

Gauß-Manin determinants for rank 1 irregular connections on curves

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Abstract. Let $f : U \rightarrow \text{Spec}(K)$ be a smooth open curve over a field $K \supset k$, where k is an algebraically closed field of characteristic 0. Let $\nabla : L \rightarrow L \otimes \Omega_{U/k}^1$ be a (possibly irregular) absolutely integrable connection on a line bundle L . A formula is given for the determinant of de Rham cohomology with its Gauß-Manin connection $(\det Rf_*(L \otimes \Omega_{U/K}^1), \det \nabla_{GM})$. The formula is expressed as a norm from the curve of a cocycle with values in a complex defining algebraic differential characters [7], and this cocycle is shown to exist for connections of arbitrary rank.

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Thus mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

Bertrand Russell

1. Introduction

Let $f : U \rightarrow \text{Spec}(K)$ be a smooth open curve over a field $K \supset k$, where k is an algebraically closed field of characteristic zero. Let $\nabla : L \rightarrow L \otimes \Omega_{U/k}^1$ be a possibly irregular absolutely integrable (or vertical, see Definition 2.16) connection on a line bundle L . The Riemann-Roch problem in this context is to describe characteristic classes for the relative de Rham cohomology $Rf_*(L \otimes \Omega_{U/K}^*)$ as a (virtual) vector space over K with an integrable connection, in terms of data on U . The 0-th characteristic class, the Euler characteristic $\dim R^0 - \dim R^1$, is well-known to be given by

$$(1.1) \quad 2 - 2g - n - \sum_i \max(0, m_i - 1)$$

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where g is the genus of the complete curve C , n is the number of missing points, and m_i is the order of the polar part of the connection at the i -th missing point. The purpose of this article is to give a formula for the first characteristic class, which is the determinant of the Gauß-Manin connection on the relative de Rham cohomology of the line bundle,

$$(1.2) \quad \left(\det(Rf_*(L \otimes \Omega_{U/K}^*)), \det \nabla_{GM} \right).$$

When $U = C$, so the connection has no poles, the formula given in [2] is

$$(1.3) \quad \left(\det(Rf_*(L \otimes \Omega_{C/K}^*)), \nabla_{GM} \right) = -f_*((L, \nabla) \cdot c_1(\Omega_{C/K}^1)).$$

Concretely, if one has $c_i \in C(K)$ with $\sum c_i$ a 0-cycle in the linear series representing $\Omega_{C/K}^1$, then the determinant is given by restricting L with its connection to each c_i and then tensoring the resulting lines with connection together.

When the connection ∇ has at worst regular singular points at the points in $D := C - U$ there is an analogous formula using linear series given by divisors of rational sections s of $\Omega_{C/K}^1(D)$ satisfying the rigidity condition $\text{res}_D(s) = 1$. Indeed, these formulas are valid also for higher rank connections. One takes the determinant at zeroes and poles of s .

In the case of irregular singular points, a similar formula is possible, but the rigidification taken must depend on the polar part of the connection. Let (\mathcal{L}, ∇) be an extension of (L, ∇) to C , $\mathcal{D} = \sum_i m_i D_i$ be a divisor with multiplicities $m_i \geq 1$ supported in $C - U$ such that the relative connection

$$(1.4) \quad \nabla_{/K} : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})$$

yields a complex quasiisomorphic to $j_*L \rightarrow j_*L \otimes \Omega_{(C-D)/K}^1$ and has poles at all points D_i . Then $\nabla_{/K}$ does not factor through

$$(1.5) \quad \nabla_{/K} : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D} - D_i)$$

for any i . Writing \mathcal{D} also for the artinian subscheme of C determined by \mathcal{D} , this implies that $\nabla_{/K}$ induces a *function linear isomorphism*

$$(1.6) \quad \nabla|_{\mathcal{D}} : \mathcal{L}|_{\mathcal{D}} \xrightarrow{\cong} \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$$

Because these maps are function linear, we may cancel the lines $\mathcal{L}|_{\mathcal{D}}$ and deduce canonical elements $\text{triv}_{\nabla} \in \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$. We view triv_{∇} as a trivialization of $\Omega_{C/K}^1(\mathcal{D})$ along \mathcal{D} . It is known ([6], Appendix B) that the coboundary of triv_{∇} in $H^1(C, \omega_{C/K}) \cong K$ is given by the degree of \mathcal{L} . Our main result is:

Theorem 1.1. *Let notation be as above. Assume all D_i are defined over K and some $m_i \geq 2$. Because we are only concerned with the cohomology over $X - \mathcal{D}$, we can take \mathcal{L} of degree 0 so that the element triv_∇ can be lifted to $H^0(C, \Omega_{C/K}^1(\mathcal{D}))$. Let s be any such lifting, and write (s) for the divisor of s as a section of $\Omega_{C/K}^1(\mathcal{D})$. Note the support of (s) is disjoint from \mathcal{D} . Then*

$$(1.7) \quad \det(Rf_*(L \otimes \Omega_{U/K}^*), \det(\nabla_{GM})) \cong \\ - f_*(L \cdot (s)) + \tau(L) \in \Omega_K^1/d \log K^*.$$

Here $\tau(L)$ is a 2-torsion term which can be written

$$\tau(L) = \sum_i \frac{m_i}{2} d \log(g_{i,0}) \in \frac{1}{2} d \log(K^\times)/d \log(K^\times)$$

where the connection $\nabla_{/K} = (g_{i,0} + g_{i,1}z_i + \dots)dz_i/z_i^{m_i}$ for a local coordinate z_i at $D_i \in \mathcal{D}$.

Note that $\Omega_K^1/d \log K^\times$ is the group of isomorphism classes of rank 1 connections on $\text{Spec}(K)$. Our assumption that points of \mathcal{D} are defined over K is made to avoid complications involving generalized jacobians in Sect. 2. We remark, of course, that part of our task will be to give a precise definition of the right hand side of the formula of the theorem. It will appear as a product followed by a trace, and this definition does not depend on the particular choice of \mathcal{L} above. In particular, this gives a formulation if we don't assume that \mathcal{L} is of degree 0, and also if we don't assume that $m_i \geq 2$ for at least one i , that is if ∇ has regular singular points (see Theorem 4.6). The precise general formulation of our theorem is in 4.8. In the case that (s) is a sum of K -points c_i , one may simply take the tensor product of the lines with connection $L|_{c_i}$. The right hand side of the formula depends only on the equivalence class of (s) in a generalized Picard (or divisor class) group of line bundles with trivializations along \mathcal{D} . Thus, by analogy with (1.3), it is natural to write formula (1.7) in the form

$$(1.8) \quad \left((\det(Rf_*(L \otimes \Omega_{U/K}^*)), \det(\nabla_{GM})) \right) \cong \\ f_* \left(L \cdot c_1(\Omega_{C/K}^1(\mathcal{D}), \text{triv}_\nabla) \right)^{-1} + \tau(L).$$

The classical Riemann-Roch pattern begins to break down in that the characteristic class $c_1(\Omega_{C/K}^1(\mathcal{D}), \text{triv}_\nabla)$ depends on more than just the geometry of $f : U \rightarrow \text{Spec}(K)$. This reflects the fact that the de Rham cohomology of an irregular connection depends on more than topology.

There is an analogy here with the case of ℓ -adic sheaves. If \mathcal{E} is an unramified ℓ -adic sheaf on a complete curve C over a finite field \mathbb{F}_q , then the global epsilon factor is given by

$$(-F_q | \det(H_{\text{ét}}^*(C_{\overline{\mathbb{F}}}, \mathcal{E}))) = \det(\mathcal{E})^{-1} \cdot c_1(\Omega_{C/\mathbb{F}_q}^1).$$

The basic result in the ramified ℓ -adic case ([8]) is that the global epsilon factor can be written as a product of local terms corresponding to points on the curve where the sheaf ramifies or where a chosen meromorphic 1-form has zeroes or poles. We suspect formula (1.8) is analogous to a classical formula for Gauß sums

$$g(c, \psi) = \sum_{a \in (\mathcal{O}/\mathfrak{f})^\times} c(a)\psi(a)$$

where $\mathfrak{f} \subset \mathcal{O}$ is an ideal in the ring of integers in a local field, c (resp. ψ) is a character of $(\mathcal{O}/\mathfrak{f})^\times$ (resp. $(\mathcal{O}/\mathfrak{f})^+$), and both c and ψ have conductor \mathfrak{f} . If the residue field of \mathcal{O} has q elements with q odd, one finds

$$g(\epsilon, \psi) = \begin{cases} q^n c(x) & \mathfrak{f} = \mathfrak{m}^{2n}, n \geq 1 \\ q^n c(x)\sigma & \mathfrak{f} = \mathfrak{m}^{2n+1}, n \geq 1. \end{cases}$$

In this formula $x \in \mathcal{O}/\mathfrak{f}$ is a suitable point, $\sigma = \zeta\sigma_0$ with $\zeta^q = 1$ and $\sigma_0^2 = \left(\frac{-1}{\mathbb{F}_q}\right)$. (Here σ_0 is a quadratic Gauß sum.)

Our proof follows the main idea of Deligne [3]. For computing the ϵ -factor associated to a rank one Galois representation on a curve, he expresses the determinant of the cohomology as the cohomology on a symmetric product of $(C - D)$ and reduces the computation to the geometry of the generalized jacobian. In the geometric situation one is further able to express the determinant Gauß-Manin connection as the connection arising by restricting a certain translation-invariant connection to one specific K -point of the generalized jacobian. The essential point seems to be that the de Rham cohomology of a connection of the form $d + \omega$ on a trivial bundle is somehow concentrated at the points where $\omega = 0$.

In Sect. 4 we reinterpret the Riemann-Roch formula in terms of a pairing (4)

$$(1.9) \quad \begin{aligned} \cup : \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*) \times \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \Omega_C^1(D)(D')) \\ \rightarrow \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2) \end{aligned}$$

and a trace map (4.7)

$$\text{Tr} : \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2) \rightarrow \Omega_K^1/d \log K^*.$$

In Sect. 5 we give an analogous “non-commutative” product formula in the higher rank case which we conjecture calculates the determinant connection in the generic situation when the connection defines local isomorphisms $E|_{\mathcal{D}} \cong E|_{\mathcal{D}} \otimes \omega_{\mathcal{D}/K}$ (see (5.3)) and the poles of the absolute connection behave well (see (5.1)). We verify the formula has the appropriate invariance properties. We also show that there is a more general higher rank product of which it is a special case. Finally, in Sect. 6 we give a general formula which calculates the group of isomorphism classes of irregular, integrable, rank 1 connections in higher dimensions on a smooth projective variety.

We apologize for not expressing our results in the modern language of \mathcal{D} -modules, but in fact for the study of Gauß-Manin determinants there is little gain in passing from connections to \mathcal{D} -modules. Also, rigidity for connections means that the Gauß-Manin determinant connection is determined by its value at the generic point on the base, so we may work with curves over a function field $\text{Spec}(K)$.

It is our pleasure to acknowledge the intellectual debt we owe in this work to P. Deligne. We are also grateful to the Humboldt foundation for financing which enabled us to work together.

2. Connections and forms on generalized jacobians

Throughout this paper C will be a smooth projective curve over a field K containing an algebraically closed subfield k of characteristic 0, and $\mathcal{D} = \sum m_i c_i$ is a divisor on C , with $c_i \in C(K)$. We write $G = J_{\mathcal{D}}$ for the generalized Jacobian parametrizing isomorphism classes of degree 0 line bundles on C with trivialization along \mathcal{D} . Fixing a K -rational point

$$c_0 \in (C - D)(K),$$

there is a cycle map $i : C - D \rightarrow J_{\mathcal{D}}$ associating to a closed point $x \in C$ with $[K(x) : K] = n$ the class of the line bundle $\mathcal{O}(x - nc_0)$ together with the trivialization $b|_{\mathcal{D}} \circ (a|_{\mathcal{D}})^{-1}$, where

$$\mathcal{O}_C \xleftarrow{a} \mathcal{O}_C(-nc_0) \xrightarrow{b} \mathcal{O}_C(x - nc_0)$$

are the natural maps.

The aim of this section is to describe invariant line bundles with connection on $J_{\mathcal{D}}$, comparing them via the cycle map i to line bundles with connection on $(C - D)$ with a certain irregularity behavior along D .

When the line bundle in question is the trivial bundle, this amounts to studying invariant (absolute) differential forms on the generalized jacobian, so we should start with that. Before doing so, however, it is necessary to understand global functions on the generalized jacobian. We write

$$(2.1) \quad G \twoheadrightarrow G_0 \twoheadrightarrow J$$

where J is the usual Jacobian of C , and G_0 is a semi-abelian variety. We have extensions

$$(2.2) \quad 0 \rightarrow T \rightarrow G_0 \rightarrow J \rightarrow 0$$

$$(2.3) \quad 0 \rightarrow \mathbb{V} \rightarrow G \rightarrow G_0 \rightarrow 0$$

Here \mathbb{V} is a vector group (isomorphic to $\text{Spec}(\text{Sym}(V^*))$) for some vector space V and T is a torus, i.e. $T_{\bar{K}} \cong \mathbb{G}_m^r$.

Lemma 2.1. *The semi-abelian variety G_0 admits a universal vectorial extension*

$$(2.4) \quad 0 \rightarrow \mathbb{W} \rightarrow \mathcal{G} \rightarrow G_0 \rightarrow 0.$$

In fact, this extension is given by the pullback to G_0 of the universal vectorial extension over J . In particular, $\mathbb{W} = \Gamma(J^\vee, \Omega_{J^\vee/K}^1) \otimes \mathbb{G}_a$.

Proof. It will suffice to show the pullback vectorial extension is universal. Since $\text{Ext}^1(T, \mathbb{G}_a) = (0) = \text{Hom}(T, \mathbb{G}_a)$, any extension of G_0 by a vector group \mathbb{W} is pulled back from a unique extension of J by \mathbb{W} . This extension of J is a pushout from the universal vectorial extension, so the same holds for the pullbacks to G_0 . \square

Lemma 2.2. *Let $\pi : G_0 \rightarrow J$ be an extension of J by T as above. There exists, possibly after a finite field extension, a quotient torus $T \twoheadrightarrow S$ and a diagram*

$$(2.5) \quad \begin{array}{ccc} T & \hookrightarrow & G_0 \\ \text{surj. } \downarrow & \swarrow a & \\ S & & \end{array}$$

such that

$$(2.6) \quad H^i(G_0, \mathcal{O}_{G_0}) \cong H^i(J, \mathcal{O}_J) \otimes_K H^0(S, \mathcal{O}_S)$$

Proof. There is a boundary map

$$(2.7) \quad \partial : \text{Hom}_{\bar{K}}(T, \mathbb{G}_m) \rightarrow \text{Ext}_{\bar{K}}^1(J, \mathbb{G}_m)$$

Define $N := \ker(\partial) \subset M := \text{Hom}_{\bar{K}}(T, \mathbb{G}_m)$. Let $S = \text{Hom}(N, \mathbb{G}_m)$ be the torus with character group N . For $m \in M$ let $L(m)$ be the line bundle on $J_{\bar{K}}$ corresponding under the map (2.7). As an $\mathcal{O}_{J_{\bar{K}}}$ -algebra

$$(2.8) \quad \pi_* \mathcal{O}_{G_0, \bar{K}} \cong \bigoplus_{m \in M} L(m) \cong H^0(\mathcal{O}_S) \otimes \left(\bigoplus_{m \in M/N} L(m) \right)$$

The map a in the diagram (2.5) comes from the above inclusion

$$H^0(\mathcal{O}_S) \otimes L(0) \subset \pi_* \mathcal{O}_{G_0}.$$

For $m \in M/N$, (as is well known, cf. [9] III 16), $L(m)$ has trivial cohomology in all degrees unless $m = 0$. The proposition follows by taking cohomology of (2.8). \square

Lemma 2.3. *Let notation be as above. Let*

$$(2.9) \quad 0 \longrightarrow H^0(J^\vee, \Omega_{J^\vee/K}^1) \otimes \mathbb{G}_a \longrightarrow \mathcal{G} \xrightarrow{p} G_0 \longrightarrow 0$$

be the universal vectorial extension. Then

$$(2.10) \quad H^0(\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \cong H^0(G_0, \mathcal{O}_{G_0})$$

Proof. The \mathcal{O}_{G_0} -algebra $p_*\mathcal{O}_G$ is filtered, with

$$(2.11) \quad gr_i p_*\mathcal{O}_G = fil_i/fil_{i-1} \cong \mathrm{Sym}^i(H^0(J^\vee, \Omega_{J^\vee/K}^1)^*) \otimes \mathcal{O}_{G_0}$$

With respect to the exact sequences

$$(2.12) \quad 0 \longrightarrow fil_{i-1} \longrightarrow fil_i \longrightarrow gr_i \longrightarrow 0$$

it suffices to show the boundary map

$$(2.13) \quad b : \mathrm{Sym}^i(H^0(J^\vee, \Omega_{J^\vee/K}^1)^*) \otimes H^0(G_0, \mathcal{O}_{G_0}) \rightarrow H^1(G_0, fil_{i-1})$$

is injective. Composing on the right with the evident map, it suffices to show the maps

$$(2.14) \quad \begin{aligned} \mathrm{Sym}^i(H^0(J^\vee, \Omega_{J^\vee/K}^1)^*) \otimes H^0(G_0, \mathcal{O}_{G_0}) \\ \rightarrow H^1(G_0, \mathcal{O}_{G_0}) \otimes \mathrm{Sym}^{i-1}(H^0(J^\vee, \Omega_{J^\vee/K}^1)^*) \end{aligned}$$

are injective. But

$$(2.15) \quad \begin{aligned} H^0(J^\vee, \Omega_{J^\vee/K}^1)^* \otimes H^0(G_0, \mathcal{O}_{G_0}) &\cong H^1(J, \mathcal{O}_J) \otimes H^0(G_0, \mathcal{O}_{G_0}) \\ &\cong H^1(G_0, \mathcal{O}_{G_0}) \end{aligned}$$

and the map in (2.14) is the map $x^i \mapsto x \otimes x^{i-1}$, which is injective. \square

Lemma 2.4. *Let $G = J_{\mathcal{D}}$ be a generalized jacobian as above. Then there exists a commutative affine algebraic group \mathbb{G} over K and a map $\psi : G \rightarrow \mathbb{G}$ such that*

$$(2.16) \quad \psi^* : H^0(\mathcal{O}_{\mathbb{G}}) \cong H^0(\mathcal{O}_G).$$

Proof. Take $\mathbb{G} = \mathrm{Spec}(H^0(G, \mathcal{O}_G))$. \square

Lemma 2.5. *Let A be the coordinate ring of a commutative affine algebraic group H over a field K of characteristic 0. Corresponding to the simplicial algebraic group BH , one has a complex*

$$(2.17) \quad A \xrightarrow{\mu^* - p_1^* - p_2^*} A \otimes_K A \xrightarrow{p_{23}^* - \mu_{12} \otimes p_3^* + p_1^* \otimes \mu_{23}^* - p_{12}^*} A \otimes A \otimes A$$

This complex is exact at the middle term.

Proof. By Proposition 4 on p. 168 of [10], the cohomology in the middle is a subgroup of the group of extensions $\mathrm{Ext}(H, \mathbb{G}_a)$. (Note, $A = \mathrm{Map}(H, \mathbb{G}_a)$.) By the classification of commutative algebraic groups in characteristic 0, this ext group vanishes (cf [10], pp. 170-172). \square

We write Ω_G^1 (resp. $\Omega_{G/K}^1$) for the sheaf of 1-forms relative to k (resp. K). Now we would like to define invariant bundles, connections, differential forms, cohomology classes of \mathcal{O}_G .

- Definition 2.6.** 1. A rank one bundle $L \in H^1(G, \mathcal{O}_G^*)$ is called invariant if $\mu^*L = p_1^*L \otimes p_2^*L \in H^1(G \times G, \mathcal{O}_{G \times G}^*)$, where $\mu : G \times G \rightarrow G$ is the multiplication and $p_i : G \times G \rightarrow G$ are the projections.
2. A global 1-form $\eta \in \Gamma(G, \Omega_G^1)$ is called invariant if $\eta(0) = 0 \in \Omega_K^1$ and $\mu^*\eta = p_1^*\eta + p_2^*\eta \in \Gamma(G \times G, \Omega_{G \times G}^1)$.
3. A rank one bundle with a connection $(L, \nabla) \in \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)$ is called invariant if $(L, \nabla)|_{\{0\}} = 0 \in \mathbb{H}^1(\text{Spec } K, \mathcal{O}_{\text{Spec } K}^* \rightarrow \Omega_{\text{Spec } K}^1) = \Omega_K^1/d \log K^*$ and $\mu^*(L, \nabla) = p_1^*(L, \nabla) \otimes p_2^*(L, \nabla) \in \mathbb{H}^1(G \times G, \mathcal{O}_{G \times G}^* \rightarrow \Omega_{G \times G}^1)$.
4. A class $s \in H^i(G, \mathcal{O}_G)$ is called invariant if $s|_{\{0\}} = 0$ and $\mu^*s = p_1^*s + p_2^*s$ in $H^i(G \times G, \mathcal{O}_{G \times G})$.

We denote by $H^1(G, \mathcal{O}_G^*)^{\text{inv}}$, $\Gamma(G, \Omega_G^1)^{\text{inv}}$, $\mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv}}$, and $H^i(G, \mathcal{O}_G)^{\text{inv}}$ the corresponding groups of invariant bundles, forms, connections and classes. One defines similarly the groups of relative invariant forms $H^0(G, \Omega_{G/K}^1)^{\text{inv}}$ and relative invariant connections $\mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}}$ without condition on the restriction to the zero section, and observe that the natural map $\Omega_G^1 \rightarrow \Omega_{G/K}^1$ takes global invariant groups to relative invariant groups.

Remark 2.7. In the above definitions, we could have defined a weaker notion of invariance by allowing constant elements. We adopt here the rigidification at the origin, keeping in mind that without this condition, the corresponding groups obtained are a direct sum of the ones obtained with the rigidification and the value of the group on the zero section. Notice, for example, that with our definition, nonzero constant functions are not invariant! Of course, for relative objects, there is no distinction.

Lemma 2.8. Let $G = J_{\mathcal{D}}$ be a generalized Jacobian as above. Let $\tau \in \Gamma(G, \Omega_{G/K}^1)^{\text{inv}}$ be an invariant relative 1-form on G , and assume τ lifts to an absolute global form. Then τ lifts to an invariant absolute form on G .

Proof. Let $\eta \in \Gamma(G, \Omega_G^1)$ be an absolute lifting. Replacing η with $\eta - \eta(0)$ we may assume $\eta(0) \in \Omega_K^1$ vanishes. Then

$$(2.18) \quad (\mu^* - p_1^* - p_2^*)(\eta) \in H^0(\mathcal{O}_{G \times K G}) \otimes \Omega_K^1$$

vanishes in $H^0(\mathcal{O}_{G \times K G \times K G}) \otimes \Omega_K^1$. Let $\psi : G \rightarrow \mathbb{G}$ be as in Lemma 2.4, so $\psi^* : A := H^0(\mathcal{O}_{\mathbb{G}}) \cong H^0(\mathcal{O}_G)$. The previous lemma implies there exists $\sigma \in H^0(\mathcal{O}_G) \otimes \Omega_K^1$ with $(\mu^* - p_1^* - p_2^*)(\sigma) = (\mu^* - p_1^* - p_2^*)(\eta)$. Then $\eta - \sigma$ is the desired invariant absolute form. \square

We next need to relate connections on the curve with invariant connections on the generalized Jacobian. Here $G = J_{\mathcal{D}}$ with $\mathcal{D} = \sum m_i c_i$. Also,

$$(2.19) \quad D := \sum c_i; \quad D' := D - D.$$

We assume $D \neq \emptyset$.

First, rather briefly, we consider the question of what poles invariant forms on G have when pulled back to $C - D$ via the jacobian map $C - D \rightarrow G$ (defined once we have a basepoint in $C - D$). For simplicity, we continue to assume the c_i are defined over K . Consider the diagram:

$$(2.20) \quad \begin{array}{ccc} G & \rightarrow & G_0 & \twoheadrightarrow & J \\ & \uparrow & i & & \uparrow & i' \\ C - D & & & \xrightarrow{j} & C \end{array}$$

We want to compute the pullbacks $i^*H^0(\Omega_G^1)^{inv}$ (resp. $i^*H^0(\Omega_{G/K}^1)^{inv}$) in $H^0(C, \Omega_C^1(*D))$ (resp. in $H^0(C, \Omega_{C/K}^1(*D))$).

Pulling back G via i' we get a torseur

$$(2.21) \quad i'^*G \xrightarrow{p} C$$

under the group $\mathcal{O}_{\mathcal{D}}^*/\mathbb{G}_m = \prod \mathcal{O}_{m_i c_i}^*/\mathbb{G}_m$. Fix ℓ and let $R = k[[t_\ell]] \subset M = k((t_\ell))$ where t_ℓ is a formal parameter at c_ℓ . Fix a splitting of the torseur

$$(2.22) \quad G_R \cong \mathcal{O}_{\mathcal{D}}^*/\mathbb{G}_m \times \text{Spec}(R).$$

Let $c_0 \in (C - D)(K)$ be the base point used to define the map $i : C - D \rightarrow G$. Let \mathcal{L} on $C \times C$ be a line bundle with $\mathcal{L}|_{\{c\} \times C} \cong \mathcal{O}(c - c_0)$. Note one has

$$(2.23) \quad \mathcal{O}_{C \times C} \leftarrow p_2^* \mathcal{O}(-c_0) \rightarrow \mathcal{L}$$

and these maps are isomorphisms on

$$(2.24) \quad \text{Spec}(M) \times \text{Spec}(\mathcal{O}_{C,D}) \subset \text{Spec}(M) \times C$$

Corresponding to $\mathcal{L}|_{\text{Spec}(M) \times C}$ and the above trivialization, one gets a map $u : \text{Spec}(M) \rightarrow G$. With respect to the above splitting, we view u as an element

$$(2.25) \quad \prod_i (u_{i0} + u_{i1}t_i + \dots + u_{i,m_i-1}t_i^{m_i-1}) \in \prod_i (\mathcal{O}_{m_i c_i} \otimes_K M)^* \pmod{M^*}$$

As described in [10] VII 4, 21, the local shape of u around c_ℓ is given by taking the rational function $(s_\ell - t_\ell)^{-1}$, where the local coordinates around (c_ℓ, c_ℓ) in $C \times C$ is (s_ℓ, t_ℓ) , and considering it as a unit in $\mathcal{O}_{m_\ell c_\ell} \otimes M \cong M[t_\ell]/\langle t_\ell^{m_\ell} \rangle$. (We change notation so s_ℓ is the local parameter in $R \subset M$.) Since u is well defined and non-vanishing in c_i for $i \neq \ell$, we have (2.24) that

$$(2.26) \quad u_{ij} \in R, u_{i0} \in R^* \text{ if } i \neq \ell, \text{ ord}(u_{\ell 0}) = 1$$

The pullbacks to $\text{Spec}(M)$ of the invariant relative differential forms on G are given by the pullback of invariant relative forms on J together with the coefficients of powers of the $T_i \bmod T_i^{m_i}$ in the formal expression

$$(2.27) \quad \sum_i (u_{i0} + u_{i1}T_i + \dots + u_{i,m_i-1}T_i^{m_i-1})^{-1} \times \\ (du_{i0} + du_{i1}T_i + \dots + du_{i,m_i-1}T_i^{m_i-1}) \\ = \sum_i \tau_{i0} + \dots + \tau_{i,m_i-1}T_i^{m_i-1}.$$

Then (2.26) implies that

$$(2.28) \quad \tau_{ij} \in \Omega_R^1 \text{ for } i \neq \ell \\ \tau_{\ell j} \in \Omega_R^1(D)(jD_\ell) - \Omega_R^1(D)((j-1)D_\ell).$$

Here we denote by $\Omega_C^1(D)$ the sheaf of absolute differential forms of degree 1 with logarithmic poles along D . (See formula 3.59 for a more precise computation).

