal{K}_n \\ &= 0 & x \in \bar{M}^{\geq m}[m], y \in \bar{M}^{\geq n}[n]. \end{aligned}$$

Here the product $d \log x \wedge y$ ($x \wedge d \log y$) means $\tau(d \log x \wedge y)$ ($\tau(x \wedge d \log y)$) for $m + \ell \geq 2(m+n)$ ($\ell + n \geq 2(m+n)$). The product being compatible with the natural product of $(\bar{M}^{\geq n})^{\geq 2n}$, the image of $p^{-1}c_n(E, \nabla)$ in $H^0(G, \bar{M}^{2n} \rightarrow \bar{M}^{2n+1} \rightarrow \dots)$ is the n -th symmetric product of the image $F(\ell_i)$ of

$$\ell_i = c_1(L_i, \tau \circ \nabla) \in \mathbb{H}^1(G, \mathcal{K}_1 \rightarrow p^*N^1)$$

in

$$\mathbb{H}^2(G, (\bar{M}^{\geq 1})^{\geq 2}) = \mathbb{H}^0(G, \frac{p^*N^2}{\mu(\Omega_{G/X}^1)} \rightarrow \dots).$$

In other words, considering as in [3], § 4, the quotient differential graded algebra

$$\Omega_{G,\tau_0}^\bullet(\log \Delta) \rightarrow (\mathcal{O}_G \rightarrow \bar{M}^{\geq 1}),$$

$F(\ell_i)$ is the curvature of the $(\bar{M}^{\geq 1})^1 = p^*N^1$ valued connection $\tau \circ \nabla$ on L_i . As in [3], (4.7.8), this is exactly $p^*P_n(\nabla^2, \dots, \nabla^2)$. Now, as N^{2n} is locally free, $H^0(X, N^{2n})$ injects into $H^0(U, N^{2n})$ for any open $\phi \neq U \subset X$. Thus we may assume that $\nabla : E \rightarrow N^1 \otimes E$ lifts to $\bar{\nabla} : E \rightarrow \Omega_X^1 \otimes E$ as $\Omega_X^1(\log D) \rightarrow N^1$ is surjective, and it is enough to prove 1) for $\bar{\nabla}$. As in [3], § 4, we may further assume that there is a morphism $\varphi : X \rightarrow T$, where T is an affine space, such that $(E, \bar{\nabla}) = \varphi^*(\mathcal{E}, \psi)$, where \mathcal{E} is the trivial bundle, and ψ is such that if $q : P \rightarrow T$ is the flag bundle of \mathcal{E} , and $(\bar{M}^{\geq 1})$

is defined on p , then the map $\Omega_T^{2n} \rightarrow p_*(\bar{M}^{\geq 1})^{2n}$ is injective ([3], Proposition 4.9.1). Since

$$\begin{aligned} AD^n(T) &= \frac{H^0(T, \Omega_T^{2n-1})}{dH^0(T, \Omega_T^{2n-2})} = H^0(X, \frac{\Omega_T^{2n-1}}{d\Omega_T^{2n-2}}) \\ &\subset H^0(T, \Omega_T^{2n}) \subset H^0(P, (\bar{M}^{\geq 1})^{2n}) \end{aligned}$$

we see that

$$\text{Im } c_n(\mathcal{E}, \psi) \text{ in } H^0(P, (\bar{M}^{\geq 1})^{2n}) = p^*P_n(F(\psi), \dots, F(\psi))$$

implies that

$$\text{Im } c_n(\mathcal{E}, \psi) \text{ in } H^0(T, \Omega_T^{2n}) = P_n(F(\psi), \dots, F(\psi)).$$

2) From the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X - D, \mathcal{H}^{2n-1}) \rightarrow H^0(X, \frac{\Omega_X^{2n-1}(\log D)}{d\Omega_X^{2n-1}(\log D)}) \\ \rightarrow H^0(X, \Omega_X^{2n}(\log D)) \end{aligned}$$

and the injectivity

$$H^0(X - D, \mathcal{H}^{2n-1}) \rightarrow H^0(U - D, \mathcal{H}^{2n-1})$$

for any open $0 \neq U \subset X$ ([4], (4.2.2)) one sees that the question is local. We may check it on (T, \mathcal{E}, ψ) as in 1). But on T , the algebraic Chern-Simons invariant is the only class mapping to $P_n(F(\psi), \dots, F(\psi))$.

