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Surface singularities dominated by smooth varieties

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Abstract. We give a version in characteristic $p > 0$ of Mumford's theorem characterizing a smooth complex germ of surface (X, x) by the triviality of the topological fundamental group of $U = X \backslash \{x\}$.

1. Introduction

Let (X, x) be a 2-dimensional normal complex analytic germ. Let $U = X \backslash \{x\}$. Mumford ([14]) showed the celebrated theorem

Theorem 1.1 (Mumford). *(X, x) is smooth if and only if the topological fundamental group of U is trivial.*

This is a remarkable theorem which connects a topological notion to a scheme-theoritic one. His theorem has been a bit refined by Flenner [7] who showed that in fact, the conclusion remains true if one replaces the topological by the étale fundamental group of U, that is by its profinite completion. Then one can replace the analytic germ by a complete or henselian germ over an algebraically closed field k of characteristic 0.

If k is an algebraically closed field k of characteristic $p > 0$, Mumford himself observed that the theorem is no longer true. As an example, while in characteristic 0, the singularity $z^2 + xy$ is the quotient of $\hat{\mathbb{A}}^2$, the completion of $\mathbb{A}^2$ at the origin, by the group $\mathbb{Z}/2$ acting via $\mathrm{diag}(-1, -1)$, in characteristic 2, it is the quotient of $\hat{\mathbb{A}}^2$ by $\mu_2 = \mathrm{Spec}\, k[t]/(t^2 - 1)$ acting via $\mathrm{diag}(t, t)$. Thus $\pi^{\mathrm{et}}(U) = \pi^{\mathrm{et}}(\hat{\mathbb{A}}^2 \backslash \{0\}) = 0$, yet $z^2 + xy$ is not smooth.

Artin asked in [3] whether, if $\pi^{\mathrm{et}}(U)$ is finite, there is always a finite morphism $\hat{\mathbb{A}}^2 \to X$. He shows this if (X, x) is a rational double point loc.cit.

The purpose of this note is to give an answer to a similar question where one replaces the étale fundamental group by the Nori one. Strictly speaking, Nori in [15], Chapter II,

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defined his fundamental group-scheme for irreducible reduced schemes endowed with a rational point. But as U has no rational point, one has to modify a tiny bit Nori's construction to make it work. This is done in subsection 2.2. While the étale fundamental group of X is trivial, Nori's one isn't. So the right notion of Nori's fundamental group is a relative one denoted by $\pi_{\mathrm{loc}}(U, X, x)$ (see Lemma 2.5). Roughly speaking, it measures the torsors on U under a finite flat k-group-scheme G which do not come by restriction from a torsor on X. We show (Theorem 4.2) that if $\pi^N_{\mathrm{loc}}(U, X, x)$ is finite, then (X, x) is a rational singularity, and if $\pi^N_{\mathrm{loc}}(U, X, x) = 0$, then there is a finite morphism $f : \hat{\mathbb{A}}^2 \to X$.

This note relies on discussions the authors had during the Christmas break 2009/10 in Ivry. They have been written down by Hélène in the night when Eckart died, as a despaired sign of love.

2. Local Nori fundamental groupscheme

2.1. Nori's construction. Let U be a scheme defined over a field k, endowed with a rational point $u \in U(k)$. In [15], Chapter II, Nori constructed the fundamental group-scheme $\pi^N(U, u)$. Let $\mathscr{C}(U, u)$ be the following category. The objects are triples $(h : V \to U, G, v)$ where G is a finite k-group-scheme, h is a G-principal bundle and $v \in V(k)$ with $h(v) = u$. Recall ([15], Chapter I, 2.2) that a G-principal bundle $h : V \to U$ is a cover in the fppf topology, together with a group action $G \times_k V \xrightarrow{\bullet} V$ such that $V \times_k G \xrightarrow{(1,\bullet)} V \times_U V$ is an isomorphism. Then

$$\mathrm{Hom}\big((h_1 : V_1 \to U, G_1, v_1), (h_2 : V_2 \to U, G_2, v_2)\big)$$

consists of the U-morphisms $f : V_1 \to V_2$ which are compatible with the principal bundle structure.