These are not all the absolutely invariant forms, however. One also has forms pulled back from J , but these are regular along D . Finally, from Lemma 2.8 one has an exact sequence

$$(2.29) \quad 0 \rightarrow H^0(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1 \rightarrow H^0(G, \Omega_G^1)^{\text{inv}} \rightarrow H^0(G, \Omega_{G/K}^1)^{\text{inv}} \\ \rightarrow H^1(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1$$

It shows one must consider invariant forms in $H^0(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1$. We will see in the proof of Proposition 2.13 below that these map to $\Omega_G^1(D')$. In sum, the above discussion shows that the maps in the following proposition are defined.

Proposition 2.9. *Pullback gives isomorphisms*

$$(2.30) \quad H^0(\Omega_{G/K}^1)^{\text{inv}} \xrightarrow{\cong} H^0(C, \Omega_{C/K}^1(D))$$

$$(2.31) \quad H^0(\Omega_G^1)^{\text{inv}} \xrightarrow{\cong} H^0(C, \Omega_C^1(D)(D'))$$

Proof. Pullback on invariant relative forms is injective, because G is generated by the image of $C - D$. It follows by dimension count that the first arrow (2.30) above is an isomorphism. For the absolute forms we may consider the diagram

$$(2.32) \quad \begin{array}{ccccccc} 0 \rightarrow & H^0(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1 & \rightarrow & H^0(\Omega_G^1)^{\text{inv}} & \rightarrow & H^0(\Omega_{G/K}^1)^{\text{inv}} & \rightarrow & H^1(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1 \\ & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 \rightarrow & H^0(\mathcal{O}_C(D')) \otimes \Omega_K^1 & \rightarrow & H^0(\Omega_C^1(D)(D')) & \rightarrow & H^0(\Omega_{C/K}^1(D)) & \rightarrow & H^1(\mathcal{O}_C(D')) \otimes \Omega_K^1 \end{array}$$

The left and right hand vertical arrows are shown to be isomorphisms in the proof of Proposition 2.13. Hence the isomorphism on invariant relative forms implies the isomorphism (2.31) on invariant absolute forms. \square

We now consider invariant connections on line bundles on G .

Lemma 2.10. *Assume the toric subquotient T of G has trivial Picard group (e.g. T split). Then the map $\mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}} \rightarrow H^1(\mathcal{O}_G^*)^{\text{inv}}$ is surjective.*

Proof. This follows because

$$(2.33) \quad \mathbb{H}^1(J, \mathcal{O}_J^* \rightarrow \Omega_{J/K}^1)^{\text{inv}} \twoheadrightarrow H^1(\mathcal{O}_J^*)^{\text{inv}} \twoheadrightarrow H^1(\mathcal{O}_G^*)^{\text{inv}}.$$

The second arrow is surjective because we have a diagram

$$(2.34) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathcal{O}_J^*)^{\text{inv}} & \longrightarrow & H^1(\mathcal{O}_J^*) & \longrightarrow & H_{DR}^2(J/K) \\ & & \downarrow & & \downarrow \text{surj.} & & \downarrow \text{inj.} \\ 0 & \longrightarrow & H^1(\mathcal{O}_G^*)^{\text{inv}} & \xrightarrow{a} & H^1(\mathcal{O}_G^*) & \xrightarrow{b} & H_{DR}^2(G/K) \end{array}$$

The bottom row is not a priori exact, but $b \circ a = 0$ (because $(N\delta)^*$ acts by N^2 on $H_{DR}^2(G/K)$.) The middle vertical arrow is onto e.g. because the Picard group of the generic fibre of $G \rightarrow J$ is zero. Indeed, $G \rightarrow J$ is rationally split, and the kernel has trivial Picard group by hypothesis. (Since the function field of the generic fibre equals the function field of G , any divisor on G can be moved by rational equivalence to avoid the generic fibre, i.e. to be a pullback from the base.)

Finally, the right hand vertical arrow is injective because, after making a base change $K \subset \mathbb{C}$, one can think of G and J as quotients of vector spaces by lattices, and the map on lattices is surjective $\otimes \mathbb{Q}$. \square

Lemma 2.11. *Let $a \in \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}}$ be an invariant connection on a line bundle on the generalized jacobian G . Suppose a lifts to an absolute connection $b' \in \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)$. Then a lifts to an invariant absolute connection $b \in \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv}}$.*

Proof. One has as in (2.17)

$$(2.35) \quad (\mu^* - p_1^* - p_2^*)(b') \in \text{Im } H^0(\mathcal{O}_{G \times G}) \otimes \Omega_K^1 \\ \subset \mathbb{H}^1(G \times G, \mathcal{O}_{G \times G}^* \rightarrow \Omega_{G \times G}^1)$$

Now

$$\text{Im } H^0(\mathcal{O}_{G \times G}) \otimes \Omega_K^1 =$$

$$H^0(\mathcal{O}_{G \times G}) \otimes \Omega_K^1/d \log \text{Ker} \{H^0(\mathcal{O}_{G \times G}^* \rightarrow H^0(\Omega_{G \times G/K}^1))\} = \\ H^0(\mathcal{O}_{G \times G}) \otimes (\Omega_K^1/d \log K^*).$$

Exactness of the sequence in (2.17) implies that there exists an element $x \in H^0(G, \mathcal{O} \otimes \Omega_K^1)$ with $(\mu^* - p_1^* - p_2^*)(x) = (\mu^* - p_1^* - p_2^*)(b')$. Take $b = b' - x$. \square

Proposition 2.12. *One has an exact sequence*

$$(2.36) \quad 0 \rightarrow H^0(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1 \rightarrow \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv}} \\ \rightarrow \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}} \rightarrow H^1(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1$$

Proof. This is immediate from the lemma. \square

Recall our notation. C is a smooth, projective, geometrically connected curve over K . $G = J_{\mathcal{D}}$ with $\mathcal{D} = \sum m_i c_i$, $D = \sum c_i$, $\mathcal{D}' = \mathcal{D} - D$.

Proposition 2.13. *There exists a diagram of exact sequences, with vertical arrows isomorphisms:*

$$(2.37) \quad \begin{array}{ccc} 0 \rightarrow & H^0(\mathcal{O}_G)^{\text{inv}} \otimes (\Omega_K^1/d \log K^*) & \rightarrow \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv}} \\ & \downarrow e & \downarrow \\ 0 \rightarrow & \left(H^0(\mathcal{O}_C(\mathcal{D}'))/H^0(\mathcal{O}_C) \right) \otimes (\Omega_K^1/d \log K^*) & \rightarrow \frac{\mathbb{H}^1(C, j_* (\mathcal{O}_{C-D}^* \rightarrow \Omega_C^1(D)(\mathcal{D}'))}{(\Omega_{K/K^\times}^1)} \\ & \downarrow & \downarrow h \\ & \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}} & \rightarrow H^1(\mathcal{O}_G)^{\text{inv}} \otimes \Omega_K^1 \\ & \downarrow & \downarrow h \\ & \mathbb{H}^1\left(C, j_* (\mathcal{O}_{C-D}^* \rightarrow \Omega_{C/K}^1(D))\right) & \rightarrow H^1(\mathcal{O}_C(\mathcal{D}')) \otimes \Omega_K^1 \end{array}$$

Proof. The first step is to compute $H^i(\mathcal{O}_G)^{\text{inv}}$ for $i = 0, 1$. Let W be a finite dimensional K -vector space, and suppose $G = J_{\mathcal{D}}$ is a vectorial extension

$$(2.38) \quad 0 \rightarrow W \otimes \mathbb{G}_a \rightarrow G \xrightarrow{p} G_0 \rightarrow 0$$

We know by Lemma 2.1 that this sequence pulls back from an extension of J by $W \otimes \mathbb{G}_a$. Let

$$(2.39) \quad 0 \rightarrow \mathcal{O}_J \rightarrow \text{fil}_1 \rightarrow W^* \otimes \mathcal{O}_J \rightarrow 0$$

be the exact sequence of functions of filtration degree ≤ 1 as in Lemma 2.3, and let $\partial : W^* \rightarrow H^1(\mathcal{O}_J)$ be the boundary map in cohomology.

Lemma 2.14. *We have*

$$(2.40) \quad H^0(\mathcal{O}_G)^{\text{inv}} \cong \ker(\partial : W^* \rightarrow H^1(\mathcal{O}_J))$$

$$(2.41) \quad H^1(\mathcal{O}_G)^{\text{inv}} \cong H^1(\mathcal{O}_J)/\partial(W^*).$$

Proof of lemma. One has a filtration $\text{fil}.p_*\mathcal{O}_G$ with $\text{fil}_0 = \mathcal{O}_{G_0}$ and $gr_r = \text{Sym}^r(W^*) \otimes \mathcal{O}_{G_0}$. The corresponding spectral sequence looks like

$$(2.42) \quad E_1^{pq} = H^{p+q}(G_0, gr_{-p}) = H^{p+q}(G_0, \text{Sym}^{-p}(W^*) \otimes \mathcal{O}_{G_0}) \Rightarrow H^{p+q}(\mathcal{O}_G).$$

Let

$$(2.43) \quad 0 \rightarrow H_0 \rightarrow G_0 \rightarrow S \rightarrow 0$$

be as in Lemma 2.2, so S is the maximal quotient torus of G_0 .

The equation (2.6) identifies $H^i(G_0, \mathcal{O}_{G_0})$ with $H^i(J \times S, \mathcal{O}_{J \times S})$, and the invariance condition might be looked at on $J \times S$. Let us write $H^0(S, \mathcal{O}_S) = K \oplus V$, $f \mapsto f(0) \oplus (f - f(0))$, where V consists of the regular functions which vanish at $\{0\} \in S$. Then $H^i(G_0, \mathcal{O}_{G_0})^{\text{inv}} = H^i(J, \mathcal{O}_J)^{\text{inv}} \oplus (H^i(J, \mathcal{O}_J) \otimes V)^{\text{inv}}$. Thus if a class $F = \sum \varphi_f \otimes f \in H^i(\mathcal{O}_J) \otimes V$ is invariant, where $\varphi_f \in H^i(J, \mathcal{O}_J)$ and the $f \in V$ are linearly independent over K , then

$$(\mu^* - p_1^* - p_2^*)(F)|(J \times J \times \{0\} \times S) = \sum (\mu^* - p_2^*)(\varphi_f) \otimes f = 0$$

thus $\mu^*\varphi_f = p_1^*\varphi_f$ and $\mu^*\varphi_f|_{\{0\} \times J} = \varphi_f = \varphi_f|_{\{0\}}$. So for $i \geq 1$, this implies that $\varphi_f = 0$ and for $i = 0$ this implies that $F \in H^0(\mathcal{O}_S)^{\text{inv}} = 0$.

In short:

$$(2.44) \quad H^*(\mathcal{O}_{G_0})^{\text{inv}} \cong \left(H^*(\mathcal{O}_J) \otimes H^0(\mathcal{O}_S) \right)^{\text{inv}} \cong H^*(\mathcal{O}_{H_0})^{\text{inv}} \cong H^*(\mathcal{O}_J)^{\text{inv}}$$

Thus, it suffices to prove the lemma with G_0 replaced by H_0 , so we may assume the quotient torus $S = (0)$. Since in this case $H^i(\mathcal{O}_{G_0}) \cong \wedge^i H^1(\mathcal{O}_J)$, one sees that pullback under the multiplication by N map, $N\delta : G \rightarrow G$ acts on E_1^{pq} by multiplication by N^q . It follows that the spectral sequence (2.42) degenerates at E_2 . In particular the eigenspace where $N\delta^*$ acts by multiplication by N on $H^1(\mathcal{O}_G)$ is

$$(2.45) \quad H^1(\mathcal{O}_J)/\partial(W^*) \cong E_2^{0,1} \cong E_\infty^{0,1} \hookrightarrow H^1(\mathcal{O}_G).$$

Note that as a quotient of $H^1(\mathcal{O}_J)$ the space $E_\infty^{0,1}$ is clearly invariant. Conversely, let $\Delta : G \rightarrow G \times G$ be the diagonal. Since $\mu \circ \Delta = 2\delta$ it follows that for $a \in H^1(\mathcal{O}_G)^{\text{inv}}$ we have

$$(2.46) \quad (2\delta)^*(a) = \Delta^*\mu^*(a) = \Delta^*(p_1^*(a) + p_2^*(a)) = 2a$$

so necessarily $a \in E_\infty^{0,1}$, proving (2.41). A similar argument on $E_\infty^{-1,1}$ proves (2.40).

We remark here that $H^0(J, \text{fil}_1)$ is in a natural way a subspace of the regular functions on G , and (2.40) takes the quotient of this by $H^0(J, \mathcal{O}_J) = K$. This is because we have forced the rigidification condition in the Definition 2.6. \square

We return to the proof of Proposition 2.13. The exact sequence

$$(2.47) \quad 0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_C(\mathcal{D}') \rightarrow \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \rightarrow 0$$

defines a map

$$(2.48) \quad \psi : W^* := H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) \rightarrow H^1(\mathcal{O}_C).$$

By Lemma 2.1, as a group extension of G_0 , the group G is defined by a unique map from $H^0(C, \Omega_{C/K}^1)$ to a vector space. We claim that this map is the dual of ψ . To see this, one identifies J and J^\vee . Then it is well known that the universal vectorextension on J^\vee is

$$0 \rightarrow H^0(C, \Omega_{C/K}^1) \rightarrow \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \Omega_{C/K}^1) \rightarrow \text{Pic}^0(C) \rightarrow 0$$

inducing the universal vectorextension

$$0 \rightarrow H^0(C, \Omega_{C/K}^1) \rightarrow \mathbb{H}^1(C, \mathcal{O}_{C,D}^* \rightarrow \Omega_{C/K}^1) \rightarrow \text{Pic}^0(C, D) \rightarrow 0$$

on

$$\text{Pic}^0(C, D) := \text{Ker}\left(H^1(C, \mathcal{O}_{C,D}^* \rightarrow H^1(C, \Omega_{C/K}^1)\right)$$

where

$$\mathcal{O}_{C,Z}^* := \text{Ker}(\mathcal{O}_C^* \rightarrow \mathcal{O}_Z^*)$$

for any subscheme $Z \subset C$. The map of complexes

$$a : \{\mathcal{O}_{C,D}^* \rightarrow 0\} \rightarrow \{\mathcal{O}_{C,D}^* \rightarrow \Omega_{C/K}^1|_{\mathcal{D}'}\}$$

induces an isomorphism on \mathbb{H}^1 . Indeed, a sends the exact sequence

$$0 \rightarrow H^0(C, (1 + \mathcal{O}_{\mathcal{D}'}(-D))) \rightarrow H^1(C, \mathcal{O}_{C,D}^*) \rightarrow H^1(C, \mathcal{O}_{C,D}^*) \rightarrow 0$$

to the exact sequence

$$0 \rightarrow H^0(C, \Omega_{C/K}^1|_{\mathcal{D}'}) \rightarrow \mathbb{H}^1(C, \mathcal{O}_{C,D}^* \rightarrow \Omega_{C/K}^1|_{\mathcal{D}'}) \rightarrow H^1(C, \mathcal{O}_{C,D}^*) \rightarrow 0,$$

so one has just to see that

$$d \log : H^0(C, (1 + \mathcal{O}_{\mathcal{D}'}(-D))) \rightarrow H^0(C, \Omega_{C/K}^1|_{\mathcal{D}'})$$

is an isomorphism. But $H^0(C, \mathcal{O}_{\mathcal{D}'}(-D)) \cong H^0(C, (1 + \mathcal{O}_{\mathcal{D}'}(-D)))$ via the exponential map and the quasiisomorphism [4]

$$\{\mathcal{O}_C(-\mathcal{D}) \rightarrow \Omega_{C/K}^1(-\mathcal{D}')\} \rightarrow \{\mathcal{O}_C(-\mathcal{D}) \rightarrow \Omega_{C/K}^1\}$$

allows to conclude.

Define a bundle \mathcal{E} on C by pullback

$$(2.49) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & W^* \otimes \mathcal{O}_C \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}(\mathcal{D}') & \longrightarrow & \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \longrightarrow 0 \end{array}$$

Because of the isomorphism $H^1(\mathcal{O}_J) \cong H^1(\mathcal{O}_C)$, the top row of the diagram (2.49) pulls back uniquely from an extension of $W^* \otimes \mathcal{O}_J$ by \mathcal{O}_J . There is a unique vectorial extension

$$(2.50) \quad 0 \rightarrow W \otimes \mathbb{G}_a \rightarrow H \xrightarrow{r} J \rightarrow 0$$

such that the above extension of vector bundles coincides with

$$(2.51) \quad 0 \rightarrow \mathcal{O}_J \rightarrow \text{fil}_1 r_* \mathcal{O}_H \rightarrow W^* \otimes \mathcal{O}_J \rightarrow 0$$

From this we get a diagram (defining t and u . Here $i : C \hookrightarrow J$)

$$(2.52) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_J & \longrightarrow & \text{fil}_1 r_* \mathcal{O}_H & \longrightarrow & W^* \otimes \mathcal{O}_J \longrightarrow 0 \\ & & \downarrow & & \downarrow t & & \downarrow u \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{O}_C(\mathcal{D}') & \longrightarrow & \mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C \longrightarrow 0 \end{array}$$

We get a diagram with exact rows

$$(2.53) \quad \begin{array}{ccccccc} 0 & \rightarrow & H^0(\mathcal{O}_J) & \rightarrow & H^0(\text{fil}_1 r_* \mathcal{O}_H) & \rightarrow & H^0(W^* \otimes \mathcal{O}_J) \xrightarrow{\partial} H^1(\mathcal{O}_J) \rightarrow \\ & & \downarrow \cong & & \downarrow t & & \downarrow \cong \\ 0 & \rightarrow & H^0(\mathcal{O}_C) & \rightarrow & H^0(\mathcal{O}_C(\mathcal{D}')) & \rightarrow & H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) \rightarrow H^1(\mathcal{O}_C) \rightarrow \\ & & & & & & \\ H^1(\mathcal{O}_H)^{\text{inv}} & \rightarrow & 0 & & & & \\ & & \downarrow v & & & & \\ H^1(\mathcal{O}_C(\mathcal{D}')) & \rightarrow & 0 & & & & \end{array}$$

The diagram (2.53) gives isomorphisms

$$(2.54) \quad e : H^0(\mathcal{O}_G)^{\text{inv}} \cong \ker(\partial) \cong \text{Ker}(H^0(\mathcal{O}_C(\mathcal{D}')/\mathcal{O}_C) \rightarrow H^1(\mathcal{O}_C))$$

$$(2.55) \quad h : H^1(\mathcal{O}_G)^{\text{inv}} \cong H^1(\mathcal{O}_C(\mathcal{D}')).$$

These are two of the desired arrows for the diagram in the proposition.

Lemma 2.15. *The natural map on relative connections*

$$(2.56) \quad \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}} \rightarrow \mathbb{H}^1\left(C, j_*(\mathcal{O}_{C-D}^* \rightarrow \Omega_{C/K}^1(\mathcal{D}))\right)$$

is an isomorphism.

Proof of lemma. We note the following facts:

1. $H^1(G, \Omega_{G/K}^1)^{\text{inv}} = \left(H^0(G, \Omega_{G/K}^1)^{\text{inv}} \otimes H^1(\mathcal{O}_G) \right)^{\text{inv}} = (0)$.
2. $H^1(\mathcal{O}_G^*)^{\text{inv}} \cong H^1(\mathcal{O}_{C-D}^*)$. Indeed, as remarked in the proof of Lemma 2.10 one has $J(K) \twoheadrightarrow H^1(\mathcal{O}_G^*)^{\text{inv}}$. One checks that the kernel is generated by divisors of degree 0 supported on D .
3. $H^0(\mathcal{O}_G^*)/\text{const.} \cong H^0(\mathcal{O}_G^*)^{\text{inv}} \cong H^0(\mathcal{O}_{C-D}^*)/\text{const.}$.
- 4.

$$\left(H^0(\Omega_{G/K}^1)/d \log(H^0(\mathcal{O}_G^*)) \right)^{\text{inv}} = H^0(\Omega_{G/K}^1)^{\text{inv}}/d \log(H^0(\mathcal{O}_G^*)^{\text{inv}})$$

(This is seen by noting $H^0(\mathcal{O}_G^*)^{\text{inv}} = \text{Hom}(G, \mathbb{G}_m)$, so one has a homomorphism $\psi : G \rightarrow \mathbb{G}_m^r$ such that ψ^* is an isomorphism on global units modulo constants. The assertion then reduces to the case $G = \mathbb{G}_m^r$, which is easy.)

We build a diagram

$$(2.57) \quad \begin{array}{ccccccc} 0 & \rightarrow & \frac{H^0(\Omega_{G/K}^1)^{\text{inv}}}{H^0(\mathcal{O}_G^*)^{\text{inv}}} & \rightarrow & \mathbb{H}^1(\mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv}} & \rightarrow & H^1(\mathcal{O}_G^*)^{\text{inv}} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{H^0(\Omega_{C/K}^1(\mathcal{D}))}{H^0(\mathcal{O}_{C-D}^*)} & \rightarrow & \mathbb{H}^1(C, j_*(\mathcal{O}_{C-D}^* \rightarrow \Omega_{C/K}^1(\mathcal{D}))) & \rightarrow & H^1(C-D, \mathcal{O}^*) \rightarrow 0 \end{array}$$

Since the left and right hand arrows are isomorphisms, it follows that the central arrow is as well, proving the lemma. \square

The assertions of the proposition follow easily from the lemma. \square

Finally, we need an analogous result for integrable connections. More precisely, we consider a slightly weaker condition.

Definition 2.16. *Let X be a variety over K . A connection $\nabla : E \rightarrow E \otimes \Omega_X^1 \otimes K(X)$ (so possibly with poles) is said to have vertical curvature if the curvature*

$$(2.58) \quad \nabla^2 : E \rightarrow E \otimes \Omega_X^2 \otimes K(X)$$

has values in the subsheaf $E \otimes \Omega_K^2 \otimes K(X) \subset E \otimes \Omega_X^2 \otimes K(X)$. The group of line bundles with vertical curvature will be denoted

$$\mathbb{H}^1(X, \mathcal{O}_X^* \rightarrow \Omega_X^1)^{\text{vert}}$$

and similarly for invariant line bundles with vertical curvature

$$\mathbb{H}^1(X, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv, vert}}.$$

Proposition 2.17. *With notation as above, we have isomorphisms*

$$(2.59) \quad \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_{G/K}^1)^{\text{inv,vert}} = \mathbb{H}^1(C, j_* \mathcal{O}_{C-D}^* \rightarrow \Omega_{C/K}^1(\mathcal{D}))^{\text{vert}}$$

$$(2.60) \quad \mathbb{H}^1(G, \mathcal{O}_G^* \rightarrow \Omega_G^1)^{\text{inv,vert}} \cong \frac{\mathbb{H}^1(C, j_* \mathcal{O}_{C-D}^* \rightarrow \Omega_C^1 \langle D \rangle (\mathcal{D}'))^{\text{vert}}}{\Omega_K^1 / K^\times}$$

Proof. For example, in the absolute case, the curvature of a line bundle with invariant absolute connection on G is a section $\eta \in H^0(\Omega_G^2 / \Omega_K^2 \otimes \mathcal{O}_G)$ satisfying $\mu^*(\eta) = p_1^*(\eta) + p_2^*(\eta)$. It is easy to see that such a section lies in the subsheaf $\Omega_{G/K}^1 \otimes \Omega_K^1$. The isomorphism (2.60) follows from Proposition 2.13 and the fact that pullback to C of invariant forms is injective by (2.30). The case of relative forms is similar and is left for the reader. \square

3. The geometric setup

We continue to work with a curve C/K and a line bundle L on C of degree 0. Let $\nabla_{/K} : L \rightarrow L \otimes \Omega_{C/K}^1(\mathcal{D})$, where $\mathcal{D} = \sum m_i c_i$. As in the previous section, write $D = \sum c_i$ and $\mathcal{D}' = \mathcal{D} - D$.

Lemma 3.1. *Assume $\nabla_{/K}|_{C-D}$ lifts to an absolute integrable connection $\nabla' : L_{C-D} \rightarrow L_{C-D} \otimes \Omega_{C-D/K}^1$. Then ∇' extends to an absolute integrable connection*

$$(3.1) \quad \nabla : L \rightarrow L \otimes \Omega_{C/K}^1 \langle D \rangle (\mathcal{D}').$$

The notation $\langle D \rangle$ refers to log poles at D as in (1.4).

Proof. Let e be a basis for L at c a point with multiplicity $m \geq 1$ in \mathcal{D} , and let x be a local parameter at c on C . Write

$$(3.2) \quad \nabla'(e) = A(x)dx + \sum_i B_i(x)d\tau_i; \quad d\tau_i \text{ basis in } \Omega_K^1, \quad x^m A(x) \in \mathcal{O}_{C,c}.$$

We must show $x^{m-1}B_i(x)$ is regular at c . But integrability of ∇' implies that $\partial A / \partial \tau_i = \partial B_i / \partial x$, from which the assertion is clear. \square

We know from Proposition 2.17 that the restriction to $C - D$ of an integrable absolute connection of the form (3.1) pulls back from a unique invariant integrable absolute connection $\mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_G^1$ on $G = J_{\mathcal{D}}$. More precisely, we fix a basepoint $c_0 \in (C - D)(K)$ and normalize our connection (3.1) to be trivial at the basepoint by tensoring with a pullback from $\text{Spec}(K)$.

We consider now the basic geometric picture of Deligne [3]

$$(3.3) \quad \pi : \text{Sym}^N(C - D) \rightarrow G_N; \quad N = 2g - 2 + \sum_i m_i$$

where G_N is the $J_{\mathcal{D}}$ -torsor of degree N line bundles trivialized along \mathcal{D} and $\pi(\sum z_i) = \mathcal{O}(\sum z_i)$ with trivialization given by restricting to \mathcal{D} the canonical (upto scalar in K) map $\mathcal{O}_C \rightarrow \mathcal{O}_C(\sum z_i)$. Note that $N = \deg(\Omega_{C/K}^1(\mathcal{D}))$ and $\dim G = g - 1 + \sum m_i = N - g + 1$. (Recall we assume $\mathcal{D} \neq \emptyset$.) We identify $G_N \cong G$ by sending the point $[\mathcal{O}(Nc_0)] \mapsto 0$, and we write \mathcal{L} for the resulting line bundle with connection on G_N . The basic remark of Deligne is

Proposition 3.2. *Assume*

$$H_{DR}^0(C - D, (L, \nabla)) = H_{DR}^2(C - D, (L, \nabla)) = 0.$$

Then

$$(3.4) \quad \det(H_{DR}^*(C - D, (L, \nabla))) = \det(H_{DR}^1((C - D, (L, \nabla)))) \\ \cong H^N(\text{Sym}^N(C - D), (\pi^*(\mathcal{L}, \nabla)))$$

as a line with connection on K .

Proof. Our hypotheses imply $H_{DR}^1(C - D, (L, \nabla))$ has dimension N . Consider the diagram

$$(3.5) \quad \begin{array}{ccc} (C - D)^N & \xrightarrow{p} & \text{Sym}^N(C - D) \\ & \searrow q \downarrow \pi & \\ & & G_N \end{array}$$

We have $q^*(\mathcal{L}) \cong L \boxtimes \cdots \boxtimes L$ (exterior tensor product on $(C - D)^N$). The Künneth formula gives

$$(3.6) \quad H^N((C - D)^N, (L \boxtimes \cdots \boxtimes L, \nabla)) \cong H_{DR}^1(C - D, (L, \nabla))^{\otimes N}.$$

There is an action of the symmetric group \mathcal{S}_N on the pair

$$((C - D)^N, L \boxtimes \cdots \boxtimes L).$$

The resulting action on $(H_{DR}^1)^{\otimes N}$ is alternating because of the odd degree cohomology, so the invariants are precisely $\det H_{DR}^1$. There is an evident map

$$(3.7) \quad p^* : H_{DR}^N(\text{Sym}^N(C - D), \pi^*\mathcal{L}) \rightarrow H_{DR}^N((C - D)^N, L \boxtimes \cdots \boxtimes L)^{\mathcal{S}_N} \\ = \det H_{DR}^1(C - D, L)$$

To show this map is an isomorphism, it suffices to remark that one has a trace map

$$(3.8) \quad p_* : H_{DR}^N((C - D)^N, L \boxtimes \cdots \boxtimes L) \rightarrow H_{DR}^N(\text{Sym}^N(C - D), \pi^*\mathcal{L})$$

Because $L \boxtimes \cdots \boxtimes L = p^*\pi^*\mathcal{L}$, the existence of such a trace follows from the projection formula and the trace in de Rham cohomology with constant coefficients. \square

Now one uses the geometry of the map π and (3.4) to compute the determinant.