□

3.3. Comparison with classes of bundles with flat connections.

Theorem 3.3.1. *Let (E, ∇, τ_0) be as in 3.1.6, with τ_0 surjective and with $\nabla^2 = 0$. Then the class $c_n(E, \nabla) \in \mathbb{H}^n(X, N^\infty \mathcal{K}_n)$ (2.0.2) are the same as the classes defined in [7] (see also [3], (3.11)).*

Proof. We consider the flag bundle $p : G \rightarrow X$, with $\Delta = p^{-1}(D)$. We have the quotient maps

$$\Omega_G^\bullet(\log \Delta) \rightarrow \Omega_{G, \tau_0}^\bullet(\log \Delta) \rightarrow p^*N^\bullet$$

when $\Omega_{G, \tau_0}^\bullet(\log \Delta)$ is as in 3.1.8, 1). We also have the filtration Φ on $\Omega_{G, \tau_0}^\bullet(\log \Delta)$ defined as in 3.1.5, for G replacing \mathbb{P} , and the corresponding complex $\Gamma(n)_\tau$. One has a quotient map

$$\Gamma(n)_\tau \rightarrow \text{cone}(\mathcal{K}_n \oplus (p^*N^{\geq 2n})[n] \rightarrow (p^*N^{\geq n})[n])[-1]$$

and the multiplication \cup_0 on $\Gamma(n)_\tau$ defined in 3.1.5 maps to

$$\begin{aligned} x \cup_0 y &= \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= \tau d \log x \wedge y & x \in \mathcal{K}_m, y \in p^* N^\ell \\ &= 0 & \text{otherwise.} \end{aligned}$$

By the Whitney product formula 3.1.8, 3), the class $c_n(E, \nabla) \in AD_{\tau_0}^n(X)$ is the n -th symmetric function of the rank one subquotients

$$(L_i, \tau \circ \nabla) \in \mathbb{H}^1(G, \mathcal{K}_1 \rightarrow (p^* N^1)_{cl}) \subset \mathbb{H}^1(G, \mathcal{K}_1 \rightarrow p^* N^1).$$

Since the multiplication on $\Gamma(n)_\tau$, restricted to the elements of $\mathbb{H}^n(G, \Gamma(n)_\tau)$ mapping to $\mathbb{H}^n(G, \mathcal{K}_n \rightarrow (p^* N^n)_{cl})$, maps to the multiplication on $(\mathcal{K}_n \rightarrow (p^* N^n)_{cl})$ defined in [7], p. 51 in order to construct the classes of flat bundles, one has that the image of $c_n(E, \nabla)$ in $\mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n)$ is the class defined in [7]. One concludes by the commutative diagram

$$\begin{array}{ccc} \mathbb{H}^n(X, N^\infty \mathcal{K}_n) & \xrightarrow{\subset} & AD_{\tau_0}^n(X) \\ \subset \downarrow & & \subset \downarrow \\ \mathbb{H}^n(G, \mathcal{K}_n \rightarrow p^* N^n \rightarrow \dots \rightarrow p^* N^{2n-1}) & \longleftarrow & \mathbb{H}^n(G, \Gamma(n)_\tau), \end{array}$$

where the left vertical injective arrow is the composition of the injection $\mathbb{H}^n(X, N^\infty \mathcal{K}_n) \rightarrow \mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n)$ as in [7], p.51, p.52, with the injection $\mathbb{H}^n(G, p^* N^\infty \mathcal{K}_n) \rightarrow \mathbb{H}^n(G, \mathcal{K}_n \rightarrow p^* N^n \rightarrow \dots \rightarrow p^* N^{2n-1})$, knowing that

$$\text{Im } c_n(E, \nabla) \text{ in } H^0(X, N_{cl}^{2n}) = P_n(\nabla^2, \dots, \nabla^2) = 0$$

3.2.1, 1).