The objects of the ind-category $\mathscr{C}^{\mathrm{ind}}(U, u)$ associated to $\mathscr{C}(U, u)$ are triples $(h : V \to U, G, v)$ where $G = \varprojlim_\alpha G_\alpha$ is a prosystem of finite k-group-schemes G_α, $h = \varprojlim_\alpha h_\alpha, h_\alpha : V_\alpha \to U$, is a pro-$G$-principal bundle and $v = \varprojlim_\alpha v_\alpha \in Y(k)$ is a pro-point with $h(v) = u$. The morphisms are the ind-morphisms $V_1 \to V_2$ over U which are compatible with the principal bundle structure and such that $f(v_1) = v_2$.

Then (U, u) has a fundamental group-scheme $\pi^N(U, u)$, which is then a k-profinite group-scheme, if by definition ([15], Chapter II, Definition 1) there is a

$$\big(\mathfrak{h} : W \to U, \pi^N(U, u), w\big) \in \mathscr{C}^{\mathrm{ind}}(U, u)$$

with the property that for any $(h : V \to U, G, v) \in \mathscr{C}^{\mathrm{ind}}(U, u)$, there is a unique map $\big(\mathfrak{h} : W \to U, \pi^N(U, u), w\big) \to (h : V \to U, G, v)$ in $\mathscr{C}^{\mathrm{ind}}(U, u)$.

Nori shows ([15], Chapter II, Lemma 1) that if G_1, G_2, G_0 are three finite k-group-schemes, $h_i : V_i \to U$ are G_i-principal bundles, and $f_i : V_i \to V_0$, $i = 1, 2$, are principal bundle U-morphisms, then $V_1 \times_{V_0} V_2 \to Z$ is a principal bundle under $G_1 \times_{G_0} G_2$, where $Z \subset U$ is a closed subscheme (no reference to the base point here). Then he shows that (U, u) has a fundamental group-scheme if and only if $Z = U$ for all $(h_i : V_i \to U, G_i, y_i)$,

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$f_i \in \mathscr{C}(U,u)$ and he concludes ([15], Chapter II, Proposition 2) that if U is reduced and irreducible, then (U,u) has a fundamental group-scheme.

2.2. Local Nori fundamental group-scheme. Let k be a field, let A be a complete normal local k-algebra with maximal ideal $\mathfrak{m}$ and residue field k. We define $X = \operatorname{Spec} A$ and $U = X \backslash \{x\}$, where $x \in X(k)$ is the rational point associated to $\mathfrak{m}$. So in particular, $U(k) = \emptyset$, and we have to slightly modify Nori's construction to define the group-scheme of U.

Let G be a finite k-group-scheme, and let $h : V \to U$ be a G-principal bundle. Recall from [10], Corollaire 6.3.2, Proposition 6.3.4, that the *integral closure* $\tilde{h} : Y \to X$ of h is the *unique* extension $\tilde{h} : Y \to X$ of h such that $Y = \operatorname{Spec} B$, B is the integral closure of A in $j_* h_* \mathcal{O}_V$, where $j : U \to X$ is the open embedding. Then $\tilde{h}$ is finite. In particular, if $h_i : V_i \to U$ are principal bundles under the finite k-group-schemes G_i, and $f : V_1 \to V_2$ is a U-morphism which respects the principal bundle structures, then it extends uniquely to a X-morphism $\tilde{f} : Y_1 \to Y_2$, which is then finite. We can now mimic Nori's construction.

Definition 2.1. The objects of the category $\mathscr{C}_{\mathrm{loc}}(U,x)$ are triples $(h : V \to U, G, y)$ where G is a finite k-group-scheme, $y \in Y(k)$ with $\tilde{h}(y) = x$, where $\tilde{h} : Y \to X$ is the integral closure of h. The morphisms $\operatorname{Hom}\big((h_1 : V_1 \to U, G_1, y_1) \to (h_2 : V_2 \to U, G_2, y_2)\big)$ consist of U-morphisms $f : V_1 \to V_2$ which respect the principal bundle structure and such that $\tilde{f}(y_1) = y_2$.