Lemma 3.3. *Let X be a smooth variety over a field of characteristic 0. Let $A \subset X$ be a smooth subvariety of codimension p . Let (E, ∇) be an integrable connection on X . Then*

$$(3.9) \quad \mathbb{H}_A^n(X, E \otimes \Omega_X^*) \cong H^{n-2p}(A, E \otimes \Omega_A^*).$$

Proof. Write $\underline{H}_A^r(F)$ for the Zariski sheaf associated to the presheaf $U \mapsto H_A^r(U, F)$ for any Zariski sheaf F on X . For F locally free, $\underline{H}_A^r(F) = (0)$ for $r \neq p$ by purity. Duality theory gives (here $X \supset A_\alpha \supset A$ runs through nilpotent thickenings)

$$(3.10) \quad E \otimes \Omega_A^m \rightarrow \text{Ext}^p(\mathcal{O}_A, E \otimes \Omega_X^{m+p}) \rightarrow \varinjlim_\alpha \text{Ext}^p(\mathcal{O}_{A_\alpha}, E \otimes \Omega_X^{m+p}) \\ \cong \underline{H}_A^p(E \otimes \Omega_X^{m+p}).$$

We want to show that this map is an isomorphism, compatible with the connection, thus yielding a quasiisomorphism of complexes

$$(3.11) \quad E \otimes \Omega_A^* \rightarrow \underline{H}_A^p(E \otimes \Omega_X^*).$$

The problem is local, so we can assume

$$A \subset A_1 \subset \dots \subset A_p = X$$

with A_i smooth of codimension $p - i$ in X . Now $\underline{H}_A^p(E \otimes \Omega_X^*)$ represents $R\underline{\Gamma}_A(E \otimes \Omega_X^*)[p]$ in the derived category of Zariski sheaves on A , and in the derived category we may write

$$(3.12) \quad R\underline{\Gamma}_A(E \otimes \Omega_X^*)[p] = R\underline{\Gamma}_A[1] \circ R\underline{\Gamma}_{A_1}[1] \circ \dots \circ R\underline{\Gamma}_{A_{p-1}}(E \otimes \Omega_X^*)[1]$$

In this way, we reduce to verifying (3.9) in the case $p = 1$. So, suppose $A : t = 0$ in $X = \text{Spec}(R)$. We have

$$(3.13) \quad \underline{H}_A^1(E \otimes \Omega_X^*) \cong E_{R[t^{-1}]} \otimes \Omega_{R[t^{-1}]}^* / E_R \otimes \Omega_R^*$$

as $H_{DR}^1(X, E) \subset H_{DR}^1(X - A, E)$.

By [4], since E has no singularity along $t = 0$, one has

$$(3.14) \quad (E_R \otimes \Omega_R^*(\log(t = 0)), \nabla) \xrightarrow{\text{q.iso.}} (E_{R[t^{-1}]} \otimes \Omega_{R[t^{-1}]}^*, \nabla|_{\text{Spec}R[t^{-1}]})$$

Thus $\text{res} : \underline{H}_A^1(E \otimes \Omega_X^*, \nabla) \rightarrow E_{R/tR} \otimes (\Omega_{R/tR}^*[-1], \nabla|_{\text{Spec}(R/tR)})$ is an isomorphism. \square

Lemma 3.4. *Let $p : G_N \rightarrow J_N$ be the projection to the corresponding torseur over the absolute jacobian. Write $[\Omega_{C/K}^1(\mathcal{D})] \in J_N$ for the point corresponding to the canonical bundle twisted by $\mathcal{O}(\mathcal{D})$. Let $a \in G_N$. We have*

$$(3.15) \quad \pi^{-1}(a) = \begin{cases} \mathbb{A}^{g-1} & p(a) \neq [\Omega_{C/K}^1(\mathcal{D})] \\ \mathbb{A}^g & p(a) = [\Omega_{C/K}^1(\mathcal{D})]; \partial(a) = 0 \\ \emptyset & p(a) = [\Omega_{C/K}^1(\mathcal{D})]; \partial(a) \neq 0 \end{cases}$$

where by definition $\mathbb{A}^{g-1} = \emptyset$ if $g = 0$. Note that if $p(a) = [\Omega_{C/K}^1(\mathcal{D})]$, then a corresponds to a trivialization $\mathcal{O}_{\mathcal{D}} \cong \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$ defined upto scalar. $\partial(a)$ in the above refers to the evident boundary of this trivialization in $H^1(\Omega_{C/K}^1) = K$ (again upto scale).

Proof. Let $p(a)$ correspond to a line bundle M of degree N , we consider the exact sequence

$$(3.16) \quad 0 \rightarrow M(-\mathcal{D}) \rightarrow M \rightarrow M|_{\mathcal{D}} \rightarrow 0$$

Suppose first $M \neq \Omega_{C/K}^1(\mathcal{D})$. Then $H^1(M(-\mathcal{D})) = (0)$, so any trivialization in $H^0(M|_{\mathcal{D}})$ lifts to $H^0(M)$, and the space of such liftings is a torseur under $H^0(M(-\mathcal{D}))$, a vector space of dimension $g - 1$. (Note this is an affine torseur, not a projective torseur.) If $M = \Omega_{C/K}^1(\mathcal{D})$, $H^0(M(-\mathcal{D}))$ has dimension g , and the image $H^0(M) \rightarrow H^0(M|_{\mathcal{D}})$ has codimension 1. \square

Remark 3.5. If we choose local parameters t_i at $c_i \in \mathcal{D}$, then

$$H^0(\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}})$$

can be identified with the space of polar parts of 1-forms with poles along \mathcal{D} , and the map ∂ is given by the residue

$$(3.17) \quad \partial\left(\sum_i \sum_{j=0}^{m_i-1} u_{ij} dt_i / t_i^{m_i-j}\right) = \sum_i u_{i, m_i-1}$$

Note the (open) condition for an element in $H^0(\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}})$ to be a trivialization is simply

$$(3.18) \quad \prod_i u_{i0} \neq 0.$$

Because $B \subset G_N$, we must factor out by the action of \mathbb{G}_m , which we can normalize away by setting $u_{10} = 1$. Thus we have

$$(3.19) \quad B := \pi(\text{Sym}^N(C - D)) \cap p^{-1}[\Omega_{C/K}^1(\mathcal{D})] = \left\{ \sum_i \sum_{j=0}^{m_i-1} u_{ij} dt_i / t_i^{m_i-j} \mid \sum_i u_{i, m_i-1} = 0; \prod_i u_{i0} \neq 0; u_{10} = 1 \right\}.$$

Define $A := \pi^{-1}(B) \subset \text{Sym}^N(C - D) =: X$. Using the localization sequence and Lemma 3.4, we get an exact sequence

$$(3.20) \quad \begin{array}{ccccc} H_{DR}^{N-2g}(A/K, \pi^*\mathcal{L}) & \rightarrow & H_{DR}^N(X/K, \pi^*\mathcal{L}) & \rightarrow & H_{DR}^N(X - A/K, \pi^*\mathcal{L}) \\ \uparrow \cong & & & & \uparrow \cong \\ H_{DR}^{N-2g}(B/K, \mathcal{L}|_B) & & & & H_{DR}^N(G_N - p^{-1}[\Omega_{C/K}^1(\mathcal{D})]/K, \mathcal{L}) \end{array}$$

where $p : G_N \rightarrow J_N$ with $J = J(C)$ the absolute Jacobian. The fact that the vertical arrows in this diagram are isomorphisms follows because the maps are maps of affine space bundles and the line bundles with connection are pulled back from the base.

To simplify the presentation, we will assume that $m_1 \geq 2$. Another proof of our formula for the de Rham determinant in the case $\mathcal{D} = D = \sum c_i$, (i.e. for regular singular points) will be given in Theorem 4.6.

Lemma 3.6. *Assume the line bundle L on C has degree 0, and that \mathcal{D} is minimal, i.e. $\nabla : L|_{\mathcal{D}} \cong L \otimes \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$. We continue to assume also that $\mathcal{D} = \sum m_i c_i$ with $m_1 \geq 2$. Then*

$$(3.21) \quad H_{DR}^*(\text{Sym}^N(C - D) - A/K, \pi^*\mathcal{L}) = (0).$$

Proof. The isomorphism on the right in (3.20) implies we must show $H_{DR}^*(G_N - p^{-1}[\Omega_{C/K}^1(\mathcal{D})]/K, \mathcal{L}) = (0)$. The assumption $m_1 \geq 2$ means we have a \mathbb{G}_a action by translation on $G_N - p^{-1}[\Omega_{C/K}^1(\mathcal{D})]/K, \mathcal{L}$, and minimality of \mathcal{D} implies that the connection is nontrivial on the fibres. The fibration is Zariski-locally trivial, so the Leray spectral sequence for de Rham cohomology reduces us to showing $H_{DR}^*(\mathbb{G}_{a,S}/S, (\mathcal{O}, \mathcal{E})) = (0)$ where \mathcal{E} is an everywhere non-zero, translation invariant, relative 1-form on $\mathcal{O}_{\mathbb{G}_{a,S}}$. In other words, for $s \in \mathcal{O}_S^\times$ we must show

$$(3.22) \quad \mathcal{O}_S[t] \xrightarrow{d+sd} \mathcal{O}_S[t]dt$$

has trivial cohomology. This is straightforward. \square

Lemma 3.7. *Assume $H_{DR}^0(C - D, L) = (0)$. Then*

$$(3.23) \quad H_{DR}^{m-2}(B/K, \mathcal{L}|_B) \cong H_{DR}^N(\text{Sym}^N(C - D)/K, \pi^*\mathcal{L}) \cong K$$

as a line with a connection over K .

Proof. Note $m - 2 = N - 2g$. Extending the top sequence in (3.20) one step to the left and using the previous lemma gives the left isomorphism. We have already seen the isomorphism on the right. \square

Our task now is to calculate $H_{DR}^{m-2}(B/K, \mathcal{L}|_B)$ with its connection. We assume that $\mathcal{L} \in \text{Pic}^0(J)$ as in Lemma 3.6. Then \mathcal{L} carries a relative invariant connection $d_{/K}$ on J , and $\nabla_{/K} = d_{/K} + \mathcal{E}$ for some invariant form $\mathcal{E} \in H^0(G, \Omega_{G/K}^1)^{\text{inv}}$. Changing the choice of $d_{/K}$ changes \mathcal{E} to $\mathcal{E} + p^*(\alpha)$, where $\alpha \in H^0(J, \Omega_{J/K}^1)^{\text{inv}}$ and $p : G_N \rightarrow J_N$ is the torseur under the affine group $\mathcal{G} := \ker(G \rightarrow J)$. In particular, $\mathcal{E}|_{p^{-1}[\Omega_{C/K}^1(\mathcal{D})]}$ does not depend on the choice of $d_{/K}$. As $p^{-1}(\Omega_{G/K}^1(\mathcal{D}))$ is isomorphic to \mathcal{G} , and we see that

$$(3.24) \quad (\mathcal{L}, \nabla_{/K})|_B = (\mathcal{O}_B, d + \mathcal{E}|_B).$$

We say that $\mathcal{E}|_B$ vanishes at a point $b \in B$ if $\mathcal{E}(b) = 0$ in $\mathfrak{m}_b/\mathfrak{m}_b^2$.

Lemma 3.8. *Let $b \in B \subset G_N$ correspond to the trivialization on $\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$ given by $\nabla|_{\mathcal{D}} : L|_{\mathcal{D}} \rightarrow L \otimes \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$. Then $\mathcal{E}|_B$ vanishes at b . $\mathcal{E}|_B$ does not vanish at any other point of B .*

Proof. Let us write $\eta := i^*\mathcal{E}|_{C-D}$, where $i : C-D \rightarrow G$ is the cycle map. Then by definition, the trivialization of $\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$ associated to $i^*(\mathcal{L}, \nabla)$ depends only on $\eta|_{\mathcal{D}}$, or equivalently only on $\mathcal{E}|_{p^{-1}[\Omega_{C/K}^1(\mathcal{D})]}$. We have

$$(3.25) \quad \pi^*\mathcal{E} = \sum_{i=1}^N \eta_i$$

where η_i is the pullback of η via the i -th projection $(C-D)^N \rightarrow C-D$.

Suppose for a moment that the divisor of η (viewed as a section of $\Omega_{C/K}^1(\mathcal{D})$) is reduced, $(\eta) = \sum e_i$; $e_i \in (C-D)(\bar{K})$. Let $e := (e_1, \dots, e_N) \in \text{Sym}^N(C-D)$ be the point corresponding to η . We have $e \in A = \pi^{-1}(B) \subset \text{Sym}^N(C-D)$ and $b = \pi(e)$. Since $A \rightarrow B$ is a projective bundle, there is a surjection on tangent spaces

$$(3.26) \quad \begin{array}{ccc} T_A(e) & \hookrightarrow & T_{\text{Sym}^N(C-D)}(e) \\ \text{surjective } \downarrow & & \downarrow \pi_* \\ T_B(b) & \hookrightarrow & T_{G_N}(b) \end{array}$$

Since $T_{\text{Sym}^N(C-D)}(e)$ is spanned by expressions $\sum \tau_i|_{e_i}$, to show $\mathcal{E}|_B$ vanishes at b , it suffices to show

$$(3.27) \quad \langle \mathcal{E}, \pi_*(\sum \tau_i|_{e_i}) \rangle = 0.$$

This expression equals

$$(3.28) \quad \langle \pi^*(\mathcal{E}), \sum \tau_i|_{e_i} \rangle = \sum \langle \eta, \tau_i|_{e_i} \rangle.$$

Each term on the right vanishes because $\eta(e_i) = 0$ in $\mathfrak{m}_{e_i}/\mathfrak{m}_{e_i}^2$. The general case (η not necessarily reduced) follows from this by a specialization argument.

We postpone until Lemma 3.10 the proof that $\mathcal{E}|_B$ doesn't vanish at any other point of B . \square

By assumption we start with an absolute, invariant, integrable connection on \mathcal{L} of degree 0 on J . Restricting to B , we get an absolute closed invariant 1-form Ψ , whose corresponding relative form is \mathcal{E} . Recall (3.19) we have coordinates u_{ij} on B with $u_{i0} = 1$, $\prod u_{i0} \neq 0$, and $u_{1,m_1-1} = -\sum_{i \geq 2} u_{i,m_i-1}$.

Lemma 3.9. *Under the assumption that*

$$\{i^*\nabla_{/K} : i^*\mathcal{L} \rightarrow i^*\mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})\} \rightarrow \{j_*i^*\mathcal{L}|(C-D) \rightarrow j_*i^*\mathcal{L} \otimes \Omega_{(C-D)/K}^1\}$$

is a quasiisomorphism, and $i^\nabla_{/K}$ has poles along all points of \mathcal{D} , we can arrange that a K -basis for*

$$H_{DR}^{m-2}(B/K, \mathcal{L}|_B) = H_{DR}^{m-2}(B/K, (\mathcal{O}_B, d + \mathcal{E}))$$

is given by the closed form

$$(3.29) \quad \theta := \prod_{\substack{(i,j) \\ m_i \geq 2}} du_{ij} \wedge \prod_i \frac{du_{i0}}{u_{i0}}$$

Proof. Recall (2.27) the relative invariant forms on $\mathcal{G} := \ker(G \rightarrow J)$ are the τ_{ij} defined by the expression

$$(3.30) \quad \sum_{j=0}^{m_i-1} \tau_{ij} T_i^j = \left(\sum_j u_{ij} T_i^j \right)^{-1} \sum_j du_{ij} T_i^j.$$

Write

$$(3.31) \quad \mathcal{E} = \sum_{i,j} \lambda_{ij} \tau_{ij}; \quad \lambda_{i,m_i-1} \neq 0$$

where the nonvanishing condition comes from the requirement that the form restricted to $C-D$ gives a trivialization along \mathcal{D} (see (2)). If we write (mod $T_i^{m_i}$)

$$(3.32) \quad \left(u_{i0} + u_{i1}T_i + \dots + u_{i,m_i-1}T_i^{m_i-1} \right)^{-1} = v_{i0} + v_{i1}T_i + \dots + v_{i,m_i-1}T_i^{m_i-1}$$

we get the table

$$(3.33) \quad \begin{aligned} \tau_{i0} &= v_{i0} du_{i0} \\ \tau_{i1} &= v_{i0} du_{i1} + v_{i1} du_{i0} \\ &\vdots \\ \tau_{i,m_i-1} &= v_{i0} du_{i,m_i-1} + \dots + v_{i,m_i-1} du_{i0} \end{aligned}$$

Note if we give u_{ij} , du_{ij} , v_{ij} all weight j , then τ_{ij} will be homogeneous of weight j . Comparing (3.33) and (3.31), it follows that if we expand $\mathcal{E}|_B$ in terms of the du_{ij} , omitting du_{10} and du_{1,m_1-1} , we find for suitable $\alpha_{ij} \neq 0$

$$(3.34) \quad \mathcal{E}|_B = \sum_i g_{ij} du_{ij} = \sum_{i,j} \left[(\alpha_{i,m_i-1} u_{i0}^{-1} - \alpha_{1,m_1-1}) du_{i,m_i-1} + \right. \\ \left. u_{i0}^{-1} \sum_{p=0}^{m_i-2} \left(\alpha_{ip} \frac{u_{i,m_i-1-p}}{u_{i0}} + \sum \text{terms at least quadratic in } \frac{u_{ik}}{u_{i0}} \right) du_{ip} \right]$$

Looking at the weights, we see that for $p \leq m_i - 2$

$$(3.35) \quad g_{ip} = \text{nonzero multiple of } \frac{u_{i,m_i-p-1}}{u_{i0}} + \\ \text{terms only involving } u_{ij}, j < m_i - p - 1$$

while

$$(3.36) \quad g_{i,m_i-1} = \alpha_{i,m_i-1} u_{i0}^{-1} - \alpha_{1,m_1-1}$$

with neither α coefficient 0.

Now generators of $H_{DR}^{m-2}(B, (\mathcal{O}, d + \mathcal{E}))$ are of the form $M\theta$ where M is a monomial in the u_{ij} , u_{i0}^{-1} and θ is as in (3.29). Relations are

$$(3.37) \quad \left(\frac{\partial}{\partial u_{ij}} + g_{ij} \right) (M)\theta = 0$$

Because of (3.35) one can use these relations to eliminate u_{ij} , $j > 0$ from M by downward induction on j , starting from u_{i,m_i-1} . We are left with the case $M = u_{2,0}^{n_2} \cdots u_{r,0}^{n_r}$ with $n_i \in \mathbb{Z}$. In this case we can apply (3.36). If $m_i \geq 2$, we get the relation

$$(3.38) \quad u_{2,0}^{n_2} \cdots u_{r,0}^{n_r} \theta \equiv \frac{\alpha_{i,m_i-1}}{\alpha_{1,m_1-1}} u_{2,0}^{n_2} \cdots u_{i,0}^{n_i-1} \cdots u_{r,0}^{n_r} \theta$$

Using this, we can get $n_i = 0$. If $m_i = 1$ and $i \geq 2$ the relation becomes

$$(3.39) \quad u_{2,0}^{n_2} \cdots u_{r,0}^{n_r} \theta \equiv \frac{n_i + \alpha_{i,0}}{\alpha_{1,m_1-1}} u_{2,0}^{n_2} \cdots u_{i,0}^{n_i-1} \cdots u_{r,0}^{n_r} \theta$$

If $\alpha_{i,0}$ is not a positive integer, we can arrange $n_i = 0$.

On the other hand, we claim that if $m_i = 1$, then $\alpha_{i,0}$ is the residue of $i^* \nabla$ along c_i . Indeed (2.27) shows that in this case, $\tau_{i0} = d \log u_{i0}$, and u_{i0} is then just the local parameter in the point c_i (see (2.26)). Thus the quasiisomorphism

$$(3.40) \quad \{i^* \nabla /_K : i^* \mathcal{L} \rightarrow i^* \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})\} \rightarrow \{j_* i^* \mathcal{L} | (C - D) \\ \rightarrow j_* i^* \mathcal{L} \otimes \Omega_{(C-D)/K}^1\}$$

forces a_i not to lie in $\mathbb{N} - \{0\}$. □

The following was left open in the proof of Lemma 3.8:

Lemma 3.10. *Let $\mathcal{E} = \sum_{i,j} \lambda_{ij} \tau_{ij}$ be as in Lemma 3.8. Then \mathcal{E} vanishes at a unique point $b \in B$.*

Proof. We have seen in the proof of Lemma 3.8 that \mathcal{E} vanishes at a point in B . We must show it vanishes at at most one point. Let $b = (\dots, b_{ij}, \dots) \in B$ be a point. Write $b = (\dots, y_{ij}, \dots)$ with respect to the coordinates v_{ij} (3.32). Staring at (3.33), the conditions that $\mathcal{E}|_b = 0$ are seen to be (recall $\sum_i du_{i,m_i-1} = 0 = du_{i0}$) for $i \geq 2$

$$(3.41) \quad \begin{aligned} \lambda_{i,m_i-1} y_{i1} + \lambda_{i,m_i-2} y_{i0} &= 0 \\ \lambda_{i,m_i-1} y_{i2} + \lambda_{i,m_i-2} y_{i1} + \lambda_{i,m_i-3} y_{i0} &= 0 \\ &\vdots \\ \lambda_{i,m_i-1} y_{i,m_i-1} + \lambda_{i,m_i-2} y_{i,m_i-2} + \dots + \lambda_{i,0} y_{i0} &= 0 \end{aligned}$$

For $i = 1$ one gets the same list but with the last line (coefficient of du_{i0}) omitted. Finally, using $v_{10} = 1$ and $du_{1,m_1-1} = -\sum_{i \geq 2} du_{i,m_i-1}$ one gets

$$(3.42) \quad \lambda_{1,m_1-1} = y_{i0} \lambda_{i,m_i-1}; \quad 2 \leq i \leq r$$

Since $\lambda_{i,m_i-1} \neq 0$, equations (3.41) and (3.42) admit a unique solution for the y_{ij} . Since we know \mathcal{E} vanishes at at least one point of B , this point must lie in B . \square

Finally, we must calculate the Gauß-Manin connection on

$$H_{DR}^{m-2}(B, (\mathcal{O}_B, d + \mathcal{E}))$$

Define Ψ to be an absolute invariant form lifting \mathcal{E} . By assumption our connection on $C - D$ comes from an absolute integrable connection which, by Proposition 2.17, comes from an absolute integrable connection on G . Restricting this connection to B gives our Ψ .

Lemma 3.11. *With notation as above, there exists $F \in \mathcal{O}_B$, $\eta \in \Omega_K^1$, and $a_i \in k$, $i \geq 2$, such that*

$$(3.43) \quad \Psi = \sum_{i=2}^r a_i \frac{du_{i0}}{u_{i0}} + dF + \eta.$$

If moreover ∇ is integrable, then η is closed.

Proof. Since \mathcal{E} is (relatively) closed on B , one can write

$$(3.44) \quad \mathcal{E} = \sum_{i \geq 2} a_i \frac{du_{i0}}{u_{i0}} + d_{/K} F; \quad a_i \in K.$$

Lifting to an absolute form forces

$$(3.45) \quad \Psi = \sum_{i \geq 2} a_i \frac{du_{i0}}{u_{i0}} + dF + \sum_j f_j \eta_j; \quad f_j \in \mathcal{O}_B, \eta_j \in \Omega_K^1.$$

Here the η_j are linearly independent in Ω_K^1 . Using $d\Psi = 0$ modulo $\Omega_K^2 \otimes \mathcal{O}_B$ and taking residues along $u_{i0} = 0$ yields $a_i \in k \subset K$. Then computing $d\Psi \bmod \mathcal{O}_B \otimes \Omega_K^2$ yields

$$(3.46) \quad 0 = \sum_j d_{/K} f_j \otimes \eta_j \in \Omega_{B/K}^1 \otimes \Omega_K^1.$$

It follows that $f_j \in K$, so $\eta := \sum f_j \eta_j \in \Omega_K^1$. Taking d again shows η is closed if ∇ is integrable. \square

We now compute the Gauß-Manin connection. We have the diagram of global sections

$$(3.47) \quad \begin{array}{ccc} \Omega_{B/k}^{m-2} & \xrightarrow{\text{onto}} & \Omega_{B/K}^{m-2} \\ & & \downarrow d+\Psi \\ \Omega_{B/K}^{m-2} \wedge \Omega_K^1 & \xrightarrow{\cong} & \frac{\Omega_{B/k}^{m-1}}{\Omega_{B/k}^{m-3} \cdot \Omega_K^2} \end{array}$$

The connection is determined by its value on θ (3.29). To calculate, one lifts θ to $\tilde{\theta} \in \Omega_{B/k}^{m-2}$ and then applies $d + \Psi$. But for $\tilde{\theta}$ one can choose the form with the same expression (3.29). This form is closed, so

$$(3.48) \quad \nabla_{GM}(\theta) = \Psi \wedge \theta = (d_K(F) + \eta) \wedge \theta$$

Here we write $F = \sum_I a_I u^I$, $a_i \in K$ and $d_K(F) := \sum da_I u^I$.

Let $b \in B$ be the point corresponding to the trivialization of $\Omega_{C/K}^1(\mathcal{D})$ given by the polar part of the original relative connection. It really lies in B since we have assumed that $\deg \mathcal{L} = 0$.

Lemma 3.12. *With notation as above, the Gauß-Manin connection on the rank 1 K -vector space*

$$H_{DR}^{m-2}(B, (\mathcal{O}, d + \Xi))$$

described by (3.48) is isomorphic to the connection on K given by

$$1 \mapsto \Psi|_b + \frac{1}{2} d \log(\kappa)$$

for a suitable $\kappa \in K^\times$.

Proof. We have seen (Lemma 3.8) that this point b is determined by the condition that $\mathcal{E}(b) = 0 \in \mathfrak{m}_{B,b}/\mathfrak{m}_{B,b}^2$. Changing Ψ by a closed form pulled back from K changes the Gauß-Manin connection and the connection at b in the same way, so we can assume $\eta = 0$, i.e. $\Psi = \sum a_i \frac{du_{i0}}{u_{i0}} + dF$. Write

$$(3.49) \quad g_{ij} = \begin{cases} \frac{\partial F}{\partial u_{ij}} & j > 0 \\ \frac{\partial F}{\partial u_{i0}} + a_i/u_{i0} & j = 0. \end{cases}$$

Write $F = \sum_I a_I u^I$. Then

$$(3.50) \quad \Psi = \sum g_{ij} du_{ij} + \sum_I u^I da_I; \quad \Psi \wedge \theta = \sum_I u^I da_I \wedge \theta$$

We have

$$(3.51) \quad g_{ij}(b) = 0, \quad j < m_i - 1; \quad g_{i,m_i-1}(b) = g_{k,m_k-1}(b); \quad \text{all } i, k.$$

Since $\sum_i du_{i,m_i-1}|_B = 0$, we see from (3.51) that

$$(3.52) \quad \Psi|_{\{b\}} = \sum_I b^I da_I$$

Thus, it will suffice to relate $u^I \theta$ and $b^I \theta$ in H_{DR}^{m-2} . Note that each monomial u^I involves u_{ij} for only one value of i , and the weight of u^I is $\leq m_i - 1$ (see the discussion after (3.33)).