□

4. THE WEIL ALGEBRA

4.1. Construction of Beilinson-Kazhdan.

1. In the unpublished note [2], Beilinson and Kazhdan construct a complex generalizing the Weil homomorphism

$$\begin{aligned} (S^n \mathcal{G}^*)^G &\rightarrow H_{DR}^{2n}(X) \\ P &\mapsto P(\nabla^2, \dots, \nabla^2) \end{aligned}$$

associated to (E, ∇) . We give a short account of their construction. Instead of considering reductive groups in general, we consider the subgroup $G_e \subset G$ of matrices fixing the flag

$$e : 0 \rightarrow k^{r''} \rightarrow k^r \rightarrow k^{r'} \rightarrow 0.$$

We denote by (E, e) and exact sequence

$$0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$$

with $\text{rank } E'' = r''$, with the convention $(E, 0) = E$, $G_0 = G$ (see 1, 10), 11)). Let

$$0 \rightarrow \Omega_X^1 \xrightarrow{\iota} \Omega_{X,(E,e)}^1 \xrightarrow{\pi} \mathcal{G}_{(E,e)}^* \rightarrow 0$$

be the (functorial) Atiyah sequence of (E, e) , whose splitting is equivalent to a Ω_X^1 valued connection on E compatible with e . Then $\Omega_{X,(E,e)}^1 = (p_* \Omega_{\mathcal{E}_e}^1)_e^G$, where $p : \mathcal{E}_e \rightarrow X$ is the total space of the G_e torsor (1, 10)) associated to (E, e) . Let $\Omega_{X,(E,e)}^\bullet$ be the sheaf of commutative differential graded algebras generated by \mathcal{O}_X in degree 0, Ω_X^1 in degree 1, such that for $f \in \mathcal{O}_X$, df is the Kähler differential in $\Omega_X^1 \subset \Omega_{X,(E,e)}^1$. Then

$$\Omega_{X,(E,e)}^n = \bigoplus_{a+b=n} \Omega_{X,(E,e)}^{a,b}$$

with

$$\Omega_{X,(E,e)}^{a,b} = \Lambda^{a-b} \Omega_{X,(E,e)}^1 \otimes S^b \mathcal{G}_{(E,e)}^*$$

with differential $d = d' + d''$,

$$d' : \Omega_{X,(E,e)}^{a,b} \rightarrow \Omega_{X,(E,e)}^{a+1,b}$$

being induced from \mathcal{E}_e via $\Lambda^i \Omega_{X,(E,e)}^1 = (p_* \Omega_{\mathcal{E}_e}^i)^G$, and

$$d'' : \Omega_{X,(E,e)}^{a,b} \rightarrow \Omega_{X,(E,e)}^{a,b+1}$$

being defined by $d''|_{\Omega_{X,(E,e)}^{10}} = d''|_{\Omega_{X,(E,e)}^1} = \pi$,

$$d''(x_1 \wedge \dots \wedge x_{a-b} \otimes \sigma) = \sum_{i=1}^{a-b} (-1)^i x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge x_{a-b} \pi(x_i) \cdot \sigma$$

for

$$x_1 \wedge \dots \wedge x_{a-b} \in \Lambda^{a-b} \Omega_{X,(E,e)}^1, \sigma \in S^b \mathcal{G}_{(E,e)}^*.$$

2. One defines the decreasing filtration

$$F^n \Omega_{X,(E,e)}^\bullet = \bigoplus_{a \geq n} \Omega_{X,(E,e)}^{a,b}$$

Then

$$(\Omega_X^\bullet, \Omega_X^{\geq n}) \rightarrow (\Omega_{X,(E,e)}^\bullet, F^n)$$

is a filtered quasi-isomorphism.

3. There is a canonical map

$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \rightarrow S^n(\mathcal{G}_{(E,e)}^*) = \Omega_{X,(E,e)}^{n,n}$$

with $\text{Im } w^n \subset (\Omega_{X,(E,e)}^{n,n})_{cl} \subset F^n \Omega_{X,(E,e)}^\bullet[2n]$ which defines a map of commutative differential graded algebras

$$w : \bigoplus_n S^n(\mathcal{G}_e^*)^{G_e}[-2n] \rightarrow \Omega_{X,(E,e)}^\bullet.$$