The objects of the ind-category $\mathscr{C}^{\mathrm{ind}}_{\mathrm{loc}}(U,x)$ associated to $\mathscr{C}_{\mathrm{loc}}(U,x)$ are triples $(h : V \to U, G, y)$ where $G = \varprojlim_\alpha G_\alpha$ is a pro-system of finite k-group-schemes, $h = \varprojlim_\alpha h_\alpha$, $h_\alpha : V_\alpha \to U$, is a pro-$G$-principal bundle, and $y = \varprojlim_\alpha y_\alpha \in \varprojlim_\alpha Y_\alpha(k)$ is a pro-point in the integral closure of V_α mapping to x.

One says that (U,x) has a *local fundamental group-scheme* $\pi^N_{\mathrm{loc}}(U,x)$, which is then a k-profinite group-scheme, if there is a $\big(\mathfrak{h} : W \to U, \pi^N_{\mathrm{loc}}(U,x), z\big) \in \mathscr{C}^{\mathrm{ind}}_{\mathrm{loc}}(U,x)$ with the property that for any $(h : V \to U, G, v) \in \mathscr{C}^{\mathrm{ind}}_{\mathrm{loc}}(U,x)$, there is a unique map $\big(\mathfrak{h} : W \to U, \pi^N_{\mathrm{loc}}(U,x), z\big) \to (h : V \to U, G, y)$ in $\mathscr{C}^{\mathrm{ind}}_{\mathrm{loc}}(U,x)$.

Proposition 2.2. *If X is reduced and irreducible, then (U,x) has a local fundamental group-scheme $\pi^N_{\mathrm{loc}}(U,x)$.*

Proof. As explained above, the condition on X implies that if

$$f_i : (h_i : V_i \to U, G_i, y_i) \to (h_0 : V_0 \to U, G_0, y_0)$$

is a morphism in $\mathscr{C}_{\mathrm{loc}}(U,x)$, then $(V_1 \times_{V_0} V_2 \to U, G_1 \times_{G_0} G_2, y_1 \times_{y_0} y_2) \in \mathscr{C}_{\mathrm{loc}}(U,x)$, so as in [15], Chapter II, p. 87, the prosystem $\varprojlim_\alpha (h_\alpha : V_\alpha \to U, G_\alpha, y_\alpha)$ over all objects $(h_\alpha : V_\alpha \to U, G_\alpha, y_\alpha)$ of $\mathscr{C}_{\mathrm{loc}}(U,x)$ is well defined. So $\pi^N_{\mathrm{loc}}(U,x) = \varprojlim_\alpha G_\alpha$. □

There is a restriction functor $\rho : \mathscr{C}(X,x) \to \mathscr{C}_{\mathrm{loc}}(U,x)$ which sends $(h : Y \to X, G, y)$ to its restriction $(h_U : Y \times_X U \to U, G, y)$, as the integral closure of X in $Y \times_X U$ is Y. This defines the k-group-scheme homomorphism

$$\rho_* : \pi^N_{\mathrm{loc}}(U,x) \to \pi^N(X,x).$$

(AutoPDF V7 14/6/10 12:17) WDG Tmath J-2219 CRELLE, () PMU: K(K) 3/6/2010 pp. 1–9 2219_6711 (p. 3)

Proposition 2.3. *The homomorphism ρ is faithfully flat.*

Proof. Faithful flatness of ρ means that if $(h : Y \to X, G, y) \in \mathscr{C}(X, x)$ is such that $(Y_U \to U, G, y) \to (U, \{1\}, x)$ factors through $(\ell : V \to U, H, y) \in \mathscr{C}_{\text{loc}}(U, x)$, where $Y_U = Y \times_X U$, then necessarily $(\ell : V \to U, H, y) = \rho(\ell_X : Z \to X, H, y)$ for some $(\ell_X : Z \to X, H, y) \in \mathscr{C}(X, x)$. Let $K = \text{Ker}(G \to H)$. Since K is a k-subgroup-scheme of G, it acts on Y. We define Z to be Y/K. Then by construction, $Z \to X$ is a $G/K = H$-torsor which factors h, it restricts to $V \to U$, and is the integral closure of $V \to U$. Thus $y \in Z$ and $(\ell_X : Z \to X, H, y) \in \mathscr{C}(X, x)$. This finishes the proof. $\square$