Suppose first the weight of u^I is strictly less than $m_i - 1$. Let j be maximal such that u_{ij} appears in u^I . From (3.35) it follows that

$$(3.53) \quad g_{i,m_i-1-j} = \alpha_{i,m_i-1-j} \frac{u_{ij}}{u_{i0}^2} + \text{terms involving only } u_{ik}; \quad k < j.$$

Here $\alpha_{i,m_i-1-j} \neq 0$. Define $u^L = u^I u_{i0}^2 u_{ij}^{-1}$. Note the weight of u^L is $< m_i - 1 - j$, so in H_{DR}^{m-2} we have (compare (3.37))

$$(3.54) \quad u^I \theta = (u^I - \alpha_{i,m_i-1-j}^{-1} (\frac{\partial}{\partial u_{i,m_i-1-j}} + g_{i,m_i-1-j}) u^L) \theta = \\ (u^I - \alpha_{i,m_i-1-j}^{-1} g_{i,m_i-1-j} u^L) \theta = \sigma_I \theta$$

where σ_I is a sum of terms of weights $< |I|$ and terms of weight $|I|$ only involving $u_{i0}, \dots, u_{i,j-1}$. Note that $b^I = \sigma_I(b)$ because $g_{i,m_i-1-j}(b) = 0$. In this way we reduce to the case $u^I = u_{i0}^p$. Our assumption on the weight implies $m_i \geq 2$, so

$$(3.55) \quad (\frac{\partial}{\partial u_{i,m_i-1}} + g_{i,m_i-1}) u_{i0}^p = g_{i,m_i-1} u_{i0}^p.$$

Together with (3.35) and $g_{ij}(b) = 0$, this enables us to reduce to $p = 0$.

Suppose now the weight of I is $m_i - 1$. If $m_i \geq 2$ we can use the above argument, except in the case $u^I = u_{ij}u_{i,m_i-1-j}u_{i0}^{-2}$. Here there are two subcases. If $j \neq m_i - 1 - j$, the α_{i,m_i-1-j} in (3.54) is a_I in the expansion $F = \sum_I a_I u^I$, so

$$(3.56) \quad u^I da_I \wedge \theta = (b^I da_I + \frac{da_I}{a_I}) \wedge \theta.$$

This completes the proof in this case because the connections $b^I da_i$ and $b^I da_I + d \log(a_I)$ are isomorphic. If, on the other hand, m_i is odd and $j = \frac{m_i-1}{2}$, the monomial $u^I = u_{ij}^2 u_{i0}^{-2}$ and dF contains the term $2a_I u_{ij} du_{ij}$. Thus, from (3.53) we conclude $\alpha_{i, \frac{m_i-1}{2}} = 2a_I$. The lefthand identity in (3.54) yields in this case

$$(3.57) \quad u^I da_I \wedge \theta = (b^I da_I + \frac{1}{2} d \log a_I) \wedge \theta.$$

In the statement of the lemma, we take κ to be the product of the corresponding a_I .

Suppose finally $m_i = 1$. In this case $\frac{\partial F}{\partial u_{i0}} = 0$, so the corresponding $a_I = 0$ and by (3.52) this term contributes nothing to $\Psi|_{\{b\}}$. Similarly, by (3.50) there is no contribution to $\Psi \wedge \theta$. \square

We give two interpretations of the 2-torsion term $\frac{1}{2} d \log(\kappa)$ occurring in the previous lemma.

Definition 3.13. *Let σ be a closed 1-form relative to K on*

$$\text{Spec}(K[[t_1, \dots, t_N]]).$$

Assume $\sigma(0) = 0 \in \mathfrak{m}/\mathfrak{m}^2$. Write $\sigma = dh$ with $h(0) = 0$, so $h = h_2 + h_3 + \dots$ with h_i homogeneous of degree i . If h_2 is nondegenerate, we may define $\text{disc}(\sigma) = \text{discriminant}(h_2) \in K^\times/K^{\times 2}$. This is well-defined independent of the choice of parameters.

Theorem 3.14. *The Gauß-Manin connection on $H_{DR/K}^{m-2}(B, (\mathcal{O}, d + \mathcal{E}))$ is isomorphic to*

$$(d + \Psi)|_{\{b\}} + \frac{1}{2} d \log(\text{disc}(\mathcal{E}|_{\widehat{\mathcal{O}}_{B,b}})).$$

(In particular, the quadratic term in $h = \int \mathcal{E}|_{\widehat{\mathcal{O}}_{B,b}}$ is non-degenerate.)

Proof. First we collect some facts about $\mathcal{E} = \sum_{i,j} \lambda_{ij} \tau_{ij} = \sum g_{ij} du_{ij}$. We have $u_{1,0} = 1$ and $v_{i,0} = u_{i,0}^{-1}$. It follows from (3.33) of the paper that u_{1,m_1-1} does not appear in the expression for \mathcal{E} and du_{1,m_1-1} only appears with constant coefficient. Restricting to $B : u_{1,m_1-1} = -\sum_{i \geq 2} u_{i,m_i-1}$ thus has the effect of suppressing the term in du_{1,m_1-1} and changing the coefficients g_{i,m_i-1} by a constant for $i \geq 2$. Expressed in this way, it follows that the coefficient of du_{ij}

in $\mathcal{E}|_B$ involves only monomials in u_{ip} for the same i . Giving u_{ij} and du_{ij} both weight j , the terms in $g_{ij}du_{ij}$ all have weight $\leq m_i - 1$. It follows from formulas (3.34)-(3.36) that, writing $U_{ij} = u_{ij} - b_{ij}$ so $U_{ij}(b) = 0$, we may write

$$\mathcal{E}|_B = \sum G_{ij}(U)dU_{ij}.$$

Here $G_{ij}(0) = 0$, giving U_{ij} and dU_{ij} weights j , all terms with first index i have weights $\leq m_i - 1$. All terms of the form

$$U_{ij}dU_{i,m_i-1-j}, \quad 0 \leq j \leq m_i - 1, \quad i \geq 2 \quad (\text{resp. } i = 1, \quad 1 \leq j \leq m_1 - 2)$$

occur with nonzero coefficient. Notice that replacing u_{ij} with $U_{ij} + b_{ij}$ introduces monomials of lower degree, but these have weight $< m_i - 1$.

It follows that $\text{disc}(\mathcal{E}|_{\hat{\mathcal{O}}_{B,b}})$ is the determinant of a matrix

$$M = \begin{pmatrix} M_1 & 0 & \dots & \dots & 0 \\ 0 & M_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & M_r \end{pmatrix}$$

where M_i is symmetric, $m_i \times m_i$ (resp. $(m_1 - 2) \times (m_1 - 2)$), and has the shape

$$\begin{pmatrix} \dots & \dots & \dots & \bullet \\ \dots & \dots & \bullet & 0 \\ \dots & \dots & \bullet & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bullet & 0 & \dots & \dots & 0 \end{pmatrix}$$

with the entries \bullet non-zero.

Mod squares, $\det(M_i)$ is 1 if m_i is even, and is given by

$$\frac{1}{2} \cdot \text{coefficient of } (U_{i, \frac{m_i-1}{2}} dU_{i, \frac{m_i-1}{2}})$$

if m_i is odd. Writing

$$(3.58) \quad \Psi|_B = \sum a_i \frac{du_{i0}}{u_{i0}} + dF$$

as just above (3.49) with $F = \sum a_I u^I$ we find

$$\frac{1}{2} d \log \text{disc}(\mathcal{E}|_{\hat{\mathcal{O}}_{B,b}}) = \frac{1}{2} \sum \frac{da_I}{a_I} = \frac{1}{2} d \log(\kappa).$$

(The sum on the right is over all I such that $u^I = (u_{i, \frac{m_i-1}{2}})^2$.)

□

Another interpretation of the 2-torsion is the following. As in (2.25) for s_i a local parameter at $c_i \in \mathcal{D}$ and t_i another copy of s_i (so $s_i - t_i$ is a local defining equation for the diagonal in $(C \times C)$), the pullback of u_{ip} to $K((s_i))$ is the coefficient $s_i^{-(p+1)}$ of t_i^p in $(s_i - t_i)^{-1}$. It follows from (3.30) that

$$(3.59) \quad \tau_{ij} \text{ pulls back to } \frac{-ds_i}{s_i^{j+1}}.$$

Write the polar part of the connection at $s_i = 0$ in the form $(g_0 + g_1 s_i + \dots) \frac{ds_i}{s_i^{m_i}}$. Since $\mathcal{E} = \sum_{i,j} \lambda_{ij} \tau_{ij}$ pulls back to this connection form, we get $g_0 = -\lambda_{i,m_i-1}$. On the other hand, again from (3.30) the coefficient of $u_{i,0}^{-2} u_{i,m_i-1} du_{i,m_i-1}$ in \mathcal{E} is $-\lambda_{i,m_i-1}$ if m_i is odd. This coefficient is the contribution to $\text{disc}(\mathcal{E}|_{\mathcal{O}_{B,b}})$ from the point $c_i \in \mathcal{D}$, so we conclude

Theorem 3.15. *Write the relative connection at $c_i \in \mathcal{D}$ in the form $(g_{i,0} + g_{i,1}s_i + \dots) \frac{ds_i}{s_i^{m_i}}$. Then the Gauß-Manin connection on $H_{DR/K}^{m-2}(B, (\mathcal{O}, d + \mathcal{E}))$ is isomorphic to*

$$(d + \Psi)|_{\{b\}} + \sum_i \frac{m_i}{2} d \log(g_{i,0}(0)).$$

Definition 3.16. *With notation as above, write*

$$\tau(L) = \sum_i \frac{m_i}{2} d \log(g_{i,0}(0)).$$

To summarize, we have proven

Theorem 3.17. *Let C/K be a complete smooth curve of genus g over a field $K \supset k$. Let $\nabla_{/K} : L \rightarrow L \otimes \Omega_{C/K}^1(\mathcal{D})$ be a connection, such that*

$$(L \rightarrow L \otimes \Omega_{C/K}^1(\mathcal{D})) \rightarrow (j_* L|(C - \mathcal{D}) \rightarrow j_* L \otimes \Omega_{C/K}^1|(C - \mathcal{D}))$$

is a quasiisomorphism. This implies that the divisor \mathcal{D} is minimal such that $\nabla|_{C-\mathcal{D}}$ extends with values in $\Omega_{C/K}^1(\mathcal{D})$ (see Sect. 4, (4.2)). We also assume that L has degree 0, and that the connection on $L|_{C-\mathcal{D}}$ lifts to an integrable, absolute (i.e. / k) connection $\tilde{\nabla}$. Then

$$(3.60) \quad \nabla|_{\mathcal{D}} : L|_{\mathcal{D}} \rightarrow L \otimes \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$$

is an $\mathcal{O}_{\mathcal{D}}$ -linear isomorphism and determines a trivialization of

$$\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$$

Write $J_{\mathcal{D}}$ for the generalized jacobian and $J_{\mathcal{D},N}$ for the torseur of divisors of degree $N := 2g - 2 + \deg \mathcal{D}$. The above trivialization corresponds to a K -point $b \in J_{\mathcal{D},N}$. Write $\pi_N : (C - \mathcal{D})^N \rightarrow J_{\mathcal{D},N}$ for the natural map, and let

$(L_N, \tilde{\nabla}_N)$ be the evident bundle and absolute connection on $(C - D)^N$. Then there exists a unique invariant, absolute connection (\mathcal{L}, Φ) on $J_{\mathcal{D}, N}$ such that $\pi_N^*(\mathcal{L}, \Phi) = (L_N, \tilde{\nabla}_N)$. Moreover, we have

$$(3.61) \quad \left(\det(H_{DR}^*(C - D, (L, \nabla)), \nabla_{GM}) \right)^{-1} \cong (\mathcal{L}, \Phi)|_{\{b\}} + \tau(L)$$

where $\tau(L)$ is as in Definition 3.16.

Remark 3.18. 2-torsion also occurs in the determinant of de Rham cohomology for the trivial connection [2]. By virtue of the following lemma this can only happen when the variety has even dimension.

Lemma 3.19. *Let X/K be a smooth projective variety of odd dimension $n = 2m + 1$ over a function field in characteristic 0. Then the Gauß-Manin determinant*

$$\det(H_{DR}(X/K), d)$$

is trivial in $\Omega_K^1/d \log K^*$.

Proof. The strong Lefschetz theorem identifies the determinant connections on H_{DR}^p and H_{DR}^{2n-p} so we need only consider the connection on $\det H_{DR}^n$. As well known, the Poincaré duality morphism

$$\varphi : H_{DR}^n(X/K) \otimes H_{DR}^n(X/K) \rightarrow H_{DR}^{2n}(X/K) = K$$

is compatible with the Gauß-Manin connection, which is trivial on $H_{DR}^{2n}(X/K) = K$. On the other hand, it is alternating, thus its determinant

$$\det(\varphi) : \det(H_{DR}^n(X/K)) \otimes \det(H_{DR}^n(X/K)) \rightarrow K$$

fulfills

$$\det(\varphi)(e \otimes e) = p^2 \cdot 1$$

where $p \in K^*$ is the Pfaffian of the determinant of φ , written in the basis e . Thus if $\nabla(e) = \alpha \otimes e$, one has

$$\det(\varphi)(\nabla(e \otimes e)) = 2\alpha p^2 \cdot 1 = 2pd(p) \cdot 1.$$

Thus $\alpha = d \log p$ and the determinant of the Gauß-Manin connection is trivial. \square

4. Product and trace

In this section, we introduce a product which is reminiscent of Deligne's product explained in [5].

We keep the notations of Sects. 2 and 3 for C/K , (L, ∇) , $j : U = C - D \rightarrow C$, $\mathcal{D} = \sum m_i c_i$, and $\mathcal{D}' = \mathcal{D} - D$. Further,

$$(4.1) \quad \nabla : L \rightarrow L \otimes \Omega_U^1$$

is an absolute connection with vertical curvature $\nabla^2(L) \subset L \otimes \Omega_K^2 \otimes K(X)$. Let $\nabla_{/K} : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})$ be an extension of $(L, \nabla_{/K})$ such that

$$(4.2) \quad \{\mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})\} \rightarrow \{j_*L \rightarrow j_*L \otimes \Omega_{U/K}^1\}$$

is a quasiisomorphism. We assume $\nabla_{/K}$ has a pole at every $c_i \in D$. Note this implies that $\nabla_{/K}$ does not factor through $\Omega_{C/K}^1(\mathcal{D} - c_i)$ for any i . Indeed, by assumption, the complex

$$(4.3) \quad j_*L/\mathcal{L} \rightarrow (j_*L/\mathcal{L}) \otimes \Omega_{C/K}^1(\mathcal{D})$$

is acyclic. Take e a local basis of \mathcal{L} at c_i and z a local parameter, and suppose the connection can be written locally as $\nabla_{/K}e = a(z)dz/z^{m-1}e$ with $m = m_i$. Then $\nabla_{/K}(z^{-1}e) = (a(z)dz/z^m - z^{-2}dz)e$. The assumption that $\nabla_{/K}$ does have a pole at c_i implies that $m \geq 2$, so $z^{-1}e$ would represent a nontrivial element in H^0 of the complex (4.3), a contradiction.

By Lemma 3.1 we know that the verticality condition implies that the absolute connection extends as

$$(4.4) \quad \nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_C^1\langle D \rangle(\mathcal{D}').$$

From now on, we fix such a (\mathcal{L}, ∇) .

As we have seen, the map $\mathcal{L}|_{\mathcal{D}} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$ is function linear. Since the connection does not factor through lower order poles, this gives a trivialization (denoted $\text{triv}(\nabla)$) of $\Omega_{C/K}^1(\mathcal{D})|_{\mathcal{D}}$. We have

$$(4.5) \quad (c_1(\Omega_{C/K}^1(\mathcal{D})), \text{triv}(\nabla)) \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_{\mathcal{D}}^*) \\ (\mathcal{L}, \nabla) \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \Omega_C^1\langle D \rangle(\mathcal{D}')).$$

The aim of this section is to define a product

$$(4.6) \quad \cup : \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_{\mathcal{D}}^*) \times \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \Omega_C^1\langle D \rangle(\mathcal{D}')) \\ \rightarrow \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2)$$

Here \mathcal{K}_2 is the Milnor sheaf associated to K_2 , and the map $\mathcal{K}_2 \rightarrow \Omega_C^2$ is the $d \log$ map $\{a, b\} \mapsto \frac{da}{a} \wedge \frac{db}{b}$. For a more detailed study of characteristic classes

for connections defined in the hypercohomology of such complexes, the reader is referred to [7]. In addition, we will define a trace

$$(4.7) \quad \text{Tr} : \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2) \rightarrow \Omega_K^1/d \log K^*$$

We write

$$(4.8) \quad A \cdot B := \text{Tr}(A \cup B)$$

so for example

$$(4.9) \quad (c_1(\Omega_{C/K}^1(D)), \text{triv}(\nabla)) \cdot (\mathcal{L}, \nabla) := \\ \text{Tr}((c_1(\Omega_{C/K}^1(D)), \text{triv}(\nabla)) \cup (\mathcal{L}, \nabla))$$

Let $\mathcal{O}_{C,D}^* = \text{Ker}(\mathcal{O}_C^* \rightarrow \mathcal{O}_D^*)$. Then

Lemma 4.1. $d \log \mathcal{O}_{C,D}^* \wedge \Omega_C^1(D)(D') \subset \Omega_C^2$

Proof. Since $\mathcal{O}_{C,D}^* \subset 1 + \mathcal{I}_D$, where \mathcal{I}_D is the ideal sheaf of D ,

$$d \log \mathcal{O}_{C,D}^* \subset \mathcal{O}_C d\mathcal{I}_D \subset \Omega_C^1(*D).$$

Also one has $d\mathcal{I}_D \subset \mathcal{I}_D \otimes_{\mathcal{O}_C} \Omega_C^1(D)$. Thus

$$d \log \mathcal{O}_{C,D}^* \wedge \Omega_C^1(D)(D') \subset \mathcal{I}_D \otimes_{\mathcal{O}_C} \Omega_C^2(D) \subset \Omega_C^2.$$

□

We define \cup by

$$(4.10) \quad \begin{aligned} \mathcal{O}_{C,D}^* \cup \mathcal{O}_C^* &\rightarrow \mathcal{K}_2 \\ (\lambda, c) &\mapsto \{\lambda, c\} \\ \mathcal{O}_{C,D}^* \cup \Omega_C^1(D)(D') &\rightarrow \Omega_C^2 \\ (\lambda, \omega) &\mapsto d \log \lambda \wedge \omega. \end{aligned}$$

Concretely, we can write the product in terms of Čech cocycles. Here \mathcal{C}^i refers to Čech cochains, δ is the Čech coboundary, and d is a boundary in the complex:

$$\begin{aligned} (\lambda_{ij}, \mu_i) &\in (\mathcal{C}^1(\mathcal{O}_C^*) \times \mathcal{C}^0(\mathcal{O}_D^*))_{d-\delta} \\ (c_{ij}, \omega_i) &\in (\mathcal{C}^1(\mathcal{O}_C^*) \times \mathcal{C}^0(\Omega_C^1(D)(D')))_{d-\delta} \end{aligned}$$

one has

$$(4.11) \quad \begin{aligned} (\lambda, \mu) \cup (c, \omega) &= (\{\lambda_{ij}, c_{jk}\}, d \log \lambda_{ij} \wedge \omega_j, -d \log \tilde{\mu}_i \wedge \omega_i) \\ &\in (\mathcal{C}^2(\mathcal{K}_2) \times \mathcal{C}^1(\Omega_C^2(D)(D'))) \times \mathcal{C}^0(\Omega_C^2(D)(D')/\Omega_C^2)_{d+\delta} \end{aligned}$$

where $\tilde{\mu}_i \in \mathcal{C}^0(\mathcal{O}_C^*)$ is a local lifting of μ_i . Note we have replaced the complex $\mathcal{K}_2 \rightarrow \Omega_C^2$ with the quasiisomorphic complex

$$\mathcal{K}_2 \rightarrow \Omega_C^2(D)(D') \rightarrow \Omega_C^2(D)(D')/\Omega_C^2.$$

Proposition 4.2. *The product \cup extends to*

$$\begin{aligned} \cup : \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*) \times \mathbb{H}^1(C, j_*\mathcal{O}_U^* \rightarrow \Omega_C^1\langle D \rangle(\mathcal{D}')) \\ \rightarrow \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2). \end{aligned}$$

Proof. The map

$$\mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \Omega_C^1\langle D \rangle(\mathcal{D}')) \rightarrow \mathbb{H}^1(C, j_*\mathcal{O}_U^* \rightarrow \Omega_C^1\langle D \rangle(\mathcal{D}'))$$

is surjective, and its kernel is the \mathbb{Z} -module generated by $(\mathcal{O}(D_i), d_i)$ where d_i is the connection with logarithmic poles along D_i with residue -1 . Let z_1 be a local coordinate around c_1 . Let U_i be a Cech covering of C , with $c_1 \in U_1 \subset V_1$, and $c_1 \notin U_i, i \neq 1$. Assume c_1 is the only zero or pole of z_1 on U_1 . Let

$$(\lambda, \mu) \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*)$$

be a Cech representative of a class in $\mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*)$. Then (c_{ij}, ω_i) with $c_{1j} = z_1^{-1}, c_{ij} = 1$ for $i \neq 1, \omega_1 = -d \log z_1, \omega_i = 0$ for $i \neq 1$ is a Cech representative of $(\mathcal{O}(D_1), d_1)$. Thus considering $Z \in \mathcal{C}^0(\mathcal{O}_C[z_1^{-1}]^*)$ with $Z_1 = z_1$ and $Z_i = 1$ for $i \neq 1$, the cocycle of (4.7) is just the coboundary

$$(d - \delta)(\{\lambda_{ij}, Z_j\}, d \log \tilde{\mu}_i \wedge d \log Z_i) \in (d - \delta)(\mathcal{C}^1(\mathcal{K}_2) \times \mathcal{C}^0(\Omega_C^2\langle D \rangle(\mathcal{D}'))).$$

(Note Z_j is invertible on U_{ij} for $i \neq j$ so the K_2 -cochain is defined.) \square

Now we define the trace. We have (with standard K -theoretic notation, [1])

$$(4.12) \quad \begin{aligned} H^2(C, \mathcal{K}_2) &= 0 \\ \text{Nm} : H^1(C, \mathcal{K}_2) &= \{\oplus_{x \in C^{(0)}} \kappa(x)^*\} / \text{Tame}(K_2(K(C))) \rightarrow K^* \\ &\sum_x \varphi_x \mapsto \prod_x \text{Nm}(\varphi_x) \end{aligned}$$

and of course $H^1(C, \Omega_C^2) = \Omega_K^1 \otimes H^1(C, \Omega_{C/K}^1) = \Omega_K^1$. This defines

$$(4.13) \quad \text{Tr} : \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2) = H^1(C, \Omega_C^2) / H^1(C, \mathcal{K}_2) \rightarrow \Omega_K^1 / d \log K^*.$$

Lemma 4.3. *The trace*

$$\begin{aligned} \text{Tr} : \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2) &= \mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2\langle D \rangle(\mathcal{D}')) \rightarrow \Omega_C^2\langle D \rangle(\mathcal{D}') / \Omega_C^2 \\ &\rightarrow \Omega_K^1 / d \log K^* \end{aligned}$$

factors through

$$(4.14) \quad \begin{aligned} \mathbb{H}^2\left(C, \mathcal{K}_2 \rightarrow \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D}) \rightarrow \Omega_K^1 \otimes (\Omega_{C/K}^1(\mathcal{D}) / \Omega_{C/K}^1)\right) \\ \cong \Omega_K^1 / d \log K^* \\ \cong \Omega_K^1 \otimes_K H^0(\mathcal{D}, \omega_{\mathcal{D}/K}) / \mathbb{H}^1(C, \mathcal{K}_2 \rightarrow \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D})) \end{aligned}$$

where $\omega_{\mathcal{D}/K}$ is the relative dualizing sheaf of the scheme \mathcal{D} , containing $K \cong \omega_{\mathcal{D}/K}$.

Proof. Note that

$$\Omega_K^1 \otimes_K \Omega_{C/K}^1(\mathcal{D}) \cong \Omega_C^2 \langle \mathcal{D} \rangle (\mathcal{D}') / (\Omega_C^2 \otimes \mathcal{O}_C(\mathcal{D}'))$$

so the complex in (4.14) is indeed a quotient. From the diagram

$$\begin{array}{ccccccc} \mathcal{K}_2 & = & \mathcal{K}_2 & & & & \\ \downarrow & & \downarrow & & & & \\ 0 \rightarrow \Omega_K^1 \otimes \Omega_{C/K}^1 & \rightarrow & \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D}) & \rightarrow & \Omega_K^1 \otimes (\Omega_{C/K}^1(\mathcal{D}) / \Omega_{C/K}^1) & \rightarrow & 0 \end{array}$$

one deduces that the left hand side of (4.14) is isomorphic to

$$\mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_K^1 \otimes \Omega_{C/K}^1) \cong \operatorname{coker}(H^1(C, \mathcal{K}_2) \rightarrow \Omega_K^1 \otimes H^1(C, \Omega_{C/K}^1)).$$

The right hand side here is identified under the norm with $\Omega_K^1 / d \log K^*$, which proves the second equality. The third one comes from the map

$$\Omega_K^1 \otimes \omega_{\mathcal{D}/K}[-2] \rightarrow \{\mathcal{K}_2 \rightarrow \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D}) \rightarrow \Omega_K^1 \otimes \omega_{\mathcal{D}/K}\}$$

and the vanishing of $\mathbb{H}^2(\mathcal{K}_2 \otimes \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D}))$. Note that this cumbersome way of writing this cohomology allows to write local contribution of a class in this cohomology group. \square

The first main result of this section is the following

Theorem 4.4. *Let (\mathcal{L}, ∇) and (\mathcal{L}', ∇') be two extensions of the vertical connection (L, ∇) on U as above satisfying the quasiisomorphism condition (4.2). Then, with notation as in (4.10),*

$$((c_1(\Omega_{C/K}^1), \operatorname{triv} \nabla) \cdot (\mathcal{L}, \nabla)) = ((c_1(\Omega_{C/K}^1), \operatorname{triv} \nabla') \cdot (\mathcal{L}', \nabla')).$$

Proof. The quasiisomorphism condition is local about each point of D , so we may assume our line bundles are $\mathcal{L}(vc) \subset \mathcal{L}$ for some $v < 0$ and $c \in D$.

Choose local coordinates z_i near c_i and a Čech covering U_i of C such that $c_i \in U_i$, $z_i \in \mathcal{O}^*(U_i - c_i)$, $c_i \notin U_j$ for $i \neq j$. Let us denote by

$$(c_1(\Omega_{C/K}^1), z_i) \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*)$$

the class defined by the local trivialization

$$\frac{dz_i}{z_i^{m_i}} : \mathcal{O}_{m_i c_i} \rightarrow \Omega_{C/K}^1(\mathcal{D}) \otimes \mathcal{O}_{m_i c_i}.$$

Let $(\mathcal{L}, \nabla) = (c_{ij}, \omega_i)$. Then $\omega_i = a_i \frac{dz_i}{z_i^{m_i}} + \frac{b_i}{z_i^{m_i-1}}$, with $a_i \in \mathcal{O}_C$ such that $a_i|_{m_i c_i} \in \mathcal{O}_{m_i c_i}^*$ and $b_i \in \Omega_K^1 \otimes \mathcal{O}_C$. Suppose $c = c_i$. We drop the index i for convenience. One has

$$(4.15) \quad ((c_1(\Omega_{C/K}^1), \operatorname{triv} \nabla) \cdot (\mathcal{L}, \nabla)) =$$

$$(c_1(\Omega_{C/K}^1), z) \cdot (\mathcal{L}, \nabla) + \left(0, 0, -d \log(a) \wedge \left(a \frac{dz}{z^m} + \frac{b}{z^{m-1}}\right)\right).$$

where the last term is a cocycle as in (4.12) or the quotient complex (4.17). For $(\mathcal{L}(vc), \nabla(vc))$ one replaces a by $a - vz^{m-1}$, leaving b and m unchanged.