This is called the *Weil homomorphism* of (E, e) and $\Omega_{X,(E,e)}^\bullet$ is called its *Weil algebra*. (Locally on X , $T_{\mathcal{E}_e/X} \simeq \mathcal{G}_e \times G_e \times X$. The gluing functions act via the adjoint representation on \mathcal{G}_e and the multiplication on G_e . Therefore $(S^n \mathcal{G}_e^*)^{G_e}$ defines functions on $T_{\mathcal{E}_e/X}$. This defines w^n . To see that $d'(\text{Im } w^n) = d''(\text{Im } w^n) = 0$, one may assume that $\mathcal{E}_e \simeq G_e \times X$, and since $\text{Im } w^n$ does not depend on X , one also may assume that $\mathcal{E}_e \simeq G_e$. Then one knows that $d' : S^n \mathcal{G}_e^* \rightarrow \mathcal{G}_e^* \otimes S^n \mathcal{G}_e^*$ vanishes on $(S^n \mathcal{G}_e^*)^{G_e}$, and $d''|_{S^n \mathcal{G}_e^*} = 0$ by definition). In particular, w^n defines a map

$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \rightarrow \mathbb{H}^{2n}(X, F^n \Omega_{X,(E,e)}^\bullet) = \mathbb{H}^{2n}(X, \Omega_X^{\geq n}).$$

4. When $k = \mathbb{C}$, one sets

$$S^n(\mathcal{G}_e^*)_{\mathbb{Z}(n)}^{G_e} = \text{Ker } S^n(\mathcal{G}_e^*)^{G_e} \rightarrow H^{2n}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)).$$

For $E_e = E_{un,e}$, the universal bundle $E_{un,e} = G_e^{\Delta_\ell} \times k^r / G_e$ on the simplicial $BG_e = G_e^{\Delta_\ell} / G_e$, w^n is an isomorphism, and one has

$$w^n : S^n(\mathcal{G}_e^*)^{G_e} \xrightarrow{\sim} \mathbb{H}^{2n}(BG_e, \Omega^{\geq n}) = \mathbb{H}^{2n}(BG_e, \Omega^\bullet),$$

the last isomorphism coming from

$$H^i(BG_e, \Omega^j) = 0 \quad i \neq j.$$

This remains true on $BG_{e,\text{an}}$. This implies that $S^n(\mathcal{G}_e^*)_{\mathbb{Z}(n)}^{G_e} \xrightarrow{\sim} H^{2n}(BG_{e,\text{an}}, \mathbb{Z}(n))$, the group $H^{2n}(BG_{e,\text{an}}, \mathbb{Z}(n))$ being torsion free.

5. When $k = \mathbb{C}$, they define the *Weil cohomology* by

$$U_{(E,e)}(n) = \text{cone } (\mathbb{Z}(n) \oplus S^n(\mathcal{G}_e^*)^{G_e}[-2n] \rightarrow \Omega_{X,(E,e)}^\bullet)[-1],$$

with the exact sequence

$$0 \rightarrow H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n)) \rightarrow H^{2n}(X_{\text{an}}, U_{(E,e)}(n)) \rightarrow S^n(\mathcal{G}_e^*)_{\mathbb{Z}(n)}^{G_e} \rightarrow 0.$$

The complexes $U_{(E,e)}(n)$ have a multiplication defined as in [1], [8] for the Deligne cohomology, compatible with the map of complexes

$$U_{(E,e)}(n) \rightarrow \text{cone } (\mathbb{Z}(n) \oplus F^n \Omega_{X,(E,e)}^\bullet \rightarrow \Omega_{X,(E,e)}^\bullet)[-1]$$

induced by w^n . This induces a map, still denoted by w^n , from the Weil cohomology to the analytic Deligne cohomology.