We denote by $\pi^{\text{et}}(U, x)$ the étale proquotient of $\pi^N_{\text{loc}}(U, x)$. From now on, we assume $k = \bar{k}$. Then $\pi^{\text{et}}(U, x)$ is identified with $\pi^{\text{et}}(U, \eta)$ where $\eta \to U$ is a geometric generic point and $\pi^{\text{et}}(U, \eta)$ is Grothendieck's étale fundamental group. The étale proquotient of $\pi^N(X, x)$ is identified with Grothendieck's fundamental group based at x, and is trivial by Hensel's lemma, as A is complete. If ℓ is a prime number (including p), we denote by $\pi^{\text{et,ab},\ell}(U, x)$ the maximal pro-ℓ-abelian quotient of $\pi^{\text{et}}(U, x)$.

Definition 2.4. One defines $\pi^N_{\text{loc}}(U, X, x) = \text{Ker}\big(\pi^N_{\text{loc}}(U, x) \xrightarrow{\rho} \pi^N(X, x)\big)$.

From the discussion, we see

Lemma 2.5. *The compositum $\pi^N_{\text{loc}}(U, X, x) \to \pi^{\text{et}}(U, x)$ is surjective. In particular, if $\pi^N_{\text{loc}}(U, X, x)$ is a finite k-group-scheme, $\pi^{\text{et}}(U, x)$ is a finite group.*

3. Construction and elementary properties of the Picard scheme for surface singularities

Let k be a field, perfect if of characteristic $p > 0$, let A be a complete normal local k-algebra with maximal ideal $\mathfrak{m}$, $X = \text{Spec}\, A$ and $U = X \setminus \{x\}$, where $x \in X(k)$ is the rational point associated to $\mathfrak{m}$. In [9], Exposé XIII, Section 5, Grothendieck initiated the construction of a pro-system of locally algebraic k-group-schemes G_n and a canonical isomomorphism $G(k) = \text{Pic}(U)$ with $G(k) = \varprojlim_n G_n(k)$. This construction is performed in [13] (see overview in [11], p. 273) and relies on Mumford's basic idea [14], Section 2, to use a desingularization of X, if it exists, so in characteristic 0 or if $\dim_k X \leqq 2$ if k has characteristic $p > 0$. We now summarize the construction and the elementary properties under the assumptions

(1) X is normal,

(2) $\dim_k X = 2$.

Let $\sigma : \tilde{X} \to X$ be a desingularization such that $\sigma^{-1}(x)_{\text{red}} = \bigcup_i D_i$ is a strict normal crossings divisor and all components D_i are k-rational. There is a linear combination $D = \sum_i m_i D_i$ with all $m_i \geqq 1$ such that $\mathcal{O}_{\tilde{X}}(-D)$ is relatively ample. We define $\tilde{X}_n$ to be scheme $\bigcup_i D_i$ with structure sheaf $\mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}\big(-(n+1)D\big)$, so $\tilde{X}_0 = D$, and we also define D_{red} with structure sheaf $\mathcal{O}_{\tilde{X}}/\mathcal{O}_{\tilde{X}}\left(-\sum_i D_i\right)$. Then the functors $\mathscr{P}ic(\tilde{X}_n/k)$ and $\mathscr{P}ic(D_{\text{red}}/k)$, taken as a Zariski, an étale or a fppf functor, are representable by locally algebraic k-group-schemes $\mathrm{Pic}(\tilde{X}_n/k)$ and $\mathrm{Pic}(D_{\mathrm{red}}/k)$, so

$$\mathrm{Pic}(\tilde{X}_n) = \mathrm{Pic}(\tilde{X}_n/k)(k), \quad \mathrm{Pic}(D_{\mathrm{red}}) = \mathrm{Pic}(D_{\mathrm{red}}/k)(k)$$