By Proposition 4.2, one has

$$(c_1(\Omega_{C/K}^1), z) \cdot (\mathcal{L}, \nabla) = (c_1(\Omega_{C/K}^1), z) \cdot (\mathcal{L}(vc), \nabla(vc))$$

Thus

$$\begin{aligned} & ((c_1(\Omega_{C/K}^1), \text{triv } \nabla) \cdot (\mathcal{L}, \nabla)) - ((c_1(\Omega_{C/K}^1), \text{triv } \nabla') \cdot (\mathcal{L}', \nabla')) \\ &= \left(0, 0, (d(a - vz^{m-1}) - d(a)) \wedge \frac{dz}{z^m} + d \log\left(\frac{a - vz^{m-1}}{a}\right) \wedge \frac{b}{z^{m-1}}\right) \\ &= \left(0, 0, \frac{v(m-1)d \log z \wedge \frac{b}{a}}{(1 - v \frac{z^{m-1}}{a})}\right). \end{aligned}$$

The nontrivial part in the last expression is computed in

$$H^0(\Omega_C^2 \langle c \rangle / \Omega_C^2) \cong \Omega_c^1.$$

Computing using the residue at c we find the above difference is

$$\left(0, 0, v(m-1) \frac{b(c)}{a(c)}\right)$$

The verticality condition for the curvature reads

$$da \wedge \frac{dz}{z^m} + \frac{db}{z^{m-1}} - (m-1) \frac{dz}{z^m} \wedge b = 0 \in \Omega_K^1 \otimes \Omega_{C/K}^1(\mathcal{D}).$$

In particular, $(da - (m-1)b)|_c = 0$. The difference of the two products is therefore $(0, 0, d \log a^v)$, which vanishes in $\Omega_K^1/d \log K^*$. \square

Remark 4.5. A version of the formula (4.15) in higher rank plays a central role in Sect. 5.

Suppose now

$$(4.16) \quad m_i = 1 \text{ for all } i.$$

In this case, the class $(c_1(\Omega_{C/K}^1(D), z_i) \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_D^*))$ as defined in the proof of Theorem 4.4 does not in fact depend on the choice of the local coordinate z_i . Indeed, the trivialization

$$\mathcal{O}_D \rightarrow (\Omega_{C/K}^1(D)/\Omega_{C/K}^1 = \mathcal{O}_D), 1 \mapsto \frac{dz_i}{z_i}$$

is just the canonical identification given by the residue along c_i . In other words, the class $(c_1(\Omega_{C/K}^1(D), z_i)$ is what is denoted by $(c_1(\Omega_{C/K}^1(D), \text{res}_D)$ in [2], and appears on the right hand side of the Riemann-Roch formula. The second main result of this section is

Theorem 4.6. *Let (\mathcal{L}, ∇) be as above, with $m_i = 1$ for all i . Then*

$$\det \left(H_{DR}^*(U, L), \text{Gauß - Manin connection} \right) = -c_1 \left(\Omega_{C/K}^1(D), \text{triv} \nabla \right) \cdot (\mathcal{L}, \nabla).$$

Proof. Given the main result of [2], and Lemma 3.19, the theorem is of course equivalent to

$$(4.17) \quad c_1(\Omega_{C/K}^1(D), \text{res}_D) \cdot (\mathcal{L}, \nabla) = c_1(\Omega_{C/K}^1(D), \text{triv} \nabla) \cdot (\mathcal{L}, \nabla).$$

Keeping the same notations as in the proof of Theorem 4.4, one has

$$c_1(\Omega_{C/K}^1(D), \text{res}_D) - c_1(\Omega_{C/K}^1(D), \text{triv} \nabla) = (0, a_i)$$

and thus

$$\begin{aligned} (c_1(\Omega_{C/K}^1(D), \text{res}_D) - c_1(\Omega_{C/K}^1(D), \text{triv} \nabla)) \cdot (\mathcal{L}, \nabla) &= \\ (0, 0, -d(a_i) \wedge d \log z_i - d \log a_i \wedge b_i) &= \\ (0, 0, -d(a_i) \wedge d \log z_i). \end{aligned}$$

This lies in $\Omega_K^1 \otimes \omega_{D/K} = \Omega_K^1$ and by Lemma 4.3, its trace factors through $\Omega_K^1 \otimes H^1(C, \Omega_{C/K}^1)$. But the image of $\gamma = \sum_i a_i d \log z_i \in \omega_{D/K}$ in $H^1(C, \Omega_{C/K}^1)$ is the relative Atiyah class $\text{at}_{/K}(\mathcal{L})$ ([6], appendix B), thus the image of $d\gamma = \sum_i d(a_i) \wedge d \log z_i \in \Omega_K^1 \otimes \omega_{D/K}$ in $\Omega_K^1 \otimes H^1(C, \Omega_{C/K}^1)$ is $d(\text{at}(\mathcal{L}))$, where $\text{at}(\mathcal{L}) \in H^1(C, \Omega_C^1)$ is the absolute Atiyah class of \mathcal{L} . Indeed, $d : H^1(C, \Omega_C^1) \rightarrow \Omega_K^1 \otimes H^1(C, \Omega_{C/K}^1)$ factors through $H^1(C, \Omega_{C/K}^1)$ by Hodge theory. On the other hand, if $c_{ij} \in \mathcal{C}^1(\mathcal{O}_C^*)$ is a cocycle for \mathcal{L} , then $d \log c_{ij} \in \mathcal{C}^1(\Omega_C^1)$ is a cocycle for $\text{at}(\mathcal{L})$, and consequently, $d(\text{at}(\mathcal{L})) = 0$. \square

We want to explain briefly a fundamental compatibility satisfied by the pairing (4.8). Let $b = \sum b_i$ be a 0-cycle on C with support disjoint from \mathcal{D} , and let $\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_C^1 \langle D \rangle (\mathcal{D}')$ be an absolute, integrable connection. We can interpret $\nabla|_{b_i} \in \Omega_{K(b_i)}^1/d \log K(b_i)^*$.

Proposition 4.7. *With notation as above, let $[b] \in \mathbb{H}^1(C, \mathcal{O}_C^* \rightarrow \mathcal{O}_{\mathcal{D}}^*)$ be the class of the 0-cycle b . Then*

$$[b] \cdot (\mathcal{L}, \nabla) = \sum_i \text{Tr}_{K(b_i)/K}(\nabla|_{b_i}) \in \Omega_K^1/d \log K^*.$$

Let $(\mathcal{L}_0, \nabla_0)$ be the invariant connection on $J_{\mathbb{D}}$ which pulls back to (\mathcal{L}, ∇) via the cycle map $i : (C - D) \rightarrow J_{\mathcal{D}}$ (Proposition 2.17). Then

$$[b] \cdot (\mathcal{L}, \nabla) = \text{Tr}_{i_0(b)/K}(\mathcal{L}_0, \nabla_0)|_{i_0(b)}.$$

Proof. One reduces easily to the case b is a single K -point. Let U_2 be a Zariski-open set containing D and b . Shrink U_2 if necessary so there exists $z \in H^0(\mathcal{O}_{U_2})$ with $z|_{\mathcal{D}} = 1$ and $(z) = b$. Let $U_1 = C - \{b\} - \mathcal{D}$ so $C = U_1 \cup U_2$. Shrinking the U_i if necessary, we can assume $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$, so (\mathcal{L}, ∇) is represented by some cocycle $(\mu_{12}, \omega_1, \omega_2)$. Then

$$\nabla|_b = \omega_2|_b \in \Omega_K^1/d \log K^*$$

On the other hand, by the Definition (4.12), $[b] \cdot (\mathcal{L}, \nabla)$ is represented by the image of the cocycle

$$d \log(z) \wedge \omega_2|_{U_{12}} \in H^1(C, \Omega_C^2) \rightarrow \Omega_K^1 \otimes_K H^1(C, \Omega_{C/K}^1) \cong \Omega_K^1.$$

Write $\omega_2 = \omega_2(b) + z\eta_2$ with η_2 regular on U_2 . Since $d \log(z) \wedge z\eta_2$ extends to U_2 , it is homologous to zero, so

$$[b] \cdot (\mathcal{L}, \nabla) = \omega_2(b)[b] \in \Omega_K^1 \otimes \mathbb{H}^1(C, \Omega_{C/K}^1) \mapsto \omega_2(b) \in \Omega_K^1.$$

Now, since one obviously has

$$\mathrm{Tr}(\nabla|_{b_i}) = \mathrm{Tr}(\nabla_0)|_{i_0(b_i)}$$

and the translation ∇_0 is invariant, the second equality is a direct interpretation of the first one. \square

Now we can formulate and prove a variant of Theorem 3.17.

Theorem 4.8. *Let $(C/K, U/K, (L, \nabla), \mathcal{D})$ be as in (4.1), (4.2), (4.3). Then*

(4.18)

$$\det(H_{DR}^*(U, L)) = -(c_1(\Omega_{C/K}^1(\mathcal{D}), \mathrm{triv}(\nabla)) \cdot (\mathcal{L}, \nabla) \in \Omega_K^1/d \log K^*$$

modulo torsion (see remark 3.18), where $\nabla_{/K} : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})$ is any extension of $(L, \nabla_{/K})$ having poles along all points of \mathcal{D} such that

$$\{\mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_{C/K}^1(\mathcal{D})\} \rightarrow \{j_*L \rightarrow j_*(L \otimes \Omega_{U/K}^1)\}$$

is a quasiisomorphism.

Proof. If all $m_i = 1$ this is just Theorem 4.6. So we assume that $m_1 \geq 2$ in the sequel. Then as in the proof of Theorem 4.4, replacing \mathcal{L} by $\mathcal{L}(-c_1)$ changes a_1 to $(a_1 + z_1^{m_1-1})$ and keeps the rest unchanged. Thus the quotient complex

$$\mathcal{L}/\mathcal{L}(-c_1) \rightarrow \mathcal{L}/\mathcal{L}(-c_1) \otimes \Omega_{C/K}^1(\mathcal{D})$$

is \mathcal{O}_{c_1} -linear and the map is the multiplication by $a_1 \in \mathcal{O}_{c_1}^*$. In particular, $\mathcal{L}(vc_1)$ fulfills (4.2) for all $v \in \mathbb{Z}$ and taking $v = -\deg \mathcal{L}$, we may assume by Theorem

4.4 that $\text{deg } \mathcal{L} = 0$. If $H_{DR}^0(U, L) \neq 0$, then there is a meromorphic section φ of \mathcal{L} verifying the flatness condition

$$d\varphi + \omega\varphi = 0.$$

This implies in particular that ω has at most logarithmic poles along D , which contradicts the condition $m_1 \geq 2$. On the other hand, $H_{DR}^2(U, L) = 0$ for dimension reasons, thus we can apply Theorem 3.17 together with Proposition 4.7 to obtain the result, after we have replaced (\mathcal{L}, ∇) by $(\mathcal{L}, \nabla) \otimes f^* \left((\mathcal{L}, \nabla)|_{c_0}^{-1} \right)$ to trivialize the connection at c_0 and applied the projection formula to this tensor connection. \square

We finish this section with an example. Let $U = \mathbb{A}_K^1$ be the affine line over $\text{Spec}K$, with parameter t , and ∇ be a connection on the trivial bundle. Then up to a twist by a form of the base, ∇ has equation $A = df$, where $f = \sum_{i=1}^{m-1} a_i t^i$, $a_i \in K$, $a_{m-1} \neq 0$. Write $df = d_K f + f' dt$ with $d_K f = \sum_{i=1}^{m-2} da_i t^i$ and $f' = \sum_{i=1}^{m-2} i a_i t^{i-1}$. Let $b_i, i = 1, \dots, (m-2)$ be the zeroes of f' (defined over K after some finite field extension), and let $N_\ell(\underline{b}) = \sum_{i=1}^{m-2} b_i^\ell$ be the Newton classes of the zeroes of f' , which of course are expressable in the a_i already on $\text{Spec}K$. Then the main theorem says

$$\det(GM)^{-1} = \sum_{i=1}^{m-2} da_i N_i(\underline{b}).$$

5. A formula in higher rank

In this section, we want to define a sort of non-commutative product of a higher rank connection with the Chern class of the dualizing sheaf of C with poles. The notations are as in the whole article: C is a curve defined over a function field K over an algebraically closed field k of characteristic 0, U is an open set such that $D = X - U = \sum_i c_i$ consists of K -rational points. Let (\mathcal{E}, ∇) be a rank r -connection on U with vertical curvature (2.58). Let m_i be the multiplicity of the relative connection at the point c_i , that is, the minimal multiplicity such ∇ factors

$$\nabla_{/K} : E \rightarrow E \otimes \Omega_{C/K}^1 \left(\sum_i m_i c_i \right).$$

Lemma 3.1 no longer holds true in the higher rank case. We say that the poles of the global connection *behave well* if

$$(5.1) \quad \nabla : E \rightarrow E \otimes \Omega_C^1 \langle D \rangle (D')$$

where $\mathcal{D} = \sum_i m_i c_i$ and $\mathcal{D}' = \mathcal{D} - D$.

Let $s = \{s_i\}$ be a trivialization of $\omega_{C/K}(\mathcal{D})|_{\mathcal{D}} \cong \omega_{\mathcal{D}/K}$. That is, $s_i \in \omega_{m_i c_i}$ and the map $1 \mapsto s_i$ is an isomorphism $\mathcal{O}_{\mathcal{D}} \cong \omega_{m_i c_i}$. For example, if z_i is a local parameter, one can take $s_i = \frac{dz_i}{z_i^{m_i}}$. We will abuse notation and write s_i also for a lifting of the trivialization to a local section of $\Omega_C^1 \langle \mathcal{D} \rangle (\mathcal{D}')$. The local matrix of the connection has the shape

$$(5.2) \quad A_i = g_i s_i + \frac{\eta_i}{z_i^{m_i-1}}$$

where g_i and η_i are $r \times r$ matrices with coefficients in \mathcal{O}_C and $\Omega_K^1 \otimes \mathcal{O}_C$ respectively. Note that the matrix of functions g_i depends only on the lifting of s_i to a section of $\omega_{C/K}(\mathcal{D})$. (Indeed, the relative connection has matrix $g s_i$.)

We assume

$$(5.3) \quad \begin{aligned} \text{Image}(g_i) &\in M(r \times r, \mathcal{O}_{\mathcal{D}}) \\ &\text{lies in } GL(r, \mathcal{O}_{\mathcal{D}}) \end{aligned}$$

Under this assumption, we define

$$(5.4) \quad \begin{aligned} \{c_1(\omega(\mathcal{D})), \nabla\} &:= \\ c_1(\omega(\mathcal{D}), s_i) \cdot \det(\nabla) - \sum_i \text{res } \text{Tr}(d g_i g_i^{-1} A_i) \\ &\in \Omega_K^1 / d \log K^* \end{aligned}$$

Conjecture 5.1. Assuming (5.1) and (5.3), we have

$$\det H_{DR}^*(U, (\mathcal{E}, \nabla))^{-1} = \{c_1(\omega(\mathcal{D})), \nabla\} \in (\Omega_K^1 / d \log K^*) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

We discuss the assumption (5.3) (see Proposition 5.6) at the end of this section. The assumption (5.1) on the poles behaving well is not very well understood. It reflects a sort of stability in all possible directions for the poles of the global connection.

First, we justify the conjecture by establishing some rather surprising invariance properties for $\{c_1(\omega(\mathcal{D})), \nabla\}$.

Lemma 5.2. Fix an index i and write the connection matrix locally in the form

$$A = g \frac{dz}{z^m} + \frac{\eta}{z^{m-1}}$$

where g is an invertible matrix of functions and η is a matrix with entries from $\Omega_K^1 \otimes \mathcal{O}_C$. Then

1. $\text{res } \text{Tr}(d g g^{-1} A) = \text{res } \text{Tr}(d g g^{-1} \frac{\eta}{z^{m-1}})$.
2. $[\eta, g] z^{1-m}$ has no pole at the point $z = 0$.

Proof. The assumption that the curvature is vertical implies

$$(5.5) \quad dA = dg \frac{dz}{z^m} + d\left(\frac{\eta}{z^{m-1}}\right) \equiv A^2 = [\eta, g] \frac{dz}{z^{2m-1}} \pmod{\Omega_K^2 \otimes \mathcal{O}_C[z^{-1}]}$$

Multiplying through by z^m and contracting against $\frac{\partial}{\partial z}$ we deduce 2.

For 1, we must show $\text{res Tr}(dg \frac{dz}{z^m}) = 0$. From 5.5, using $\text{Tr}[g, \eta] = 0$ we reduce to showing $\text{res Tr}d(\frac{\eta}{z^{m-1}}) = 0$. Since η has entries Ω_K^1 , one has

$$\text{res Tr}d\left(\frac{\eta}{z^{m-1}}\right) = \text{res Tr}d_{C/K}\left(\frac{\eta}{z^{m-1}}\right).$$

And the residue of an exact form is vanishing. \square

Lemma 5.3. $\{c_1(\omega(\mathcal{D})), \nabla\}$ is independent of the choice of the trivializations s_i .

Proof. First we show independence of the choice of lifting of s . As remarked above, g is determined by the local lifting of s to $\omega(\mathcal{D})$, so $\{c_1(\omega(\mathcal{D})), \nabla\}$ depends only on that choice. If s and s' are two such local liftings, with $gs = g's'$, we have

$$dgg^{-1} = dg'g'^{-1} + d \log\left(\frac{s'}{s}\right)I = dg'g'^{-1} + z^m h$$

for some $h \in M_r(\mathcal{O}_C)$. It follows immediately that

$$\text{res Tr}(dgg^{-1}A) = \text{res Tr}(dg'g'^{-1}A)$$

as desired.

Next we show independence of the choice of trivializations themselves. Let f be a rational function on C whose divisor (f) is disjoint from the singular locus of ∇ . It will suffice to show that s and fs as trivializations give rise to the same invariant, i.e.

$$(5.6) \quad c_1(\omega(\mathcal{D}), s) \cdot \det(\nabla) - \sum_i \text{res Tr}(dg_i g_i^{-1} A_i) = \\ c_1(\omega(\mathcal{D}), fs) \cdot \det(\nabla) - \sum_i \text{res Tr}((dg_i g_i^{-1} - dff^{-1}I)A_i)$$

Recall we can calculate $c_1(\omega(\mathcal{D}), s) \cdot \det(\nabla)$ by choosing δ a divisor in the linear series $\omega(\mathcal{D})$ compatible with the rigidification s and then restricting $\nabla|_\delta$ and taking the norm to $\text{Spec}(K)$. Associated to the trivialization fs we may take the divisor $\delta + (f)$. It follows that

$$c_1(\omega(\mathcal{D}), fs) \cdot \det(\nabla) - c_1(\omega(\mathcal{D}), s) \cdot \det(\nabla) = \text{Norm det } \nabla|_{(f)}$$

(To get this relation, one could have taken the formula (4.15) as well). On the other hand, since the formula depends only on the local behavior of f near \mathcal{D} ,

by suitably choosing f , we may assume ∇ is defined by A in a neighborhood of (f) and that $f \equiv 1$ modulo some large power of the maximal ideal at the finite set of points where the connection is not given by A . We can interpret $\text{Tr}(dff^{-1}A) \in \Omega_K^1 \otimes \omega_{k(C)}$, so the sum of the residues over all closed points of C will vanish in Ω_K^1 . Thus

$$\sum_i \text{res Tr}(dff^{-1}A_i) = - \sum_{(f)} \text{res Tr}(dff^{-1}A) = \text{Tr}(A|_{(f)})$$

Since the connection matrix for the determinant connection is the trace of the connection matrix, the contributions to (5.6) cancel. \square

Now consider what happens to the expression

$$(5.7) \quad \{c_1(\omega(\mathcal{D})), \nabla\}$$

under a change of coordinates given by a matrix h of functions. We have

$$(5.8) \quad A \mapsto A' := hAh^{-1} + dh h^{-1} = g' \frac{dz}{z^m} + \frac{\eta'}{z^{m-1}}$$

Note that h is regular, so $dh h^{-1}$ does not contribute to the polar part of the connection, i.e.

$$(5.9) \quad g' = hgh^{-1} + z^m a; \quad \eta' = h\eta h^{-1} + z^{m-1} b$$

with a and b regular.

We compute

$$(5.10) \quad \begin{aligned} dg' g'^{-1} &= d(hgh^{-1})hg^{-1}h^{-1} + ez^m + fz^{m-1}dz = \\ &= dh h^{-1} + hdg g^{-1}h^{-1} - hgh^{-1}dhg^{-1}h^{-1} + ez^m + fz^{m-1}dz \end{aligned}$$

with e and f regular. Thus

$$(5.11) \quad \begin{aligned} \text{res Tr}(dg' g'^{-1} A') &= \text{res Tr}(dg' g'^{-1} h A h^{-1}) = \\ &= \text{res Tr}(h^{-1} dh A) + \text{res Tr}(dgg^{-1} A) - \text{res Tr}(h^{-1} dh g^{-1} A g). \end{aligned}$$

Note

$$(5.12) \quad A - g^{-1} A g = z^{1-m}(\eta - g^{-1} \eta g) = z^{1-m} g^{-1} (g \eta - \eta g)$$

From Lemma 5.2,2, this expression is regular. Plugging into (5.11), we conclude.

Lemma 5.4. $\{c_1(\omega(\mathcal{D})), \nabla\}$ is independent of the choice of basis for the bundle E .

Remark 5.5. 1. The definition of $\{c_1(\omega(\mathcal{D})), \nabla\}$ was inspired by the calculations in Sect. 4 (cf. formula (4.15)) for a rank one connection. The formula

$$(5.13) \quad \det H_{DR}^*(U, (\mathcal{E}, \nabla))^{-1} = \{c_1(\omega(\mathcal{D})), \nabla\}$$

when applied to the rank 1 case, gives back the main theorem of this article.

2. When $m_i = 1$ for all c_i , that is when ∇ has regular singularities, then the argument from Theorem 4.6 (slightly modified in the higher rank case) gives

$$(5.14) \quad \{c_1(\omega(D)), \nabla\} = c_1(\omega(D), \text{res}_D) \cdot \det(\nabla)$$

where, as in theorem 4.6, res_D refers to the natural trivialization coming from the residue.

3. Finally, twisting ∇ by $f^*\alpha$, where $\alpha \in \Omega_K^1$ comes from the base, changes the right hand side of the formula by

$$(2g - 2 + n - \sum_i m_i)r\alpha = -\chi(H_{DR}^*(U, (\mathcal{E}, \nabla/K)))\alpha,$$

as it should. Here $r = \text{rank } E$.

Proposition 5.6. *Let $\mathcal{D} \subset C$ be a divisor on a smooth curve, and let $f \in \mathcal{O}_{C, \mathcal{D}}$ be a local defining equation for \mathcal{D} . Let*

$$\nabla : E \rightarrow E \otimes \omega(\mathcal{D})$$

be a connection, and write

$$g = \nabla|_{\mathcal{D}} : E/E(-\mathcal{D}) \rightarrow (E(\mathcal{D})/E) \otimes \omega$$

Let $j : E - \mathcal{D} \hookrightarrow E$ be the inclusion and consider the connection

$$j_*j^*\nabla : j_*j^*E \rightarrow j_*j^*E \otimes \omega.$$

There is a natural map

$$\iota : \{E \rightarrow E \otimes \omega\} \rightarrow \{j_*j^*\nabla : j_*j^*E \rightarrow j_*j^*E \otimes \omega\}.$$

The map ι is a quasi-isomorphism if and only if for any $n \geq 0$ the natural map

$$g - n \cdot \text{id} \otimes \frac{df}{f} : E(n\mathcal{D})/E((n-1)\mathcal{D}) \rightarrow (E((n+1)\mathcal{D})/E(n\mathcal{D})) \otimes \omega$$

given by $f^{-n}e \mapsto f^{-n}g(e) - nf^{-n-1}e \otimes df$ is a quasi-isomorphism.

In particular, if, every point of \mathcal{D} has multiplicity ≥ 2 , then g is an isomorphism if and only if ι is a quasi-isomorphism.

Proof. The usual exact sequence reduces us to showing the condition is equivalent to

$$j_*j^*E/E \xrightarrow{\cong} (j_*j^*E/E) \otimes \omega(\mathcal{D}).$$

Writing as usual $\mathcal{D} = \mathcal{D}' + D$ where D is the reduced divisor with support equal to the support of \mathcal{D} , we have a commutative square

$$\begin{array}{ccc} E/E(-\mathcal{D}) & \xrightarrow{g-nid \otimes \frac{df}{f}} & (E(\mathcal{D}')/E(-D)) \otimes \omega(D) \\ \cong \downarrow \text{“} \cdot f^{-n} \text{”} & & \cong \downarrow \text{“} \cdot f^{-n} \text{”} \\ E(n\mathcal{D})/E((n-1)\mathcal{D}) & \xrightarrow{g} & (E(n\mathcal{D} + \mathcal{D}')/E((n-1)\mathcal{D} + \mathcal{D}')) \otimes \omega(D) \end{array}$$

The map ι is a quasi-isomorphism if and only if for all $n \geq 0$ the map g on the bottom line of the above square is an isomorphism, and this will hold only if the top line is.

In particular, since $\frac{df}{f}$ has poles of order 1, if all multiplicities are ≥ 2 , then ι is a quasiisomorphism if and only if g is an isomorphism. \square

We close those remarks by a numerical computation for $E = \oplus_1^r \mathcal{O}$ on $U = \mathbb{A}_K^1$, with parameter t . There is only one singular point at ∞ . Let us write

$$A = \sum_{i=0}^{m-1} B_i t^i + \sum_{i=0}^{m-2} C_i t^i dt$$

where the B_i and the C_i are matrices with coefficients in Ω_K^1 respectively K . The assumption 5.3 means that $C_{m-2} \in GL(r, K)$. We consider the cases $m = 2, 3$. For $m = 2$ both sides of the formula are 0, and for $m = 3$ they are equal to

$$(5.15) \quad \text{Tr}(B_0 - B_1 C_0 C_1^{-1} + B_2 C_0 C_1^{-1} C_0 C_1^{-1})$$

Note that $-C_0 C_1^{-1}$ is the zero of the “polynomial” $C_0 + C_1 t$, where t is a “variable” of matrices, and thus the formula could also be written as

$$(5.16) \quad \text{Tr} A|_{\text{zero}(C_0 + C_1 t)}$$

if it made sense. For higher m , the right hand side should be a sort of restriction of ∇ to the “Newton” classes of $\sum_{i=0}^{m-2} C_i t^i$.

In the remaining part of this section, we show that the product $\{c_1(\omega(\mathcal{D})), \nabla\}$ is a particular case of a more general product between higher rank connections and a larger class of trivializations along \mathcal{D} .

We consider the tuples $\{E, L, \nabla, \mathcal{D}, g\}$, where

$$(5.17) \quad \nabla : E \rightarrow E \otimes \Omega_C^1 \langle D \rangle (D')$$

is a connection on a rank r vector bundle E . We denote by $\nabla|_{\mathcal{D}}$ the \mathcal{O}_C -linear map

$$(5.18) \quad \nabla : E \rightarrow E \otimes \left(\Omega_C^1 \langle D \rangle (\mathcal{D}') / \Omega_C^1 \right).$$

Further, L is a rank 1 bundle, g is a trivialization

$$(5.19) \quad g : E|_{\mathcal{D}} \cong E \otimes L|_{\mathcal{D}},$$

which fulfills

$$(5.20) \quad 0 = [g, \nabla|_{\mathcal{D}}] : E|_{\mathcal{D}} \rightarrow E \otimes L|_{\mathcal{D}}$$

By Lemma 5.2, if $L = \omega_C(\mathcal{D})$ and g is the trivialization $E|_{\mathcal{D}} \rightarrow E \otimes (\omega_C(\mathcal{D})/\omega_C)$ arising from the principal part of $\nabla_{C/K}$, then the condition 5.20 is fulfilled.