6. A Ω_X^1 valued connection $\nabla : (E, e) \rightarrow \Omega_X^1 \otimes (E, e)$ on (E, e) is a splitting of the $\Omega_{X,(E,e)}^1$ sequence, or, equivalently, a quotient map of commutative differential graded algebras

$$\nabla : \Omega_{X,(E,e)}^\bullet \rightarrow \Omega_X^\bullet$$

which is a quasi-isomorphism inverting $\Omega_X^\bullet \rightarrow \Omega_{X,(E,e)}^\bullet$. (Then $\nabla^2 = 0$ if and only if ∇ induces a quasi-isomorphism of filtered complexes). Thus ∇ maps $U_{(E,e)}(n)$ to

$$C(n)^{\text{an}}|X = \text{cone} (\mathbb{Z}(n) \oplus \Omega_X^{\geq 2n} \rightarrow \Omega_X^\bullet)[-1]$$

via

$$S^n(\mathcal{G}_e^*)^{G_e}[-2n] \xrightarrow{w^n} F^{2n}\Omega_{X,(E,e)}^\bullet \xrightarrow{\nabla} \Omega_X^{\geq 2n}.$$

This defines classes of (E, ∇, e) in

$$\mathbb{H}^{2n}(X_{\text{an}}, C(n)^{\text{an}}|X)$$

(2.0.2, 2)), mapping to the classes in the analytic Deligne cohomology (but not in $D^n(X)$ if X is not proper). The image of this class is $P_n(\nabla^2, \dots, \nabla^2)$ (see 1, 10)), since ∇ maps $\mathcal{G}_{(E,e)}^*$ to Ω_X^2 via the curvature. In particular, if $\nabla^2 = 0$, then those classes are in $H^{2n-1}(X_{\text{an}}, \mathbb{C}/\mathbb{Z}(n))$. The authors claim that this is exactly $\hat{c}_n(E, \nabla)$ (without saying why) for $e = 0$.

4.2. Algebraic construction. We want to make the construction of Beilinson-Kazhdan algebraic.

Remark 4.2.1. 1) *The construction of the filtered quasi-isomorphism*

$$(\Omega_X^\bullet, \Omega_X^{\geq n}) \rightarrow (\Omega_{X,E}^\bullet, F^n \Omega_{X,E}^\bullet)$$

is algebraic, as well as the Weil morphism w^n . If (E, ∇, τ_0) is as in 1, 4), 6), one first defines $\Omega_{X,E}^\bullet(\log D)$ by replacing the Atiyah extension by the logarithmic Atiyah extension

$$0 \rightarrow \Omega_X^1(\log D) \rightarrow \Omega_{X,E}^1(\log D) \rightarrow \mathcal{G}_E^* \rightarrow 0.$$

Then one defines $\Omega_{X,E,\tau_0}^\bullet(\log D)$ by replacing the logarithmic Atiyah extension by the N^1 valued Atiyah extension

$$0 \rightarrow N^1 \rightarrow \Omega_{X,E,\tau_0}^1(\log D) \rightarrow \mathcal{G}_E^* \rightarrow 0$$

obtained by push forward through $\Omega_X^1(\log D) \rightarrow N^1$ of the logarithmic extension. Thus

$$\Omega_{X,E,\tau_0}^n(\log D) = \frac{N^n \oplus \Omega_{X,E}^n(\log D)}{\Omega_X^n(\log D)}$$

This defines a filtered quasi-isomorphism

$$(N^\bullet, N^{\geq n}) \rightarrow (\Omega_{X,E,\tau_0}^\bullet(\log D), F^n \Omega_{X,E,\tau_0}^\bullet(\log D))$$

where F^n is defined as in 4.1 2):

$$\begin{aligned} F^n \Omega_{X,E,\tau_0}^\bullet(\log D) &= \bigoplus_{a \geq n} \Omega_{X,E,\tau_0}^{a,b}(\log D) \\ \Omega_{X,E,\tau_0}^{a,b}(\log D) &= \Lambda^{a-b} \Omega_{X,E,\tau_0}^1(\log D) \otimes S^b(\mathcal{G}_E^*). \end{aligned}$$

We still denote by w^n the induced Weil homomorphism

$$\begin{aligned} (S^m \mathcal{G}^*)^G \rightarrow S^n(\mathcal{G}_E^*) \rightarrow (\Omega_{X,E,\tau_0}^{n,n}(\log D))_{cl} \\ \rightarrow F^n \Omega_{X,E,\tau_0}^1(\log D)[2n]. \end{aligned}$$

When $\pi : Y \rightarrow X$ is a morphism of (simplicial) smooth varieties, such that $D' = \pi^{-1}(D)$ is a normal crossing divisor, then one has a map

$$\pi^{-1} : \Omega_{X,E,\tau_0}^\bullet(\log D) \rightarrow R\pi_* \Omega_{Y,\pi^*E,\tau_0}^\bullet(\log D')$$

induced by

$$\pi^{-1} : \Omega_{X,\tau_0}^\bullet(\log D) \rightarrow R\pi_* \Omega_{Y,\tau_0}^\bullet(\log D')$$

(3.1.1).