(see [11], p. 273, [13], Theorem 1.2). On the other hand, for all $n \geqq 0$, and all k-algebras R, one has $\mathrm{Pic}(\tilde{X}_n \otimes_k R) = H^1(\tilde{X}_n \otimes_k R, \mathcal{O}^\times)$. As the relative dimension of σ is 1, this implies that the transition homomorphisms $\mathrm{Pic}(\tilde{X}_{n+1}) \to \mathrm{Pic}(\tilde{X}_n) \to \mathrm{Pic}(\tilde{X}_0) \to \mathrm{Pic}(D_{\mathrm{red}})$ are all surjective, and that $\mathrm{Ker}\big(\mathrm{Pic}(\tilde{X}_{n+1}) \to \mathrm{Pic}(\tilde{X}_n)\big) = H^1\big(\tilde{X}_0, \mathcal{O}_{\tilde{X}_0}\big(-(n+1)D\big)\big)$. Since $-D$ is a relatively ample divisor on $\tilde{X}$, there is an $n_0 \geqq 0$ such that the transition homomorphisms $\mathrm{Pic}(\tilde{X}_n) \to \mathrm{Pic}(\tilde{X}_{n_0})$ are all isomorphisms for $n \geqq n_0$. Since the 1-component $\mathrm{Pic}^0(D_{\mathrm{red}})$ of $\mathrm{Pic}(D_{\mathrm{red}})$ is a semi-abelian variety, so in particular smooth, and the fibers $\mathrm{Pic}(\tilde{X}_n) \to \mathrm{Pic}(D_{\mathrm{red}})$ are smooth, affine [16], p. 9, Corollaire, $\mathrm{Pic}(\tilde{X}_{n_0})$ is smooth. One defines

$$\mathrm{Pic}(\tilde{X}) = \mathrm{Pic}(\tilde{X}_{n_0}). \tag{3.1}$$

It is thus a locally algebraic smooth k-group-scheme. It is an extension of $\bigoplus_i \mathbb{Z}[D_i]$ by its 1-component. Its 1-component $\mathrm{Pic}^0(\tilde{X}) \subset \mathrm{Pic}(\tilde{X})$ is an extension of a semiabelian variety by a smooth, connected commutative unipotent algebraic group over k.

Let $\langle D \rangle \subset \mathrm{Pic}(\tilde{X})$ be the subgroup-scheme spanned by those divisors with support in D. Since the intersection matrix $(D_i \cdot D_j)$ is negative definite, $\langle D \rangle$ is a discrete subgroup-scheme of $\mathrm{Pic}(\tilde{X})$ which intersects $\mathrm{Pic}^0(\tilde{X})$ only in the origin. Thus

$$\mathrm{Pic}(U) = \mathrm{Pic}(\tilde{X})/\langle D \rangle \tag{3.2}$$

is a smooth group-scheme of finite type. By definition, the k-points of U are in bijection with isomorphism classes of line bundles on U.

The Zariski tangent space at 1 is

$$H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^1(\tilde{X}_n, \mathcal{O}_{\tilde{X}_n}) = \mathrm{Ker}\big(\mathrm{Pic}(\tilde{X}_n[\varepsilon]) \to \mathrm{Pic}(\tilde{X}_n)\big) \tag{3.3}$$

for $n \geqq n_0$, where $\tilde{X}_n[\varepsilon] := \tilde{X}_n \times_k k[\varepsilon]/(\varepsilon^2)$. Since $\mathrm{Pic}(\tilde{X})$ is smooth,

$$\dim_k H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \dim \mathrm{Pic}^0(\tilde{X}) = \mathrm{Pic}^0(U). \tag{3.4}$$

The last equality comes from the fact that $\langle D \rangle \subset \mathrm{Pic}(\tilde{X})$ is a discrete étale subgroup.