Let us introduce the cocycles of the tuple $\{E, L, \nabla, \mathcal{D}, g\}$, as in Sect. 4. If E has cocycle $c_{ij} \in GL(r, \mathcal{O}_C)$, L has cocycle $\lambda_{ij} \in \mathcal{O}_C^*$, g has cocycle $\mu_i \in GL(r, \mathcal{O}_{\mathcal{D}})$, and ∇ has cocycle $\omega_i \in M(r \times r, \Omega_C^1 \langle D \rangle (\mathcal{D}'))$ then one has

$$(5.21) \quad dc_{ij}c_{ij}^{-1} = \omega_i - c_{ij}\omega_jc_{ij}^{-1}$$

$$(5.22) \quad \mu_i = c_{ij}\mu_jc_{ij}^{-1}\lambda_{ij}.$$

The commutativity condition 5.20 then reads

$$(5.23) \quad [\mu_i, \omega_i] = 0.$$

Let $\tilde{\mu}_i \in GL(r, \mathcal{O}_C)$ be a local lifting of $\mu_i \in GL(r, \mathcal{O}_{\mathcal{D}})$. Then one has

Theorem 5.7. *The cochain*

$$(5.24) \quad \left\{ \{\lambda_{ij}, \det(c_{jk})\}, d \log \lambda_{ij} \wedge \text{Tr}(\omega_j), \text{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i) \right\}$$

is a cocycle in

$$(5.25) \quad \mathcal{K}_2 \rightarrow \Omega_X^2 \langle D \rangle (\mathcal{D}') \rightarrow \left(\Omega_X^2 \langle D \rangle (\mathcal{D}') / \Omega_X^2 \right)$$

and defines a cohomology class $\{c_1(L), g, \nabla\}$ in

$$(5.26) \quad \mathbb{H}^2 \left(C, \mathcal{K}_2 \rightarrow \Omega_X^2 \langle D \rangle (\mathcal{D}') \rightarrow \left(\Omega_X^2 \langle D \rangle (\mathcal{D}') / \Omega_X^2 \right) \right)$$

Proof. First

$$(5.27) \quad \delta(d \log(\lambda) \wedge \text{Tr}(\omega))(ijk) = \\ d \log(\lambda_{ij}) \wedge \text{Tr}(\omega_j) + d \log(\lambda_{jk}) \wedge \text{Tr}(\omega_k) - d \log(\lambda_{ik}) \wedge \text{Tr}(\omega_k) = \\ d \log(\lambda_{ik}) \wedge \text{Tr}(\omega_j - \omega_k) - d \log(\lambda_{jk}) \wedge \text{Tr}(-\omega_k + \omega_j) = \\ d \log(\lambda_{ij}) \wedge \text{Tr}(\omega_j - \omega_k) = d \log(\lambda_{ij}) \wedge d \log(\det(c_{jk})).$$

Next computing mod the ideal of \mathcal{D} and so ignoring the tilde,

$$(5.28) \quad \delta(\text{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i))(ij) = \text{Tr}(-d\tilde{\mu}_j\tilde{\mu}_j^{-1} \wedge \omega_j + d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i) = \\ \text{Tr}\left((d\lambda_{ij}\lambda_{ij}^{-1} \cdot I + c_{ij}^{-1}dc_{ij} - c_{ij}^{-1}d\mu_i\mu_i^{-1}c_{ij} - c_{ij}^{-1}\mu_idc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij}) \wedge \omega_j + \right. \\ \left. + d\mu_i\mu_i^{-1} \wedge \omega_i\right) = \text{Tr}\left((d\lambda_{ij}\lambda_{ij}^{-1} \cdot I + c_{ij}^{-1}dc_{ij} - c_{ij}^{-1}d\mu_i\mu_i^{-1}c_{ij} \right. \\ \left. - c_{ij}^{-1}\mu_idc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij}) \wedge (c_{ij}^{-1}\omega_ic_{ij} - c_{ij}^{-1}dc_{ij}) + d\mu_i\mu_i^{-1} \wedge \omega_i\right)$$

By our commutation assumption 5.20

$$(5.29) \quad \text{Tr}(c_{ij}^{-1}dc_{ij} \wedge c_{ij}^{-1}\omega_ic_{ij}) = \text{Tr}(c_{ij}^{-1}\mu_idc_{ij}c_{ij}^{-1}\mu_i^{-1}c_{ij} \wedge c_{ij}^{-1}\omega_ic_{ij})$$

Also terms with no poles (i.e. terms not involving ω) can be ignored. We get

$$(5.30) \quad \delta(\text{Tr}(-d\tilde{\mu}_i\tilde{\mu}_i^{-1} \wedge \omega_i))(ij) = d\lambda_{ij}\lambda_{ij}^{-1} \wedge \text{Tr}(\omega_i) = d\lambda_{ij}\lambda_{ij}^{-1} \wedge \text{Tr}(\omega_j)$$

which is what we want. Note the right hand equality holds because we are computing mod forms regular along \mathcal{D} , and $\text{Tr}(\omega_i) \equiv \text{Tr}(\omega_j)$ mod regular forms by the cocycle condition 5.21.

It remains to show that our cocycle 5.24 does not depend on the choice of the cocycles $\{c, \omega, \lambda, \mu\}$.

If λ_{ij} is replaced by $\lambda_{ij}\delta(v)_{ij}$, then the new 5.24 differs from the old one by the cochain

$$\delta\left(\{v_i, \det(c_{ij})\}, d \log v_i \wedge \text{Tr}(\omega_i)\right).$$

If c_{ij} is replaced by $\gamma_i c_{ij} \gamma_j^{-1}$, then ω_i is replaced by $d\gamma_i \gamma_i^{-1} + \gamma_i \omega_i \gamma_i^{-1}$, and μ_i is replaced by $\gamma_i \mu_i \gamma_i^{-1}$. The commutativity relation implies $d\gamma_i \mu_i \gamma_i^{-1} - \gamma_i \mu_i \gamma_i^{-1} d\gamma_i \gamma_i^{-1} = 0$. Then the new 5.24 differs from the old one by

$$(5.31) \quad \left\{ \{\lambda_{ij}, \det(\delta(\gamma))_{jk}\}, d\lambda_{ij} \wedge \text{Tr} d \log(\gamma_i), \right.$$

$$\left. \text{res Tr}\left(\gamma_i^{-1} d\gamma_i \omega_i d\mu_i \mu_i^{-1} \gamma_i^{-1} d\gamma_i - \mu_i \gamma_i^{-1} d\gamma_i \mu_i^{-1} \gamma_i^{-1} d\gamma_i - \mu_i \gamma_i^{-1} d\gamma_i \mu_i^{-1} \omega_i\right) \right\}$$

Using the commutativity relation 5.20 as well, we see that this is this expression is the cochain

$$(5.32) \quad \delta\left(\{\lambda_{ij}, \det \gamma_j\}, 0, 0\right).$$

□

Now we consider the image under the map $f : C \rightarrow \text{Spec}(K)$:

$$(5.33) \quad f_*\{c_1(L), g, \nabla\} \in \Omega_K^1/d \log K^*$$

of $\{c_1(L), g, \nabla\}$. We want to study closedness for forms in the image of this map.

Lemma 5.8. *Let $\{a, b, c\}$ be a cocycle as in (5.24) representing a class in $\mathbb{H}^2(C, \mathcal{K}_2 \rightarrow \Omega_C^2)$, with $db = 0$. Then $df_*\{a, b, c\} = \text{res}_{\mathcal{D}}(dc) \in \Omega_K^2$.*

Proof. Another representative of $\{a, b, c\}$ in

$$\mathbb{H}^1(C, \Omega_C^2 \langle D \rangle (\mathcal{D}') \rightarrow \Omega_C^2 \langle D \rangle (\mathcal{D}')/\Omega_C^2)/d \log(H^1(C, \mathcal{K}_2))$$

is of the shape $\{0, b + d \log \alpha, c\}$, thus its derivative in

$$\mathbb{H}^1(C, \Omega_C^3 \langle D \rangle (\mathcal{D}') \rightarrow \Omega_C^3 \langle D \rangle (\mathcal{D}')/\Omega_C^3)$$

is of the shape $\{0, 0, dc\}$ since $db = 0$. Then one applies the commutativity of the diagramm

$$(5.34) \quad \begin{array}{ccc} \mathbb{H}^1(C, \Omega_C^2 \langle D \rangle (\mathcal{D}') \rightarrow \Omega_C^2 \langle D \rangle (\mathcal{D}')/\Omega_C^2) & \xrightarrow{d} & \mathbb{H}^1(C, \Omega_C^3 \langle D \rangle (\mathcal{D}') \rightarrow \Omega_C^3 \langle D \rangle (\mathcal{D}')/\Omega_C^3) \\ \downarrow & & \downarrow \\ \Omega_K^1 & \xrightarrow{d} & \Omega_K^2 \end{array}$$

□

Theorem 5.9. *If $\nabla^2 = 0$, that is if ∇ is flat, then $f_*\{c_1(L), g, \nabla\} \in \Omega_{K, \text{clsd}}^1/d \log K^*$, that is the image is flat as well. In particular, this is true for $\{c_1(\omega(\mathcal{D})), \nabla\}$, as predicted by conjecture 5.1.*

Proof. Since, under the integrability assumption, one has in particular $d(d \log \lambda_{ij} \wedge \text{Tr} \omega_j) = 0$, one can apply Lemma 5.8. One has to compute

$$(5.35) \quad \begin{aligned} \gamma &= -\text{res Tr } d(d\mu\mu^{-1}\omega) \\ &= -\text{res Tr } (d\mu\mu^{-1})^2\omega + \text{res Tr } d\mu\mu^{-1}d\omega \end{aligned}$$

We omit the indices since we compute only with one index. Note in the calculations which follow μ is regular and ω has poles along \mathcal{D} . We write $a \equiv b$ to indicate that the polar parts of a and b coincide.

The condition 5.20 implies

$$(5.36) \quad d\mu\omega + \mu d\omega \equiv d\omega\mu - \omega d\mu$$

thus

$$(5.37) \quad d\mu\mu^{-1}\omega + \mu d\omega\mu^{-1} \equiv d\omega - \omega d\mu\mu^{-1}$$

which implies

$$(5.38) \quad (d\mu\mu^{-1})^2\omega + d\mu d\omega\mu^{-1} \equiv d\mu\mu^{-1}d\omega - d\mu\mu^{-1}\omega d\mu\mu^{-1}$$

So taking the trace, one obtains

$$(5.39) \quad 2\text{Tr}((d\mu\mu^{-1})^2\omega) \equiv \text{Tr}((d\mu\mu^{-1} - \mu^{-1}d\mu)d\omega)$$

Now, under the integrability assumption

$$(5.40) \quad d\omega = \omega\omega$$

(using (5.23)) one obtains

$$(5.41) \quad 2\text{Tr}((d\mu\mu^{-1})^2\omega) = 0$$

Now we consider the other term

$$(5.42) \quad \gamma' = \text{Tr}(d\mu\mu^{-1}(\omega)^2)$$

Choosing a local basis of the bundle E , we write μ as a matrix M , and ω as matrix Ω . Then the condition 5.20 reads

$$(5.43) \quad M\Omega = \Omega M$$

and one has

$$(5.44) \quad \gamma' = \text{Tr}(dMM^{-1}\Omega^2)$$

The condition 5.43 implies

$$(5.45) \quad \Omega^2 = M^{-1}\Omega^2M$$

thus

$$(5.46) \quad \gamma' = \text{Tr}(dM\Omega^2M^{-1}) = \text{Tr}(M^{-1}dM\Omega^2)$$

On the other hand, differentiating the condition 5.43, one obtains

$$(5.47) \quad \begin{aligned} M^{-1}dM\Omega^2 + d\Omega\Omega &= \\ M^{-1}d\Omega M\Omega - M^{-1}\Omega dM\Omega &= \\ M^{-1}d\Omega\Omega M - \Omega M^{-1}dM\Omega & \end{aligned}$$

So taking the trace, one obtains

$$(5.48) \quad \gamma' = \text{Tr}(M^{-1}dM\Omega^2) = -\text{Tr}(M^{-1}dM\Omega^2) = -\gamma'$$

Thus $\gamma' = 0$.

□

6. Rank 1 irregular connections in arbitrary dimension on projective manifolds

Let X be a smooth, projective variety in characteristic 0, and let $D \hookrightarrow X$ be a normal crossings divisor. Given $m_i \geq 0$, define $C(X, D, \underline{m})$ to be the group of isomorphism classes of line bundles \mathcal{L} on X together with an integrable connection

$$\nabla : \mathcal{L} \rightarrow \mathcal{L} \otimes \Omega_X^1 \langle D \rangle (\underline{D})$$

where $\underline{D} = \sum_i m_i D_i$. Define for $m_i \geq 1$

$$\text{Irreg}(X, D, \underline{m}) := C(X, D, \underline{m}) / C(X, D, \underline{0}).$$

Theorem 6.1. *With notation as above, there is a canonical isomorphism*

$$\text{Irreg}(X, D, \underline{m}) \cong \Gamma(X, \mathcal{O}_X(\underline{D}) / \mathcal{O}_X),$$

where $\mathcal{O}_X(\underline{D}) / \mathcal{O}_X$ is the sheaf of principal parts of degree $\leq m_i$ along D_i .

Lemma 6.2. *Exterior differentiation induces an isomorphism*

$$d : \mathcal{O}_X(\underline{D}) / \mathcal{O}_X \xrightarrow{\cong} \Omega_X^1 \langle D \rangle (\underline{D})_{\text{clsd}} / \Omega_X^1 \langle D \rangle_{\text{clsd}}$$

Proof of lemma. We first check injectivity. Let $x = 0$ be a local defining equation for D . Suppose for some n with $1 \leq n \leq m$ we had

$$d\left(\frac{a}{x^n}\right) = \frac{1}{x^n} \left(da - na \frac{dx}{x} \right) \in \Omega_X^1 \langle D \rangle$$

Multiplying by x^n and taking residue along D , it would follow that $a|_D = 0$, i.e. $\frac{a}{x^n} = \frac{b}{x^{n-1}}$.

To show surjectivity, write a local section of $\Omega^1 \langle D \rangle (nD)_{\text{clsd}}$ (here $1 \leq n \leq m$) in the form

$$(6.1) \quad \omega = \frac{adx}{x^{n+1}} + \frac{B}{x^n}$$

where B does not involve dx . Replacing ω by $\omega + d\left(\frac{a}{nx^n}\right)$, we can assume

$$\omega = \frac{adx + B}{x^n}.$$

Then

$$0 = d\omega = \frac{da \wedge dx + dB - n \frac{dx}{x} \wedge B}{x^n}.$$

Multiplying by x^n and taking residue along D , we see $B|_D = 0$. Since B does not involve dx , it follows that $B = xC$ and ω can be written

$$\omega = \frac{adx}{x^n} + \frac{C}{x^{n-1}}$$

Comparing with (6.1), we have lowered the order of pole by 1. This process continues until ω has log poles. \square

Proof of Theorem 6.1. Using the lemma, we get a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathcal{O}_X^* & = & \mathcal{O}_X^* & \rightarrow & 0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \Omega_X^1 \langle D \rangle_{\text{clsd}} & \rightarrow & \Omega_X^1 \langle D \rangle (\mathcal{D})_{\text{clsd}} & \rightarrow & \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X & \rightarrow & 0 \end{array}$$

We view this as a diagram of complexes written vertically. Using the standard hypercohomological interpretation of line bundles with connection, this yields an exact sequence

$$\begin{aligned} 0 \rightarrow C(X, D, 0) \rightarrow C(X, D, \underline{m}) \rightarrow H^0(X, \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X) \\ \xrightarrow{\delta} \mathbb{H}^2(X, \mathcal{O}_X^* \rightarrow \Omega_X^1 \langle D \rangle_{\text{clsd}}) \end{aligned}$$

We claim the map δ above is zero. In the derived category, δ factors

$$\begin{aligned} \mathcal{O}_X(\mathcal{D})/\mathcal{O}_X \cong \frac{\Omega_X^1 \langle D \rangle (\mathcal{D})_{\text{clsd}}}{\Omega_X^1 \langle D \rangle_{\text{clsd}}} \xrightarrow{\partial} \Omega_X^1 \langle D \rangle_{\text{clsd}}[1] \\ \rightarrow \{\mathcal{O}_X^* \rightarrow \Omega_X^1 \langle D \rangle_{\text{clsd}}\}[2]. \end{aligned}$$

We have a factorization of ∂ :

$$H^0(\mathcal{O}_X(\mathcal{D})/\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_X) \xrightarrow{\tilde{d}} H^1(\Omega_{X,\text{clsd}}^1) \rightarrow H^1(\Omega_X^1 \langle D \rangle_{\text{clsd}}),$$

so it suffices to show the map \tilde{d} is zero. By Hodge theory, the composition

$$H^1(\mathcal{O}_X) \xrightarrow{\tilde{d}} H^1(\Omega_{X,\text{clsd}}^1) \xrightarrow{\iota} \mathbb{H}^1(\Omega_X^1 \rightarrow \Omega_X^2)$$

is zero, and the map ι is injective as the complex $\{\Omega_X^1/\Omega_{X,\text{clsd}}^1 \rightarrow \Omega_X^2\}$ is quasi-isomorphic to the complex $\{0 \rightarrow \Omega_X^2/\Omega_{X,\text{exact}}^2\}$, and in particular starts in degree 1. \square

In his letter dated February 8, 1974 and addressed to J.-P. Serre, P. Deligne develops a geometric method in order to compute the ϵ -factor of a character of the Galois group on a curve C over a finite field, ramified over a divisor D . He expresses the determinant of cohomology as the cohomology on the symmetric product of $(C \setminus D)$, and identifies it with a class on the generalized Jacobian. The proof of our formula for the determinant of Gauß-Manin cohomology in rank 1 uses the whole strength of this geometric idea.

It was Serre's idea and wish to publish Deligne's letter as an appendix, for the reader's convenience. We thank him, first for giving us a copy of the letter, and later for insisting on the publication of the letter. We thank Deligne for accepting. We are grateful to the IHES' staff for texing the manuscript and to Serre for screening through the typed manuscript.

Hélène Esnault and Spencer Bloch

Bures-sur-Yvette, le 8 février 1974

Cher Serre,

Je t'avais dit il y a deux mois que je comprenais un peu mieux les constantes, pour un faisceau de rang un . Voici la lettre promise. Elle consiste de parties assez disjointes :

abcdf: constantes

e: corps de classe géométrique

gh: sommes de Gauss, Jacobi et Grössencharaktere

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a. Position du problème

Soient A un anneau commutatif local fini, d'ordre premier à p , \mathbb{F}_q un corps fini à q éléments de caractéristique p , X_0 une courbe lisse sur \mathbb{F}_q , \mathcal{F}_0 un faisceau localement constant de A -modules libres sur X_0 , $\bar{\mathbb{F}}_q$ une clôture algébrique de \mathbb{F}_q et \mathcal{F}/X déduit de \mathcal{F}_0/X_0 par extension des scalaires de \mathbb{F}_q à $\bar{\mathbb{F}}_q$.

Dans l'équation fonctionnelle de $L(X_0, \mathcal{F}_0)$ apparaît

$$\det(-Ft, H_c^*(X, \mathcal{F})) \quad [\text{abus de notation ; cf. ci-dessous}].$$

Dans certains cas, on sait écrire cette "constante" comme un produit étendu aux points de \bar{X}_0 . On veut de même écrire canoniquement le A -module de rang un $\det H_c^*(X, \mathcal{F})$ comme produit tensoriel, étendu aux points de X , de A -modules de rang un "locaux" (presque tous trivialisés).

Je commets ici un abus de notations : on s'intéresse non pas à $H_c^*(X, \mathcal{F})$ mais à $R\Gamma_c(X, \mathcal{F})$, un complexe fini de Λ -modules libres, bien défini à homotopie près. Si K^* représente $R\Gamma_c(X, \mathcal{F})$, on veut calculer

$$\bigotimes^i (\det K^i)^{(-1)^i}$$

à isomorphisme *unique* près.

b. A -modules stables

1. Soit A un anneau. On définit comme suit la catégorie des A -modules (projectifs) stables

α) *effectifs* : objets : A -modules projectifs

flèches : $\varinjlim_n \text{hom}(M \oplus A^n, N \oplus A^n) / \text{commutateurs de } \text{GL}(N \oplus A^n)$.

On dispose d'un foncteur évident

$\underline{\chi}$: (A -modules projectifs, isomorphismes) \longrightarrow (A -modules stables effectifs),

d'une notion de \oplus (associative et commutative, avec 0) dans les modules stables et $\underline{\chi}$ est compatible à \oplus .

NB : dans (A -modules stables), l'isomorphisme de symétrie $M \oplus M \rightarrow M \oplus M$ n'est en général pas l'identité.

β) *quelconques* : le foncteur $N \mapsto M \oplus N$ est pleinement fidèle. On peut ajouter formellement des inverses aux objets de (A -modules stables effectifs) ; on obtient la catégorie avec \oplus (A -modules stables).

On suppose maintenant A commutatif.

γ) les opérations \otimes et λ^n passent aux A -modules stables.

On a dans $\mathcal{A} = (A\text{-modules stables}, \oplus, \otimes, \lambda)$ les données et compatibilités suivantes (la liste contient ce qu'on utilisera – non tout ce qu'on peut rêver).

- (a) C'est une catégorie où tout morphisme est un isomorphisme. Elle est non vide.
- (b) $\oplus : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ est muni de données de commutativité et d'associativité. On a
 - (α) – $\forall M, N \rightarrow M \oplus N$ est une équivalence $\mathcal{A} \rightarrow \mathcal{A}$
 - (β) – $M \oplus N \rightarrow N \oplus M \rightarrow M \oplus N$ est l'identité. Pentagone. Hexagone.
- (c) $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ est un foncteur biadditif muni de données de commutativité et d'associativité. On a

(β) comme ci-dessus pour \otimes .

Il existe 1 tel que $M \mapsto M \otimes 1$ soit une équivalence $\mathcal{A} \rightarrow \mathcal{A}$ et que $1 \otimes 1$ soit isomorphe à 1.

[D’après “biadditif”, il est donc donné $A \otimes (B \oplus C) \sim A \otimes B \oplus A \otimes C$ et $(A \oplus B) \otimes C \sim \dots$; ces isomorphismes fonctoriels sont compatibles aux isomorphismes de symétrie et d’associativité de \oplus – échangés l’un en l’autre par la symétrie de \otimes – compatibles à l’associativité de \otimes .]

(d) On a $\lambda^n : \mathcal{A} \rightarrow \mathcal{A}$, muni de

$$\lambda^n(A \oplus B) \sim \bigoplus_{i+j=n} \lambda^i(A) \otimes \lambda^j(B),$$

compatible à l’associativité et à la commutativité de \oplus (via celles de \otimes).

2. a) Chaque objet K de $D_{\text{parf}}^b(\Lambda)$ définit $\chi(K)$ dans (Λ -module stable).

b) Pour K dans $DF_{\text{parf}}^b(\Lambda)$ (catégorie dérivée filtrée), on a un isomorphisme naturel

$$\chi(K) \sim \bigoplus_i \chi(Gr^i K)$$

c) $\chi(K \overset{\mathbb{L}}{\otimes} L) \sim \chi(K) \otimes \chi(L)$, compatible à ...

d) Pour K dans $D_{\text{parf}}^{\text{tor} \geq 0}$, (resp. $D^{\text{tor} \leq 0}$), on a

$$\overset{n}{\wedge} \chi(K) \sim \chi(L \overset{n}{\wedge} K)$$

($L \overset{n}{\wedge} =$ foncteur dérivé de $\overset{n}{\wedge}$).

On a $\chi(K) \sim (-1)^n \chi(K[n])$.

e) Rappelons que $L \overset{n}{\wedge} (K[1]) \sim (L \Gamma^n K)[n]$. (Quillen).

[$\Gamma =$ tenseurs symétriques].

3. Je n’utiliserai pas les modules stables, mais leur introduction rend plus naturelles les propriétés de la catégorie quotient suivante.

1^e définition : comme en 1., sauf que, dans le cas effectif, les flèches sont définies par

$$\varinjlim_n \text{hom}(M \oplus A^n, N \oplus A^n) / \text{éléments de déterminant 1 dans } \text{GL}(N \oplus A^n).$$

2^e définition : A -modules gradués inversibles.

Equivalence des définitions : (Pour simplifier, on suppose $\text{Spec}(A)$ connexe.)

α) Pour μ et ν deux A -modules gradués inversibles, posons

$$\mu \underline{\otimes} \nu = \mu \otimes \nu \quad (\text{produit tensoriel gradué}).$$

On définit l'isomorphisme de commutativité par la règle de Koszul, l'isomorphisme d'associativité de façon évidente.

β) On a un foncteur

$$(A\text{-modules projectifs ; } \oplus) \longrightarrow (A\text{-modules gradués})$$

par

$$M \mapsto \det M \text{ en degré } \dim M .$$

La règle de Koszul mange le problème de signe :

$$\begin{array}{ccc} \det(M \oplus N) & \xleftarrow{\det(\text{symétrie})} & \det(N \oplus M) \\ \Downarrow & + & \Downarrow \\ \det(M) \otimes \det(N) & \xleftarrow{\text{Koszul}} & \det(N) \otimes \det(M) ; \end{array}$$

il se factorise par un \oplus -foncteur :

$$\underline{\delta} : (A\text{-module stable, } \oplus) \longrightarrow (A\text{-module gradué inversible, } \underline{\otimes}).$$

On vérifie de plus que \otimes et λ^n passent au quotient.

4. Calcul de $\underline{\otimes}$ et $\underline{\lambda}$ dans (A -module gradué). (On suppose A ne pas être un produit, pour simplifier.)

Pour μ dans A -module gradué, soit $|\mu|$ le module inversible sous-jacent, et $\dim(\mu)$ le degré où se place μ .

a) $\mu \underline{\otimes} \nu = |\mu|^{\otimes n} \otimes |\nu|^{\otimes m}$ en degré nm , $n = \dim \nu$, $m = \dim \mu$

$$\underline{\delta}(M \otimes N) \sim \underline{\delta}(M) \underline{\otimes} \underline{\delta}(N) :$$

$$\begin{aligned} & (e_1 \otimes f_1) \wedge \dots \wedge (e_1 \otimes f_n) \wedge (e_2 \otimes f_1) \wedge \dots \wedge \dots \wedge (e_m \otimes f_n) \\ & \leftrightarrow (e_1 \wedge \dots \wedge e_m)^{\otimes n} \otimes (f_1 \wedge \dots \wedge f_n)^{\otimes m} . \end{aligned}$$

Ceci impose les signes pour les isomorphismes d'associativité, etc. de $\underline{\otimes}$; de même pour λ . On a

$$|\underline{\lambda}^i(\mu)| = |\mu|^{\otimes \binom{m-1}{i-1}} \text{ si } m = \dim \mu .$$

$$\left[\text{On pose } \binom{x}{-1} = 0 \text{ (pour } i = 0) \text{ et en général } \binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!} \right] .$$

Corollaire 1 : si $\dim(\mu) = m > 0$, alors $|\mu| = |\underline{\lambda}^m(\mu)|$.

Corollaire 2 : si $\dim(\mu) = m \geq 2$, alors $\bigoplus_{i=0}^m (-1)^i \underline{\lambda}^i(\mu) = \underline{0}$ canoniquement.

c. Méthode utilisée

Le problème posé est un affaiblissement de celui de déterminer le Λ -module stable $\chi(R\Gamma_c(X, \mathcal{F}))$ (à isom. unique près). Cet affaiblissement nous permet d'utiliser les corollaires 1 et 2 de b.