- 2) Then a N^1 valued connection on E defines a quotient of commutative graded algebras

$$\nabla : \Omega_{X,E,\tau_0}^\bullet(\log D) \rightarrow N^\bullet$$

which is an inverse to the quasi-isomorphism

$$N^\bullet \rightarrow \Omega_{X,E,\tau_0}^\bullet.$$

Then $\nabla^2 = 0$ is equivalent to ∇ being a filtered quasi-isomorphism.

Definition 4.2.2. Let E be a rank r vector bundle and τ_0 be as in 1 4). We define

$$\begin{aligned} ABK_{E,\tau_0}^n &= \text{cone}(\mathcal{K}_n \otimes S^n(\mathcal{G}^*)^G[-n] \\ &\xrightarrow{d \log \oplus -w^n} F^n \Omega_{X,E,\tau_0}^\bullet(\log D)[n])[-1]. \end{aligned}$$

For $\tau_0 = \text{identity}$, $D \neq \emptyset$ we write simply ABK_E^n .

Lemma 4.2.3. For $X = BG$, $E = E_{un}$ (4.1 4)) one has

$$\mathbb{H}^n(BG, ABK_{E_{un}}^n) = H^n(BG, \mathcal{K}_n).$$

For $X = BG_e$, and $E_{un,e} = E_{un}|_{BG_e}$ with its canonical filtration coming from this flag, one has

$$\mathbb{H}^n(BG_e, ABK_{E_{un,e}}^n) = H^n(BG_e, \mathcal{K}_n).$$

Proof. We just write the exact sequence

$$\begin{aligned} \rightarrow \mathbb{H}^{2n-1}(X, N^{\geq n}) &\rightarrow \mathbb{H}^n(X, ABK_{E, \tau_0, \epsilon}^n) \\ &\rightarrow H^n(X, \mathcal{K}_n) \oplus S^n(\mathcal{G}_e^*)^G \rightarrow \mathbb{H}^{2n}(X, N^{\geq n}) \end{aligned}$$

and apply 4.1, 4). \square

Definition 4.2.4. Let (E, τ_0) be as in 4.2.2. We define

$$c_n^{ABK}(E) \in \mathbb{H}^n(X, ABK_{E, \tau_0}^n(\log D))$$

as the inverse image under a map $[E] : X_\bullet \rightarrow BG$ defined on a simplicial model of X by the transition functions of E , of

$$c_n(E_{un}) \in H^n(BG, \mathcal{K}_n) = \mathbb{H}^n(BG, ABK_{E_{un}}^n).$$

As usual, all possible classes of $[E]$ are homotop and this definition does not depend on the choice of $[E]$. However, one has to apply here the functoriality of the complexes $ABK_{E, \tau_0}^n(\log D)$ as explained in 4.2.1 1).

Let (E, ∇, τ_0) be now as in 1, 4), 6). Then ∇ induces a map, still denoted by ∇ , from $ABK_{E, \tau_0}^n(\log D)$ to

$$C(n)_{\tau_0} = \text{cone}(\mathcal{K}_n \oplus N^{\geq 2n}[n] \rightarrow N^{\geq n}[n])[-1]$$

(4.2.1 2)).

Definition 4.2.5.

$$c_n^{ABK}(E, \nabla) = \nabla c_n^{ABK}(E) \in AD_{\tau_0}^n(X).$$

Theorem 4.2.6. One has

$$c_n^{ABK}(E, \nabla) = c_n(E, \nabla) \in AD_{\tau_0}^n(X).$$

Proof. We want to apply the characterization of the classes 3.1.8.

- 1) The functoriality of the class $c_n^{ABK}(E)$ is clear. This implies the functoriality of the classes $c_n^{ABK}(E, \nabla)$.
- 2) Let E be of rank 1. Then $\mathcal{G}_E = \mathcal{O}_X$, and $(\mathcal{G}^*)^G = k$. One has

$$\mathbb{H}^1(X, ABK_E^1) = \mathbb{H}^1(X, \mathcal{K}_1 \xrightarrow{d \log} \Omega_{X, E}^1 \longrightarrow \Lambda^2 \Omega_{X, E}^1 \oplus \frac{\mathcal{O}_X}{k}).$$

One considers the principal \mathbb{G}_m bundle $p : \mathcal{E} = E - \{0\} \rightarrow X$ with local trivialization $\mathcal{E}|_{U_i} = \mathbb{G}_m \times U_i$, and gluing $t_i = \xi_{ij} t_j$, t_i parameter of \mathbb{G}_m and $\xi_{ij} \in \mathcal{O}_X^*(U_i \cap U_j)$. This defines a $\Omega_{X, E}^1 = (p_* \Omega_{\mathcal{E}}^1)^{\mathbb{G}_m}$ connection on E via $\frac{dt_i}{t_i} - \frac{dt_j}{t_j} = \frac{d\xi_{ij}}{\xi_{ij}}$, with local form $\frac{dt_i}{t_i}$. Then $d'(\frac{dt_i}{t_i}) = 0$, while $d''(\frac{dt_i}{t_i})$, which is the image of $\frac{dt_i}{t_i}$ in \mathcal{O}_X , is the residue of $\frac{dt_i}{t_i}$ along the zero section of E . Therefore, this is

$1 \in k \subset \mathcal{O}$. Thus $(d' + d'')(\frac{dt_i}{t_i}) = 0$ in $\frac{\mathcal{O}_X}{k}$ and $(\xi_{ij}, \frac{dt_i}{t_i})$ defines the class of E in $\mathbb{H}^1(X, ABK_E^1)$. Now if ∇ is a τ_0 connection on E , it maps $\frac{dt_i}{t_i}$ to the local form α_i of the connection, Thus

$$\nabla(\xi_{ij}, \frac{dt_i}{t_i}) = (\xi_{ij}, \alpha_i) = c_1(E, \nabla) \in AD_{\tau_0}^1(X).$$

- 3) In order to understand the Whitney product formula, one has first to introduce a product on $ABK_{E,\tau_0}^n(\log D)$. Again, one takes the same definition as in 2.1.1:

$$ABK_{E,\tau_0}^m(\log D) \times ABK_{E,\tau_0}^n(\log D) \xrightarrow{\cup_\alpha} ABK_{E,\tau_0}^{m+n}(\log D)$$

is given by

$$\begin{aligned} x \cup_\alpha y &= \{x, y\} & x \in \mathcal{K}_m, y \in \mathcal{K}_n \\ &= 0 & x \in \mathcal{K}_m, y \in S^n \\ &= (1 - \alpha)d \log x \wedge y & x \in \mathcal{K}_m, y \in F^n \\ &= 0 & x \in S^m, y \in \mathcal{K}_n \\ &= x \wedge y & x \in S^m, y \in S^n \\ &= (-1)^{\deg x} \alpha x \wedge y & x \in S^m, y \in F^n \\ &= \alpha x \wedge d \log y & x \in S^m, y \in \mathcal{K}_n \\ &= (1 - \alpha)x \wedge y & x \in F^m, y \in S^n \\ &= 0 & x \in F^m, y \in F^n, \end{aligned}$$

where we shortened the notations by

$$\begin{aligned} S^m &= S^m(\mathcal{G}^*)^G[-m] \\ F^m &= F^m \Omega_{X,E,\tau_0}^\bullet(\log D)[m]. \end{aligned}$$

Then a N^1 valued connection ∇ maps $ABK_{E,\tau_0}^n(\log D)$ to $C(n)_{\tau_0}$ (4.2.1 2)) compatibly to the product.