Si \mathcal{F} est de caractéristique d'Euler-Poincaré $-N$:

$$\text{EP}(R\Gamma_c(X, \mathcal{F})) = -N$$

avec $N \geq 1$, on a

$$\begin{aligned} \det R\Gamma_c(X, \mathcal{F}) &\sim (\det R\Gamma_c(X, \mathcal{F})[1])^{\text{dual}} \sim \det(L \wedge^N (R\Gamma_c(X, \mathcal{F})) [1])^{\text{dual}} \\ &\sim \det(L \Gamma^N (R\Gamma_c(X, \mathcal{F})) [N])^{\text{dual}} \sim (\det L \Gamma^N (R\Gamma_c(X, \mathcal{F})))^{(-1)^{N+1}}. \end{aligned}$$

Par ailleurs, d'après SGA 4 XVII 5.5,

$$L \Gamma^N (R\Gamma_c(X, \mathcal{F})) \sim R\Gamma_c(\text{Sym}^N(X), \Gamma_{\text{ext}}^N(\mathcal{F}))$$

où le faisceau $\Gamma_{\text{ext}}^N(\mathcal{F})$ sur $\text{Sym}^N(X)$ se déduit du faisceau \mathfrak{S}_N -équivariant $\boxtimes_1^N \mathcal{F} = \otimes_1^N \text{pr}_i^* \mathcal{F}$ sur X^N par passage aux invariants. Pour \mathcal{F} de rang un, la fibre de \mathcal{F} en le point de $\text{Sym}^N(X)$ correspondant au diviseur ΣP_i est $\otimes \mathcal{F}_{P_i}$.

Pour \mathcal{F} de rang un, nous saurons utiliser l'isomorphisme déduit de ce qui précède

$$\det R\Gamma_c(X, \mathcal{F}) \sim (\det R\Gamma_c(\text{Sym}^N(X), \Gamma_{\text{ext}}^N(\mathcal{F})))^{(-1)^{N+1}}$$

pour calculer le membre de gauche à l'aide de la relation entre $\text{Sym}^N(X)$ et la jacobienne.

d. Le cas non ramifié

X courbe projective et lisse sur k algébriquement clos – de genre $g \geq 2$ – \mathcal{F} localement constant de rang un sur Λ .

Ici, Euler-Poincaré est $2 - 2g$; il faut calculer

$$\det R\Gamma_c(\text{Sym}^{2g-2}(X), L \Gamma_{\text{ext}}^{2g-2}(\mathcal{F})).$$

Soit π la projection de $\text{Sym}^{2g-2}(X)$ sur la composante de degré $2g - 2$ de $\text{Pic}(X)$

$$\pi : \text{Sym}^{2g-2}(X) \longrightarrow J^{2g-2}.$$

Si \mathcal{L} est un faisceau inversible universel sur $X \times J$, on a

$$\begin{array}{ccc} X \times J^{2g-2} & & \\ \text{pr}_2 \downarrow & \text{Sym}^{2g-2}(X) \sim \mathbb{P}(R^1 \text{pr}_{2*}(\Omega_X^1 \otimes \mathcal{L}^{-1})) & \\ J^{2g-2} & & \end{array}$$

et $R^1 \text{pr}_{2*}(\dots)$ est localement libre, sauf en la classe du diviseur canonique, où sa fibre est de rang un de plus.

Le morphisme π est propre, et ses fibres 1-connexes ; le faisceau localement constant $L \Gamma_{\text{ext}}^{2g-2}(\mathcal{F})$ est donc image inverse d'un faisceau localement constant \mathcal{G} sur J^{2g-2} , unique à isomorphisme unique près. On verra que les fibres de $R^i \pi_* R \Gamma_{\text{ext}}^{2g-2} \mathcal{F} = R^i \pi_* \pi^* \mathcal{G}$ sont des Λ -modules projectifs ; par b. 2) b) on a donc

$$\det R \Gamma_c(\pi^* \mathcal{G}) = \otimes_i \det (R \Gamma_c(R^i \pi_* \pi^* \mathcal{G}))^{(-1)^i}$$

(“suite spectrale de Leray de π ”).

Calcul des $R^i \pi_* \pi^* \mathcal{G}$

On a vu que Sym^{2g-2} est un fibré projectif $\mathbb{P}(\dots)$ sur J^{2g-2} . On dispose donc de $\eta \in H^0(R^2 \pi_* \mathbb{Z}_\ell(1))$, et on vérifie par fibre que

$$R^i \pi_* \mathbb{Z}_\ell = 0 \text{ si } i \text{ est impair}$$

$R^{2i} \pi_* \mathbb{Z}_\ell(i)$ est le faisceau \mathbb{Z}_ℓ sur le fermé de J^{2g-2} où la dimension de la fibre de π est $\geq i$, prolongé par zéro ailleurs (base : η^i).

Dès lors,

$$R^i \pi_* \pi^* \mathcal{G} = \begin{cases} 0 & \text{si } i \text{ est impair} \\ \text{le faisceau localement constant } \mathcal{G}(-\frac{i}{2}) & \text{si } i \text{ est pair,} \\ & \text{et } i \leq 2(g-2) \\ 0 & \text{si } i \geq 2g \end{cases}$$

et $R^{2g-2} \pi_* \pi^* \mathcal{G} = \iota_! \iota^* \mathcal{G}(-g+1)$ pour $\iota : p^L \rightarrow J^{2g-2}$ l'inclusion du point représentatif de la classe canonique.

Le résultat suivant va nous débarrasser des $R^i \pi_*$ pour $i \neq 2g-2$:

Construction 1. Soit S une extension d'un schéma abélien (dim. a) par un tore (dim. t), et soit \mathcal{F} un faisceau localement constant modéré de Λ -modules projectifs sur S . Si $2a+t \geq 2$, on a

$$\det R \Gamma(S, \mathcal{F}) \sim 1 \quad \text{canoniquement.}$$

Soit \tilde{S} le revêtement premier à p universel de S (point base l'origine de S). Puisque $H^i(\tilde{S}, \mathcal{F}) = 0$ ($i > 0$), on a

$$H^*(S, \mathcal{F}) = H^*(\pi, \mathcal{F})$$

où

$\pi = \prod_{\ell \neq p} T_\ell(S)$ est un $\widehat{\mathbb{Z}}_{p'}$ -module libre de rang > 2

F est la fibre de \mathcal{F} en 0.

On a pareil dans la catégorie dérivée des Λ -modules.

Il y a intérêt à oublier π : on dispose de Λ , d'un anneau R sur Λ , isomorphe à $\Lambda[[t_1, \dots, t_n]]$ ($n \geq 2$), et d'une augmentation $\varepsilon : R \rightarrow \Lambda$ (" $t_i \rightarrow 0$ "). On a aussi un R -module F projectif sur Λ , et on s'intéresse à

$$\underline{\delta}_\Lambda(\text{Ext}_R^*(\Lambda, F)).$$

Soit L un R -module libre, muni de $L \xrightarrow{\omega} \text{Ker}(\varepsilon)$ induisant

$$L \otimes \Lambda \xrightarrow{\sim} \text{Ker}(\varepsilon) \otimes_R \Lambda.$$

On en déduit une résolution de Koszul de Λ , et un complexe

$$(\wedge_R^i L^* \otimes_R F)$$

qui calcule $\text{Ext}_R^*(\Lambda, F)$. Choisissons de plus une base $(e_i)_{1 \leq i \leq n}$ de L . On a alors un isomorphisme

$$(1) \quad \underline{\delta}(\text{Ext}_R^*(\Lambda, F)) = \bigoplus (-1)^i \underline{\delta}(\wedge_R^i L^* \otimes_R F) = \left(\sum (-1)^i \binom{n}{i}\right) \underline{\delta}(F) = \underline{0},$$

dépendant a priori de $(L, \omega, \underline{e})$, i.e. de $\omega : R^n \rightarrow \text{Ker}(\varepsilon)$. Changeons de ω , et soit \underline{a} rendant commutatif le diagramme

$$\begin{array}{ccc} R^n & & \omega \\ & \searrow & \downarrow \\ \underline{a} \downarrow & & R \\ R^n & \nearrow & \omega' \end{array}$$

L'isomorphisme canonique identifiant Ext^* (dans D_{part}) calculé avec ω ou ω' est donné par $\wedge^i \underline{a}^{-1}$.

Pour voir que (1) est indépendant des choix faits, il suffit donc de vérifier que

$$\prod \det_\Lambda(\wedge^i \underline{a} \otimes_R F)^{(-1)^i} = 1.$$

Appliquons la formule de calcul des déterminants par blocs :

Rappel : Pour R une Λ -algèbre commutative, F un R -module projectif sur Λ , M un R -module libre et a un endomorphisme de M , on a

$$\det_\Lambda(\text{endomorphisme } a \otimes 1 \text{ de } M \otimes_R F) =$$

$$\det_\Lambda(\text{endomorphisme de multiplication par } \det(a) \text{ de } F).$$

Ici, ceci nous ramène à vérifier que $\prod \det (\wedge^i a)^{(-1)^i} = 1$ dans R , ce qui est vrai pour toute matrice inversible $n \times n$ ($n \geq 2$).

Revenons au calcul principal de ce n^o. Si K est la classe canonique, et notant $\langle \mathcal{F}, K \rangle$ la fibre de \mathcal{G} en K

$$(\text{si } K \sim \Sigma P_i, \langle \mathcal{F}, K \rangle \sim \otimes \mathcal{F}_{P_i}),$$

on a

$$\det R \Gamma (X, \mathcal{F}) = (\det R \Gamma (\text{Sym}^{2g-2}(X), \pi^* \mathcal{G}))^{(-1)} = (\langle \mathcal{F}, K \rangle (1 - g))^{(-1)}$$

$$\det R \Gamma (X, \mathcal{F}) = \langle \mathcal{F}, -K \rangle (g - 1) \\ (X \text{ projectif lisse, } \mathcal{F} \text{ de rang un) genre } g \geq 2$$

Remarque : On a encore de tels isomorphismes pour $g = 0$ ou 1 ($g = 1$: par la construction ci-dessus).

e. Corps de classe géométrique

Ce n^o est une digression – qui a bien retardé cette lettre. Pour la suite, on n’a besoin que de ton article. J’ai néanmoins voulu comprendre la lettre de Grothendieck (9/8/60). Notes que p. 2 il charrie un peu : il dit que la définition de J entraîne que (extensions de J par G) \rightarrow (G -torseurs sur X) est pleinement fidèle. Seul fidèle est clair. Pleinement fidèle requiert de plus

$$\text{Ext}^1(J, G) \hookrightarrow H^1(X, G)$$

(d’où en particulier $\text{Ext}^1(J, G) \hookrightarrow H^1(J, G)$: tout 2-cocycle symétrique $J \times J \rightarrow G$ doit être un cobord).

J’ai aussi incorporé une idée de SGA 7 XVIII § 1 pour pouvoir prendre X complet, $G = \mathbb{G}_m$.

Soient X une courbe lisse irréductible sur k algébriquement clos, \bar{X} sa complétée projective, et $S = \bar{X} - X$. On note m un module à support dans S : $m = \Sigma n_i s_i$; on l’identifie à un sous-schéma de \bar{X} , concentré en S ($\coprod_s \text{Spec} (\mathcal{O}_{X,s}/(\text{idéal max})^{n_i})$).

Pour $s \in S$, on note \bar{X}_s le spectre de $\widehat{\mathcal{O}}_{\bar{X},s}$ (complété de \bar{X} en s) et X_s le spectre du corps des fractions de \bar{X}_s

$$\begin{array}{ccc} X_s & \hookrightarrow & \bar{X}_s & (\text{local}) \\ \downarrow & & \downarrow & \\ X & \hookrightarrow & \bar{X} & (\text{global}) \end{array}$$

On va d’abord construire les objets en termes desquels s’énonce le corps de classe géométrique.

Construction 1. $J(X) = \varprojlim_m \text{Pic}_m(\bar{X})$, où $\text{Pic}_m(\bar{X})$ classifie les faisceaux inversibles sur \bar{X} trivialisés le long de m .

(Intuitivement : 1-cohomologie à support propre de X à coefficients dans \mathbb{G}_m .)

On dispose de

$$\xi : X \rightarrow J(X)$$

$x \mapsto \xi_x$ classe de $\mathcal{O}(x)$, muni de la trivialisatoin 1 à l’infini.

Pour $S = \emptyset$, il y a lieu de modifier ces définitions. On considère plutôt $\underline{J}(X)$, la catégorie des faisceaux inversibles sur X . On la considère au choix comme :

- a) un champ de Picard *fppf* sur (Sch/k) ;
- b) un “smooth stack” au sens d’Artin, muni d’une structure de groupe ;
- c) un objet de la catégorie dérivée de celle des k -groupes algébriques. Sous cette forme, il est bien concret :

(α) Le choix d’un point base x permet d’identifier $\underline{J}(X)$ au complexe à différentielle nulle

$$\mathbb{G}_m \xrightarrow{0} \text{Pic}(X) (= \text{faisceaux trivialisés en } x)$$

(β) Si on change x , on obtient deux complexes qui représentent $\underline{J}(X)$ (deux fois le même, d’ailleurs). Les deux représentations sont liées par le quasi-isomorphisme suivant

$$\begin{array}{ccccc}
 x : & & \mathbb{G}_m & \xrightarrow{0} & \text{Pic}(X) \\
 & \swarrow \lambda & \nearrow & & \nwarrow \\
 (x, y) & & \mathbb{G}_m \times \mathbb{G}_m & \xrightarrow{\varphi} & \text{Pic}_{x+y}(X) : (\lambda, \mu) \rightarrow \mathcal{O}, \\
 & \swarrow \lambda, \mu & \nearrow \lambda, \mu & & \text{avec les triv. } \lambda, \mu \text{ en } x, y. \\
 & \searrow \mu & \nwarrow & & \swarrow \\
 y : & & \mathbb{G}_m & \xrightarrow{0} & \text{Pic}(X)
 \end{array}$$

On dispose encore de

$$\xi : X \rightarrow \underline{J}(X) : x \mapsto \mathcal{O}(x).$$

Pour $S \neq \emptyset$, \underline{J} sera synonyme de J (un faisceau inversible rigidifié n’a pas d’automorphisme non trivial).

Construction 2. $J(X_s) = K_s^*$

L’objet canonique qui relie X_s et $J(X_s)$ est un $J(X_s)$ -torseur $\bar{\xi}$ sur \bar{X}_s , trivialisé sur $\{s\}$ et sur X_s . Une application

$$\xi : X_s \rightarrow J(X_s)$$

définit un tel toseur $\bar{\xi}_s$: on prend le toseur trivial sur \bar{X}_s , trivialisé

- (a) de façon évidente sur $\{s\}$

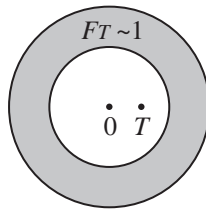
(b) par la section ξ sur X_s

$$\text{(cf. } H^0(X_s, G) \xrightarrow{\delta} H^1_{\{s\}}(\bar{X}_s, G) \text{)}.$$

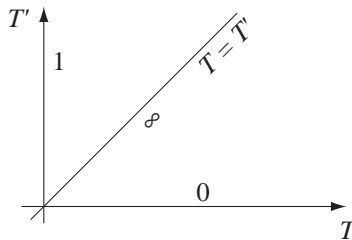
Si ξ_1 et ξ_2 sont deux telles applications, et que $\xi_1 = \sigma \xi_2$ pour $\sigma : \bar{X}_s \rightarrow J(X_s)$ valant 1 en s , alors $\bar{\xi}_1$ et $\bar{\xi}_2$ sont canoniquement isomorphes : l'isomorphisme $\bar{\xi}_1 \rightarrow \bar{\xi}_2$ est la multiplication par σ . Le problème est donc de définir une classe d'application $\xi : X_s \rightarrow J(X_s)$, avec l'arbitraire dit.

Pour dire comment je vois la classe voulue de ξ , je veux être naïf, et voir X_s comme un disque épointé autour de s , avec une coordonnée courante T . On note t cette coordonnée, vue comme élément de K_s^* . ξ est une application $T \rightarrow F_T$, F_T fonction sur le dit disque. On veut que

- (a) la fonction $F_T(x)$ a un pôle simple en $x = T$; elle n'a pas d'autre zéro et pôle dans le disque, sauf peut-être pour $x = 0$.
 - (b) loin de 0, $F_T \sim 1$
- [(a) + (b) impose à F_T d'avoir en 0 un zéro simple].



La fonction de deux variables $F(T, T') = F_T(T')$ doit avoir les valeurs 0, ∞ , 1 sur les diviseurs indiqués



Exemple : On peut prendre $F_T = \frac{t}{t-T} = -\sum_1^{\infty} T^{-i} t^i = -(t T^{-1})(1 - t T^{-1})^{-1}$ (série formelle en t à coefficients dans K_s).

Remarque : (i) La figure montre que ξ est bien défini à multiplication près par $\sigma : \bar{X}_s \rightarrow J(X_s)$ valant 1 $\in J(X_s)$ en $s \in \bar{X}_s$.

(ii) Dans l'exemple, j'ai changé le signe par rapport à Grothendieck.

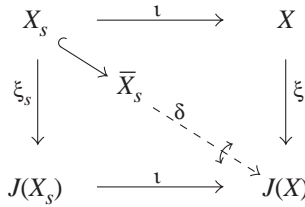
C'est nécessaire.

Construction 3. $\iota : J(X_s) \rightarrow J(X)$

$f \in K_s^* \mapsto \mathcal{O}(v_s(f) \cdot s)$, rigidification 1 en $S - s$, et en s rigidification $\mathcal{O}(v_s(f) \cdot s) \xrightarrow{f \cdot} \mathcal{O}$.

4. Compatibilité local/global ($S \neq \emptyset$)

Choisissons ξ_s comme dans la construction 2. Son lien avec le ξ global est le suivant :



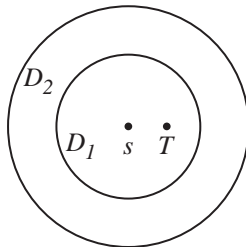
$(\xi_2)(\iota \xi_s)^{-1} : X_s \rightarrow J(X)$ se prolonge en $\delta : \bar{X}_s \rightarrow J(X)$ valant 1 en s .

Ici se testent que les définitions sont les bonnes (cf. Remarque à la construction 2, (ii)).

Je me permets encore de raisonner naïvement, en terme de $T \in$ disque épointé X_s

$$\begin{array}{l}
 T \rightarrow T \in X \rightarrow \mathcal{O}(T), \text{ rigidification 1 en } S \\
 \downarrow \\
 F_T \rightarrow \mathcal{O}(s), \text{ rigidification } \begin{cases} 1 \text{ en } S-s \\ \mathcal{O}(s) \xrightarrow{F_T} \mathcal{O} \text{ en } s \end{cases}
 \end{array}$$

d'où pour $\delta(T) : \mathcal{O}((s) - (T))$, rigidifié par $\begin{cases} 1 \text{ en } S-s \\ \xrightarrow{F_T} \mathcal{O} \text{ en } s \end{cases}$ qu'on peut décrire ainsi : recollement de \mathcal{O} , avec la rigidification 1, hors d'un disque D_1 et de $[\mathcal{O}((s) - (T))$ dans D_2 , rigidification F_T], isomorphe par $\xrightarrow{F_T}$ à $[\mathcal{O}$ dans D_2 , rigidification 1 en s], la fonction de recollement sur la couronne $D_2 - D_1$ étant dans cette nouvelle présentation F_T . Puisque $F_T \rightarrow 1$ pour $T \rightarrow 0$, $\delta(T)$ tend vers $(\mathcal{O}$, rigidification 1) pour $T \rightarrow 0$. D'où l'assertion.



Traduisons ceci en terme de toiseurs

(a) Pour \mathcal{G} un faisceau sur \bar{X} , et f une section de \mathcal{G} sur X , on définit δf , un \mathcal{G} -torseur trivialisé sur X et sur S^1 par :

δf est le tosseur \mathcal{G} , muni des trivialisations f sur X et 1 sur S [notant $\tilde{H}_S^1(\bar{X}, \mathcal{G})$ le groupe de ces tosseurs, on a

$$(\alpha) \quad H^0(X, \mathcal{G}) \xrightarrow{\delta} \tilde{H}_S^1(\bar{X}, \mathcal{G})$$

$$(\beta) \quad H_S^0(\bar{X}, \mathcal{G}) \longrightarrow \underline{H^0(S, \mathcal{G} | S)} \longrightarrow \tilde{H}_S^1(\bar{X}, \mathcal{G}) \longrightarrow H_S^1(\bar{X}, \mathcal{G}) \\ \longrightarrow H^1(S, \mathcal{G} | S)]$$

(b) La compatibilité local/global est un isomorphisme canonique

$$\begin{array}{c} \text{restriction à } \bar{X}_s \text{ de } \partial \xi : \text{ un } J(X)\text{-torseur trivialisé sur } X_s \text{ et en } s \\ \updownarrow \\ \iota(\bar{\xi}_s) \end{array}$$

$$\begin{array}{ccc} \xi \in H^0(X, J(X)) & & \\ \downarrow & & \\ \tilde{H}_S^1(\bar{X}, J(X)) & \quad H^0(X_s, J(X_s)) \ni \xi_s & \\ \downarrow & \downarrow & \\ \tilde{H}_{\{s\}}^1(\bar{X}_s, J(X)) & \longleftarrow \tilde{H}_{\{s\}}^1(\bar{X}_s, J(X_s)) \ni \bar{\xi}_s & \end{array}$$

Ces tosseurs très rigidifiés n’ayant pas d’automorphismes, l’énoncé de compatibilité est :

$\xi \in H^0(X, J(X))$ et $\bar{\xi}_s \in \tilde{H}_{\{s\}}^1(\bar{X}_s, J(X_s))$ ont même image dans $\tilde{H}_{\{s\}}^1(\bar{X}_s, J(X))$.

5. Corps de classe géométrique Soit G un groupe algébrique (commutatif) / k . Nous aurons à considérer

- (a) $H^i(X, G)$: il s’agit de cohomologie *fppf*
- (b) $\tilde{H}_{\{s\}}^i(\bar{X}_s, G)$: on travaille aussi en cohomologie *fppf*. On peut aussi l’écrire $H^i(\bar{X}_s \text{ modulo } X_s \text{ et } \{s\}, G)$; on a une suite exacte longue

$$\dots H^i(\bar{X}_s \text{ mod } \dots) \longrightarrow H^i(\bar{X}_s) \longrightarrow H^i(X_s) \oplus H^i(\{s\}) \xrightarrow{\partial} \dots$$

Puisque $H^*(\{s\}, G) = H^0(\{s\}, G) = G$, ils diffèrent peu des groupes $H_{\{s\}}^i(\bar{X}_s, G) = H^i(\bar{X}_s \text{ modulo } X_s, G)$ (égalité pour $i \neq 1$).

On a

$$\tilde{H}_S^0(\bar{X}_s, G) = 0$$

$$\tilde{H}_S^1(\bar{X}_s, G) = \text{groupe déjà défini.}$$

¹ sur S signifie : le $\mathcal{G} \otimes_X S$ tosseur sur S (réduit) ...

Si G est lisse, un G -torseur sur \bar{X}_s est trivial, et ∂ est un isomorphisme

$$\partial : G(K_s)/\text{Ker}(G(\mathcal{O}_s) \rightarrow G(k_s)) \xrightarrow{\sim} \tilde{H}_s^1(\bar{X}_s, G) \quad (G \text{ lisse}).$$

(c) Ext : il s’agira de Ext à la Yoneda (i.e. dans la catégorie dérivée de celle des groupes algébriques $/k$).

La théorie du corps de classe géométrique (en égale caractéristique) est que les lignes horizontales du diagramme commutatif suivant sont des isomorphismes, pour tout groupe algébrique (commutatif) G sur k

$$\begin{array}{ccc} \text{Ext}^i(\underline{J}(X), G) & \xrightarrow[\xi]{\sim} & H^i(X, G) \\ \downarrow \iota & & \downarrow \delta \\ \text{Ext}^i(J(X_s), G) & \xrightarrow[\xi]{\sim} & \tilde{H}_{\{s\}}^{i+1}(X_s, G) \end{array}$$

Signification

(a) global, $i = 0$: c’est le théorème de Rosenlicht

$$\text{Hom}(J(X), G) \xrightarrow{\sim} \{\text{applications } X \rightarrow G\}$$

(b) local, $i = 0$: pour G lisse, l’assertion est que les $\xi : X_s \rightarrow J(X_s)$ définis en la construction 2 induisent des isomorphismes

$$\text{Hom}(J(X_s), G) \xrightarrow{\sim} G(K_s)/ \underbrace{\text{Ker}(G(\mathcal{O}_s) \rightarrow G(k_s))}_{\text{“}G(m)\text{”, } m: \text{ idéal maximal}}$$

(c) global, $i \leq 1$: cela signifie que ξ^* est une équivalence de catégorie

$$(\text{extensions de } J(X) \text{ par } G) \xrightarrow{\cong} (G\text{-torseurs sur } X)$$

(d) local, $i \leq 1$: pour G lisse, l’assertion est que ξ induit une équivalence

$$\begin{aligned} (\text{extensions de } J(X_s) \text{ par } G) &\xrightarrow{\cong} \\ (G\text{-torseurs sur } X_s, \text{ les flèches étant modulo } G(m)) & \end{aligned}$$

Dévissages

(a) Si on a une suite exacte $0 \rightarrow G' \rightarrow G \rightarrow G'' \rightarrow 0$, et si le théorème est vrai pour deux des groupes, il l’est pour le 3^e

(b) soient les énoncés

$$(X, G ; \text{ global}) \quad (X - s, G ; \text{ global}) \quad (X_s, G ; \text{ local})$$

si deux de ces énoncés sont vrais, idem pour le 3^e.

Je n’ai pas fait une vérification soigneuse. Il y a deux points (en supposant G lisse, ce qui est permis)

- (a) comparer X_s à un hensélisé
 (b) noter que la seconde ligne équivaut à des isomorphismes

$$\mathrm{Ext}^i(J^0(X_s), G) \xrightarrow{\sim} H_s^{i+1}(X_s, G),$$

où J^0 est défini par

$$0 \longrightarrow J^0 \longrightarrow J \xrightarrow{v} \mathbb{Z} \longrightarrow 0$$

(le \mathbb{Z} correspond au \sim), et utiliser la suite exacte longue de cohomologie versus

$$0 \longrightarrow \underline{J}(X) \longrightarrow \underline{J}(X - s) \longrightarrow J^0(X_s) \longrightarrow 0$$

- (c) Un recouvrement étale $U \rightarrow X$ donne une suite exacte

$$0 \longrightarrow J(X) \longrightarrow J(U) \longrightarrow J(U \times_X U) \longrightarrow \dots$$

la suite des Ext correspondante se compare à la suite spectrale de Leray pour ce recouvrement.

Preuve : (global) Par définition, toute classe dans un $\mathrm{Ext}^i(J(X), G)$ ($i > 0$) s'efface par $G \rightarrow G'$, avec G' lisse si on veut. D'autre part, pour G lisse, toute classe dans un $H^i(X, G)$ ($i > 0$) s'efface dans un $H^i(U, G)$ avec $U \rightarrow X$ étale surjectif. Ces propriétés d'effacement ramènent le théorème au cas $i = 0$, i.e. à Rosenlicht. Enfin, global \Rightarrow local.

Pour formaliser l'argument, le mieux peut être de regarder $\mathrm{Ext}^i(J(X), G)$ et $H^i(X, G)$ comme des foncteurs cohomologiques de deux arguments

- α) un faisceau \mathcal{F} de la forme $\mathbb{Z}[U]$ (U/X étale),
 β) un groupe algébrique G/k .

Ces foncteurs sont effaçables et coïncident pour $i = 0$, donc coïncident – (il faut les deux arguments à la fois pour effacer H).

6. Remarques

- (1) Pour $S = \emptyset$, il est essentiel de travailler avec \underline{J} et non J . Dans la catégorie dérivée, on a une suite exacte (je veux dire : un triangle)

$$0 \longrightarrow \underbrace{\mathbb{G}_m[1]}_{\text{degré } -1} \longrightarrow \underline{J} \longrightarrow J \longrightarrow 0,$$

d'où

$$0 \longrightarrow \mathrm{Ext}^1(J, G) \longrightarrow \mathrm{Ext}^1(\underline{J}, G) \longrightarrow \mathrm{Hom}(\mathbb{G}_m, G) \longrightarrow \mathrm{Ext}^2(J, G) \dots$$

qui pour $G = \mathbb{G}_m$ correspond à la suite

$$0 \longrightarrow \mathrm{Pic}^0 \longrightarrow \mathrm{Pic} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

C'est de même requis pour avoir

$$0 \longrightarrow \underline{J}(\bar{X}) \longrightarrow J(\bar{X} - \{s\}) \longrightarrow \underbrace{J^0(X_s)}_{\substack{\text{y compris le} \\ \text{quotient}_{\mathbb{G}_m}} \longrightarrow 0.$$

(2) Dans certains cas, les Ext^i sont horribles, mais sont les groupes de cohomologie d'un joli complexe.

Exemple. Si P est extension d'une variété abélienne par un tore,

$$0 \longrightarrow T \longrightarrow P \longrightarrow A \longrightarrow 0$$

les $\text{Ext}^i(P, \mathbb{G}_m)$ ($i = 0, 1$) sont les noyaux et conoyaux de

$$\text{Hom}(T, \mathbb{G}_m) \longrightarrow A^* .$$

7. Constructions inverses

$i = 0$

global : pour $f : X \rightarrow G$ donné, $\tilde{f} : J(X) \rightarrow G$ se calcule par

$$\tilde{f}(\mathcal{O}(\Sigma x_i)) = \Sigma \tilde{f}(x_i) ;$$

que cela marche est l'existence d'un module.

local : l'intuition, pour moi, est la suivante (mais je n'ai rien défini ou démontré).

On part de

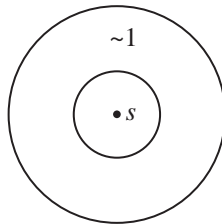
$$f : X_s \rightarrow G .$$

On doit construire $\langle f, g \rangle : K_s^* \rightarrow G$.

Regardons X_s comme un disque. On calcule ainsi

(α) trouver g' , avec g' très congru à g , et tel que

$$g' \sim 1 \text{ dans une couronne autour de } s$$



(β) faire tendre le bord intérieur de la couronne vers s . On aura

$$f(-\text{div}(g') + v_s(g)(s)) \longrightarrow \langle f, g \rangle .$$

Exemple : si f se prolonge à \bar{X}_s , il n'est pas nécessaire que g' soit très congru à g ; prenons $g' = 1$, on aura simplement $\langle f, g \rangle = f(v_s(g)(s)) = f(s)^{v_s(g)}$. Pour un autre choix, noter que g' a autant de 0 que de pôles dans le disque ; noter aussi que, avec la notation de la construction 2, cela donne

$$\left\langle f, \frac{t}{t-T} \right\rangle \sim f(T) \quad \text{comme voulu.}$$

$i = 1$: La même image s'applique – un peu moins bien –

global : à T , G -torseur sur X , on doit associer une loi qui, à \mathcal{L} , faisceau inversible sur \bar{X} très rigidifié à l'infini, associe un G -torseur $\langle T, \mathcal{L} \rangle$ sur k (la fibre en \mathcal{L} de l'extension, correspondant à T , de \underline{J} par G). On veut $\langle T, \mathcal{O}(\sum x_i) \rangle = \sum T_{x_i}$. Pour $S \neq \emptyset$, s'il n'y a pas d'application non constante de l'espace affine dans G , c'est encore une question d'existence d'un module : pour \mathcal{L} assez ample et assez rigidifié, d'espace de sections valant 1 à l'infini L , on veut que le torseur suivant sur l'espace affine L soit trivial :

$$\text{fibre en } \ell \in L : G\text{-torseur } \sum_{\ell(x)=0} T_x$$

local : pour G lisse, à T , G -torseur sur X_s , on veut associer une loi

$$g \in K_s^* \rightarrow \text{un } G\text{-torseur } \langle T, g \rangle$$

... ? ...

8. Le cas modéré

Soit \bar{X}_s local complet comme ci-dessus. Le "cas modéré" est celui où on peut remplacer $J(X_s)$ par son quotient $J(X_s)/(1+m)^*$. Ce quotient est le même pour \bar{X}_s et pour le complété à l'origine de l'espace tangent de Zariski de \bar{X}_s en s ; de plus, les revêtements modérés dans ce cas s'identifient aux revêtements modérés de la droite affine épointée.

Des arguments analogues valent en inégale caractéristique "Pour K un corps local à corps résiduel algébriquement clos, et pour T l'espace tangent de Zariski m/m^2 , il "revient au même" de se donner une extension modérément ramifiée de K ou un revêtement modéré de $T - \{0\}$."

Dès lors, il revient au même de se donner un faisceau étale localement constant modérément ramifié sur X_s ou sur $T - \{0\}$.

Notation : \natural désigne le passage à cette approximation linéaire ($\bar{X}_s^\natural = T$, $X_s^\natural = T - \{0\}$, $\mathcal{F}^\natural =$ le faisceau ... sur X_s^\natural).

Remarque : il revient au même de se donner une extension de $J(X_s)/(1+m)^*$ par G , ou de se donner sur X_s^\natural un torseur T muni de la structure additionnelle suivante :

$$T_{x_{11}} - T_{x_{10}} - T_{x_{01}} + T_{x_{00}} \sim 1 \quad \text{pour } \frac{x_{00} x_{11}}{x_{01} x_{10}} = 1$$

(tel que. . .)

[pour avoir des conventions cohérentes, il faut normaliser le “il revient au même” par : extension $E \mapsto$ toseur (en $x \in X_s^{\natural}$, si π est l’uniformisante telle que $\langle d\pi, x \rangle = 1$, la fibre du toseur est l’image inverse dans E de $-\pi \in K^*/(1+m)^*$]

$$\left\langle -\pi \text{ car } \frac{t}{t-T} = \frac{-t}{T} + \dots \right\rangle.$$

f. Cas général

X courbe projective, lisse, irréductible sur k algébriquement clos – genre g –

U ouvert non vide de X ; $j : U \hookrightarrow X$; $S = X - U$

\mathcal{F} faisceau localement constant de Λ -modules de rang un sur U .

On veut calculer

$$\det R \Gamma_c(U, \mathcal{F}).$$

Pour $s \in S$, soit $a_s = 1 +$ conducteur de Swan de \mathcal{F} en s . On a

$$\chi = 2 - 2g - \sum a_s.$$

Supposons que $2 - 2g - \sum a_s < 0$, et $S \neq \emptyset$ (on s’y ramènerait en agrandissant S). Il s’agit alors de calculer

$$\det R \Gamma_c(\text{Sym}^{2g-2+\sum a_s}(U), L \Gamma_{\text{ext}}^{2g-2+\sum a_s}(\mathcal{F})).$$

Soit m le conducteur $\sum_{s \in S} a_s \cdot s$, et J_m la jacobienne généralisée classifiant les

faisceaux inversibles m -trivialisés sur X . D’après la théorie du corps de classe géométrique, le faisceau $L \Gamma_{\text{ext}}^{2g-2+\sum a_s}(\mathcal{F})$ est image réciproque par l’application canonique $\pi : \text{Sym}^{2g-2+\sum a_s}(U) \rightarrow J_m^{(2g-2+\sum a_s)}$ d’un faisceau localement constant \mathcal{G} . Nous allons utiliser la suite spectrale de Leray de π sous la forme

$$\begin{aligned} &\det R \Gamma_c(\text{Sym}^{2g-2+\sum a_s}(U), L \Gamma_{\text{ext}}^{2g-2+\sum a_s}(\mathcal{F})) \\ &= \otimes_i \det R \Gamma_c(J_m^{(2g-2+\sum a_s)}, R^i \pi_! \pi^* \mathcal{G})^{(-1)^i}. \end{aligned}$$

Calcul des $R^i \pi_! \pi^* \mathcal{G}$

Soient $N = 2g - 2 + \sum a_s$, $\omega \in J^{(N)}$ la classe de $\Omega^1(\sum a_s s)$ et $\varphi : J_m^{(N)} \rightarrow J^{(N)}$. La fibre de π en $\ell \in J_m^{(N)}$, représentée par un faisceau inversible de degré N sur X , noté \mathcal{L} , muni d’une trivialisations α sur m , est l’ensemble des sections du $\mathcal{L}(-m)$ -torseur suivant : le faisceau des sections locales de m congrues à 1 mod m (relativement à α).

Pour $\varphi(\ell) \neq \omega$, on a $H^1(\mathcal{L}(-m)) = 0$, et $\dim H^0(\mathcal{L}(-m)) = g - 1$. Sur l’ouvert complémentaire de $\varphi^{-1}(\omega)$, π est donc une fibration de fibre un espace affine de dimension $g - 1$, et

$$R^i \pi_! \pi^* \mathcal{G} = \begin{cases} \mathcal{G}(1 - g) & \text{si } i = 2(g - 1) \\ 0 & \text{sinon.} \end{cases} \quad (\text{hors } \varphi^{-1}(\omega))$$

Pour $\varphi(\ell) = \omega$, deux cas sont à distinguer. Cette fois, on peut prendre $\mathcal{L} = \Omega^1(m)$, et α s'identifie à un système $(\omega_s)_{s \in S}$ de parties polaires de formes différentielles (avec ω_s d'ordre a_s)

- (a) Si $\Sigma \text{Res}(\omega_s) \neq 0$, $\pi^{-1}(\ell) = \emptyset$
 (b) Si $\Sigma \text{Res}(\omega_s) = 0$, $\pi^{-1}(\ell)$ est un espace affine de dimension g .

Sur le lieu (b), π est un brave espace fibré. Dès lors

$$R^i \pi_! \pi^* \mathcal{G} = \begin{cases} 0 & \text{si } i \neq 2(g-1), 2g \\ \mathcal{G}(1-g) & \text{sur l'ouvert } \varphi(\ell) \neq \omega, \text{ prolongé par zéro ailleurs,} \\ & \text{si } i = 2(g-1) \\ \mathcal{G}(-g) & \text{sur le fermé } \varphi(\ell) = \omega, \Sigma \text{Res} = 0, \\ & \text{prolongé par zéro ailleurs si } i = 2g. \end{cases}$$

Lemme 1. Soient G un groupe algébrique commutatif, E une extension de G par A (A fini) et $\chi : A \rightarrow \Lambda^*$. Soit $r : \Lambda \rightarrow \Lambda / \text{idéal maximal}$ la réduction. On suppose que $r \chi(E)$ est une extension non triviale de G par $(\Lambda/m)^*$. Soit \mathcal{G}_χ le faisceau de Λ -modules de rang un associé à Λ . On a

$$H^*(G, \mathcal{G}_\chi) = 0 \quad (\text{idem pour } H_c).$$

Preuve. 1) On peut supposer E connexe (remplacer A par $A \cap E^0$)

2) E agit sur (G, \mathcal{G}_χ) , donc sur $H^*(G, \mathcal{G}_\chi)$. Étant connexe, il agit trivialement. En particulier, l'action de A par multiplication par $\chi(a)$ est triviale. On conclut en notant que $\chi(a) - 1$ est parfois inversible.

Revenons au calcul virtuel de $H_c^*(R \pi_! \pi^* \mathcal{G})$. Le corps de classe dit que \mathcal{G} ne provient pas d'une extension de $J_{m'}$ par Λ^* , pour $m' < m$ de même support que m (i.e. S). Si un des $a_s \geq 2$, on conclut du lemme (appliqué à des groupes additifs) que, notant ψ la projection

$$\psi : J_m^{(N)} \longrightarrow J_{\sum_{s \in S} s}^{(N)},$$

on a $R \psi_!(R^{2g-2} \pi_! \pi^* \mathcal{G}) = 0$. Si les a_s sont tous un, on applique la construction 1 pour trouver que $\det(H^*(J_m^{(N)}, \mathcal{G}(1-g))) \sim 1$ (on a $2g + |S| \geq 2$ dans ce cas ($\tilde{m} > 2$) par l'hypothèse $\chi < 0$).

Au total, notant la suite exacte

$$\begin{aligned} 0 &\rightarrow (\mathcal{G}(1-g) \text{ sur } \varphi(\ell) \neq \omega, \text{ prolongé par } 0) \rightarrow \mathcal{G}(1-g) \\ &\rightarrow (\mathcal{G}(1-g) \text{ sur } \varphi(\ell) = \omega) \rightarrow 0, \end{aligned}$$

ceci fournit

$$\det(R \Gamma_c(\text{Sym}^{2g-2+\Sigma a_s}(U)), L \Gamma_{\text{ext}}^{2g-2+\Sigma a_s}(\mathcal{F})) \sim \det R \Gamma_c(U, \mathcal{F})^{(-1)^{N+1}} \sim \frac{\det(R \Gamma_c(\text{le fermé } \varphi(\ell) = \omega, \Sigma \text{ Res} = 0 \text{ de } J_m^{(N)}, \mathcal{G}(-g)))}{\det(R \Gamma_c(\text{le fermé } \varphi(\ell) = \omega \text{ de } J_m^N, \mathcal{G}(1-g)))}$$

(où le dénominateur est canoniquement 1 si un des $a_s \geq 2$ – ou si $|S| \geq 2$).

Le numérateur correspond à une somme de Jacobi – qu'on peut si on veut exprimer comme un produit de sommes de Gauss, qui seront les facteurs locaux. C'est ce que je veux maintenant expliquer.

g. Sommes de Gauss et de Jacobi

(A) Récapitulation des épisodes antérieurs

Soient $X, U, \mathcal{F}, S, (a_s)_{s \in S}, m = \Sigma a_s, N = 2g + \Sigma a_s - 2$ comme en f. Soit

a) pour $s \in S$:

J'_s = espace des parties polaires en s de formes différentielles de valuation $-a_s$ (exactement)

$$= H^0(\Omega(a_s(s))/\Omega) \setminus H^0(\Omega(a_s(s) - (s))/\Omega)$$

b)

$$J'_S = \prod_{s \in S} J'_s$$

c) sur J'_S , on dispose d'un Λ -module de rang un \mathcal{G} défini ainsi : soit $\bar{\omega} = (\bar{\omega}_s)_{s \in S} \in J'_S$. Si ω est une forme différentielle méromorphe, ayant en $s \in S$ la partie polaire $\bar{\omega}_s$, alors, la fibre $\mathcal{G}_{\bar{\omega}}$ de \mathcal{G} en $\bar{\omega}$ est

$$\mathcal{G}_{\bar{\omega}} = \bigotimes_{t \notin S} (\mathcal{F}_t)^{\otimes v_t(\omega)} \quad - \text{indépendamment du choix de } \omega.$$

d) Le groupe \mathbb{G}_m agit diagonalement sur $J'_S ((\bar{\omega}_s) \mapsto (\lambda \bar{\omega}_s))$. Cette action se prolonge à \mathcal{G} , d'où un faisceau (encore noté \mathcal{G}) sur $J''_S = J'_S/\mathbb{G}_m$, dont \mathcal{G} soit image réciproque.

e) On définit $\text{Res} : J'_S \rightarrow \mathbb{G}_a : (\bar{\omega}_s) \mapsto \sum_{s \in S} \text{Res}_s(\bar{\omega}_s)$. Soit $\Sigma' \subset J'_S$ défini par l'équation $\text{Res} = 0$; Σ' est stable sous \mathbb{G}_m et définit donc par passage au quotient $\Sigma'' \subset J''_S$.

f) On a

$$\det R \Gamma_c(U, \mathcal{F})^{(-1)^{N+1}} \sim \det(R \Gamma_c(\Sigma'', \mathcal{G}(-g))) \otimes \det(R \Gamma_c(J''_S, \mathcal{G}(1-g)))^{\otimes -1}$$

(pour $N > 0$).

(B) Faisceau de Jacobi

Soit ψ un caractère non trivial $\mathbb{Z}/p \rightarrow \Lambda^*$ [on ajoute à Λ $\sqrt[p]{1}$ si requis]. Soit \mathcal{A}_ψ le faisceau de rang un sur J'_S défini par

$$\begin{cases} \psi \text{ et le revêtement (de groupe } \mathbb{Z}/p) \\ Z^p - Z = \text{Res}(\bar{\omega}) \end{cases}$$

Soit $\tau : J'_S \rightarrow J''_S$.

Calculons $R\tau_!(\mathcal{A}_\psi)$. J'_S est un \mathbb{G}_m -torseur sur J''_S . Soit $\bar{J}'_S \xrightarrow{\bar{\tau}} J''_S$ le fibré en \mathbb{G}_a associé (via l'action de \mathbb{G}_m sur \mathbb{G}_a)

$$J' \hookrightarrow \bar{J}' = J' \cup \text{section } 0 \xrightarrow{\bar{\tau}} J''.$$

On prolonge Res pour qu'il vaille 0 en la section 0, ce qui prolonge \mathcal{A}_ψ en $\bar{\mathcal{A}}_\psi$. La suite exacte $0 \rightarrow \partial_! \mathcal{A}_\psi \rightarrow \bar{\mathcal{A}}_\psi \rightarrow (\Lambda \text{ sur la section } 0) \rightarrow 0$ fournit un triangle

$$0 \rightarrow R\tau_! \mathcal{A}_\psi \rightarrow R\bar{\tau}_! \bar{\mathcal{A}}_\psi \rightarrow \Lambda \rightarrow 0.$$

On vérifie que $R^i \bar{\tau}_! \mathcal{A}_\psi = 0$ si $i \neq 2$, et que $R^2 \bar{\tau}_! \mathcal{A}_\psi$ est $\Lambda(-1)$ sur Σ'' , prolongé par zéro ailleurs.

Les faisceaux de cohomologie de $R\tau_! \mathcal{A}_\psi$ sont donc

$$\begin{cases} i \neq 1, 2 : 0 \\ i = 1 : \Lambda \\ i = 2 : \Lambda(-1) \text{ sur } \Sigma'', \text{ prolongé par zéro} \end{cases}$$

Notons \mathcal{J} le complexe de faisceaux $R\bar{\tau}_! \mathcal{A}_\psi$. A isomorphisme canonique près, il ne dépend pas du choix de ψ : si $\psi' = \psi(\lambda x)$, ($\lambda \in \mathbb{Z}/p^*$), la multiplication par λ définit un isomorphisme $(J''_S, \mathcal{A}_\psi) \leftrightarrow (J''_S, \mathcal{A}_{\psi'})$, d'où $R\tau_! \mathcal{A}_\psi \leftrightarrow R\tau_! \mathcal{A}_{\psi'}$. En étant plus soigneux, on doit pouvoir en déduire une définition de \mathcal{J} même si Λ ne contient pas $\sqrt[p]{1}$. Il me semble clair que \mathcal{J} est aussi loin que possible d'être somme de ses faisceaux de cohomologie [l'invariant de sa classe d'isomorphie est dans $\text{Ext}^2(\Lambda(-1) \text{ sur } \Sigma, \Lambda) = \Lambda$].

La conclusion (A) f) se reformule en

$$(\det R\Gamma_c(U, \mathcal{F}))^{(-1)^{N+1}} \sim \det(R\Gamma_c(J''_S, \mathcal{G}(1-g) \otimes \mathcal{J})).$$

Le point de cette reformulation est que $R\Gamma_c(J''_S, \mathcal{G}(1-g) \otimes \mathcal{J})$ n'a qu'un groupe de cohomologie non nul – en degré Σa_s – et que ce dernier est de dimension un. Puisque $2g-2$ est pair, on trouve donc un isomorphisme canonique

Théorème 1.

$$(\det R\Gamma_c(U, \mathcal{F}))^{\otimes (-1)} \sim H_c^{\Sigma a_s}(J''_S, \mathcal{G}(1-g) \otimes \mathcal{J})$$

(expression de la constante comme une somme de Jacobi).

Pour prouver le théorème, on va “écrire une somme de Jacobi comme un produit de sommes de Gauss”, en partant de l’isomorphisme

$$R \Gamma_c(J_S'', \mathcal{G}(1-g) \otimes \mathcal{J}) \simeq R \Gamma_c(J_S', \mathcal{G}(1-g) \otimes \mathcal{A}_\psi).$$

On a $J_S' = \prod_{s \in S} J_s'$ et \mathcal{A}_ψ est le produit tensoriel externe de faisceaux $\mathcal{A}_{s,\psi}/J_s'$ analogues. Pour appliquer Künneth, il reste à décomposer $\mathcal{G}(1-g)$ de même. Ceci va nous être donné par la relation entre corps de classes global et local. On a

- (a) \mathcal{F} correspond à un tore sur U de groupe Λ^* , et à une extension E de J_m par Λ^* . On a $J_S'' \hookrightarrow J_m$ ($\bar{\omega} \mapsto$ le faisceau $\Omega(\Sigma a_s \cdot s)$ rigidifié par les $\bar{\omega}_s$) et \mathcal{G} sur J_S'' se déduit du Λ^* -torseur sur J_S'' induit par E .
- (b) On a $\prod K_s^* \rightarrow J_m$, et l’extension de $\prod K_s^*$ par Λ^* induite par E se déduit d’extensions des K_s^* par Λ^* , qui sont de nature locale (en s).
- (c) Soit \mathcal{G}_s le faisceau sur $(\mathcal{O}_s/m^{a_s})^*$ défini par cette extension locale. Si $\bar{\omega} \in J_S'$, le choix de $\bar{\omega}$ identifie J_S'' au produit des $(\mathcal{O}_s/m^{a_s})^*/\mathbb{G}_m$, et J_S' à $\prod (\mathcal{O}_s/m^{a_s})^*$. \mathcal{G} devient le produit tensoriel de
 - α) le produit tensoriel externe des \mathcal{G}_s
 - β) le faisceau constant de valeur $\otimes_{t \notin S} (\mathcal{F}_t)^{\otimes v_t(\omega)}$, pour ω une forme différentielle méromorphe de parties polaires $\bar{\omega}_s$ en $s \in S$.

On a donc

$$R \Gamma_c(J_S'', \mathcal{G}(1-g) \otimes \mathcal{J}) \sim \left(\otimes_{s \in S} R \Gamma_c((\mathcal{O}_s/m^{a_s})^*, \mathcal{G}_s(1-g) \otimes \mathcal{A}_\psi) \right) \otimes \otimes_{t \notin S} \mathcal{F}_t^{\otimes v_t(\omega)}$$

(décomposition en produit des sommes de Jacobi).

Il reste à prouver que les facteurs sont concentrés en une dimension. On va le faire par un argument parallèle à celui qui montre que les constantes locales de caractères sauvages sont “élémentaires”.

Théorème. Soient K_v un corps local à corps résiduel algébriquement clos k_v , $a \geq 1$, les groupes algébriques sur k_v $((\mathcal{O}_v/m^a)^*, \cdot)$ et $(\mathcal{O}_v/m^a, +)$, et des extensions

$$0 \rightarrow \Lambda^* \rightarrow E \rightarrow (\mathcal{O}_v/m^a)^* \rightarrow 0$$

$$0 \rightarrow \Lambda^* \rightarrow F \rightarrow (\mathcal{O}_v/m^a) \rightarrow 0$$

définissant des faisceaux de rang un \mathcal{G} et \mathcal{A} . On suppose que

- (α) Si $a > 1$, E ne se déduit pas d’une extension de $(\mathcal{O}_v/m^{a-1})^*$ par Λ^*

(β) F ne se déduit pas d'une extension de \mathcal{O}_v/m^{a-1} par Λ^* .

Alors,

$$H_c^*((\mathcal{O}_v/m^a)^*, \mathcal{G} \otimes \mathcal{A}) \begin{cases} = 0 & \text{si } * \neq a \\ \text{est libre de rang un} & \text{si } * = a \end{cases}$$

(c'est plus général que requis en ce qu'on ne suppose pas égale caractéristique)

1) $a = 1$ la base est de dimension 1, $H_c^0 = H_c^2 = 0$ de façon évidente, et $\chi = 2 - 1 - 2 = -1$

$a > 1$ et :

2) $a = 2b$ on emploie la suite spectrale de Leray pour

$$(\mathcal{O}_v/m^a)^* \rightarrow (\mathcal{O}_v/m^b)^* ;$$

pour toutes les fibres, sauf une, le faisceau est non constant et la cohomologie nulle [on invoque le lemme de f.].

3) $a = 2b + 1$ on emploie la suite spectrale de Leray pour

$$(\mathcal{O}_v/m^a)^* \rightarrow (\mathcal{O}_v/m^{b+1})^* ;$$

on a une seule image non nulle ($R^{2(a-b-1)}$), concentrée en une classe latérale mod m^b . Là encore, on est ramené à une situation de dimension 1. Je n'ai pas vérifié les détails. Il doit être facile d'obtenir $H^0 = H^2 = 0$, $\chi = 2 - 3 = -1$.

h. Jacobi sums as “Größencharaktere”

On se place sur l'anneau $\mathbb{Z}[1/m, \zeta]$, pour ζ une racine m ième de l'unité. Sur le spectre de cet anneau, on dispose d'une extension

$$(1) \quad 0 \rightarrow \mathbb{Z}/m \rightarrow \mathbb{G}_m \xrightarrow{x^m} \mathbb{G}_m \rightarrow 0.$$

Soit E un corps de nombres. Pour λ une place de E , et $\chi : \mathbb{Z}/m \rightarrow E^*$, on dispose via (1) d'un E_λ -faisceau \mathcal{F}_χ sur \mathbb{G}_m .

Soient I un ensemble fini, et Σ la partie de $\mathbb{G}_m^I/\mathbb{G}_m$ image de la partie de \mathbb{G}_m^I d'équation $\sum_i x_i = 0$. Soient $(\chi_i)_{i \in I}$ une famille de caractères $\chi_i : \mathbb{Z}/m \rightarrow E^*$, telle que

$$\prod_i \chi_i = 1.$$

Cette condition assure que $\otimes \text{pr}_i^* \mathcal{F}_{\chi_i}$ est image réciproque d'un faisceau $\mathcal{F}_{\underline{\chi}}$ sur $\mathbb{G}_m^I/\mathbb{G}_m$.

Notation :

$$|I| = r + 1$$

$R a_!(\Sigma, \)$: foncteurs image directe à support propre $\Sigma \rightarrow \mathbb{Z}[1/m, \zeta]$

ℓ : caractéristique résiduelle de λ .

Supposons qu'un des χ_i soit non trivial. Alors, $R a_1(\mathbb{G}_m^I/\mathbb{G}_m, \mathcal{F}_{\underline{\chi}})$ est 0 et la note avant le théorème 1 dans g. montre que

Proposition 1. Sur $\mathbb{Z}[1/m, \zeta, 1/\ell]$, $R^i a_1(\Sigma, \mathcal{F}_{\underline{\chi}}) = 0$ pour $i \neq r - 1$, et $R^{r-1} a_1(\Sigma, \mathcal{F}_{\underline{\chi}})$ est un E_{λ} -faisceau de rang un.

La formule des traces montre que le Frobenius en p agit par multiplication par la somme de Jacobi que considère Weil dans l'article cité [il suffit de calculer sa trace ; le $(-1)^{r+1} = (-1)^{r-1}$ de Weil correspond au fait qu'on est en dimension $(-1)^{r-1}$; noter que Weil identifie Σ au sous-espace de \mathbb{G}_m^r d'équation $\Sigma x_i = -1$ via $\mathbb{G}_m^r \xrightarrow{\sim} \mathbb{G}_m^{r+1}/\mathbb{G}_m : (x_1 \dots x_r) \rightarrow (1, x_1, \dots, x_r)$].

La proposition donne que les dites sommes de Jacobi sont les valeurs propres de Frobenius dans un système compatible de représentations λ -adiques – et comment les construire –. Ceci est beaucoup plus faible que ce qu'obtient Weil. Je ne vois pas comment faire plus par voie géométrique.

Bien à toi,

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