- 4) Let $e : 0 \rightarrow E'' \rightarrow E \rightarrow E' \rightarrow 0$ be an exact sequence of bundles, with $\text{rank } E'' = r''$, $\text{rank } E' = r'$, $r = r' + r''$. Let $\mathcal{G}_{E,e} \xrightarrow{\rho} \mathcal{G}_E$ be the endomorphisms of E respecting the extension e , and $\mathcal{G}_{E,e} \xrightarrow{p' \oplus p''} \mathcal{G}_{E'} \oplus \mathcal{G}_{E''}$ be the restrictions of the compatible endomorphisms to those of E' and of E'' . This defines maps

$$\mathcal{G}_E \xrightarrow{\rho^*} \mathcal{G}_{E,e}^* \xleftarrow{(p' \oplus p'')^*} \mathcal{G}_{E'}^* \oplus \mathcal{G}_{E''}^*$$

and maps

$$ABK_E^n \xrightarrow{\rho^*} ABK_{E,e}^n \xleftarrow{p'^* \oplus p''^*} ABK_{E'}^n \oplus ABK_{E''}^n$$

where one defines $ABK_{E,e}^n$ in the following way. Let $G_e \subset G$ be the matrices of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, A \in GL(r''), B \in GKL(r'),$$

and $\mathcal{G}_e \subset \mathcal{G}$ be the corresponding algebra. Then by restriction of the structure group of E from G to G_e , one considers \mathcal{E}_e the corresponding G_e torsor and one does the same construction as in 4.1 with \mathcal{E}_e replacing \mathcal{E} . Now $E_{un}|_{BG_e}$ is the extension

$$e_{un} : 0 \rightarrow E''_{un} \rightarrow E_{un}|_{BG_e} \rightarrow E'_{un} \rightarrow 0$$

and one has

$$\begin{aligned} H^n(BG, \mathcal{K}_n) &= \mathbb{H}^n(BG, ABK_{E_{un}}^n) \\ &\xrightarrow{\rho^*} H^n(BG_e, ABK_{(E_{un}, e_{un})}^n) = H^n(BG_e, \mathcal{K}_n) \end{aligned}$$

(Lemma 4.2.3) and

$$H^n(BG_e, \mathcal{K}_n) = \bigoplus_{a=0}^n p'^* H^a(BG', \mathcal{K}_a) \cup p''^* H^{n-a}(BG'', \mathcal{K}_{n-a}).$$

Since the product on ABK is compatible with the product on \mathcal{K} , this implies

$$\begin{aligned} H^n(BG_e, ABK_{(E_{un}, e_{un})}^n) &= \\ &= \bigoplus_{a=0}^n p'^* \mathbb{H}^a(BG', ABK_{E'_{un}}^a) \cup p''^* \mathbb{H}^{n-a}(BG'', ABK_{E''_{un}}^{n-a}) \end{aligned}$$

and one has the decomposition

$$\rho^* c_n^{ABK}(E_{un}) = \bigoplus_{a=0}^n p'^* c_a^{ABK}(E'_{un}) \cup p''^* c_{n-a}^{ABK}(E''_{un}).$$

By functoriality, one obtains the similar relation for the classes of E :

$$\rho^* c_n^{ABK}(E) = \bigoplus_{a=0}^n p'^* c_a^{ABK}(E') \cup p''^* c_{n-a}^{ABK}(E'').$$

If E now has a τ_0 connection ∇ , the map $\nabla : \Omega_{X,E,\tau_0}^\bullet(\log D) \rightarrow N^\bullet$ (4.2.1 2)) factors through

$$\Omega_{X,E,\tau_0}^\bullet(\log D) \xrightarrow{\rho^*} \Omega_{X,E,e,\tau_0}^\bullet(\log D) \xrightarrow{\nabla_\rho} N^\bullet$$

when ∇ is compatible with the exact sequence. This shows the Whitney product formula. \square

Remark 4.2.7. *The construction of the tautological $\Omega_{X,E}^1$ valued connection on E of rank r goes as in the second section of the proof of 4.2.6 for $r = 1$: One considers a local trivialization $\mathcal{E}|_{U_i} \simeq G \times U_i$ of the principal G bundle \mathcal{E} . Then the gluing is given by $g_i = g_{ij}g_j$, $g_{ij} \in G(U_i \cap U_j)$, where g_i is the tautological G valued function on $G : g_i(x) = x$. Then*

$$dg_i g_i^{-1} - g_{ij} dg_j g_j^{-1} g_{ij}^{-1} = dg_{ij} g_{ij}^{-1}$$

is the equation of the connection.

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