Recall that the surface singularity (X, x) is said to be *rational* if $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$. The definition does not depend on the choice of the resolution $\sigma : \tilde{X} \to X$ of singularities of (X, x).

One has

Lemma 3.1. *The following conditions are equivalent*:

(1) *The surface singularity* (X, x) *is rational.*

(2) $\mathrm{Pic}^0(\tilde{X}) = 0$.

(3) $\mathrm{Pic}(U)$ *is finite.*

Proof. The equivalence of (1) and (2) is given by (3.4). As $\langle D \rangle \subset \mathrm{Pic}(\tilde{X})$ is discrete, the definition (3.2) shows that (3) implies (2). Vice-versa, assume (2) holds. Then $\mathrm{Pic}(\tilde{X})$ is a discrete group-scheme. Let $L \in \mathrm{Pic}(\tilde{X})$. Since the intersection matrix $(D_i \cdot D_j)$ is negative definite (but not necessarily unimodular), there is an $m \in \mathbb{N} \setminus \{0\}$ such that $L^{\otimes m} \in \langle D \rangle \subset \mathrm{Pic}(\tilde{X})$. Thus any $L \in \mathrm{Pic}(\tilde{X})$ has finite order in $\mathrm{Pic}(U)$. Since $\mathrm{Pic}(U)$ is of finite type, this shows (3). □

4. The theorems

Throughout this section, we assume k to be a field, perfect if of characteristic $p > 0$, A to be a complete normal local k-algebra with maximal ideal $\mathfrak{m}$, of Krull dimension 2 over k. We set $X = \mathrm{Spec}\, A$, $U = X \setminus \{x\}$, where $x \in X(k)$ is the rational point associated to $\mathfrak{m}$. We say (X, x) is a *surface singularity* over k. We denote by $\sigma : \tilde{X} \to X$ a desingularization such that $\sigma^{-1}(x)_{\mathrm{red}} = \bigcup_i D_i$ is a strict normal crossings divisor. We define $H^i\big(Z, \mathbb{Z}_\ell(1)\big) := \varprojlim_n H^i(Z, \mu_{\ell^n})$ for a k-scheme Z.

Theorem 4.1. *Let (X, x) be a surface singularity over an algebraically closed field k. The following conditions are equivalent*:

(1) $H^1\big(\tilde{X}, \mathbb{Z}_\ell(1)\big) = 0$.

(2) $H^1\big(\tilde{U}, \mathbb{Z}_\ell(1)\big) = 0$.

(3) *There is a prime number ℓ, different from p if* $\mathrm{char}(k) = p > 0$, *such that* $\pi^{\mathrm{et,ab},\ell}(U, x)$ *is finite.*

(4) *For all prime numbers ℓ,* $\pi^{\mathrm{et,ab},\ell}(U, x)$ *is finite and if* $\mathrm{char}(k) = p > 0$, *then* $\pi^{\mathrm{et,ab},\ell}(U, x) = 0$.

(5) $\mathrm{Pic}^0(\tilde{X}) = \mathrm{Pic}^0(U)$ *is a smooth, connected commutative unipotent algebraic group-scheme over k.*

(6) *D is a tree of $\mathbb{P}^1$s.*

(7) $\mathrm{Pic}^0(D_{\mathrm{red}}) = 0$.

Proof. We firt make general remarks. For any surface singularity, one has the localization sequence

$$(4.1) \qquad H^1\big(\tilde{X}, \mathbb{Z}_\ell(1)\big) \to H^1\big(U, \mathbb{Z}_\ell(1)\big) \to H^2_{D_{\mathrm{red}}}\big(\tilde{X}, \mathbb{Z}_\ell(1)\big) \to H^2\big(\tilde{X}, \mathbb{Z}_\ell(1)\big)$$
$$\to H^2\big(U, \mathbb{Z}_\ell(1)\big) \to H^3_{D_{\mathrm{red}}}\big(\tilde{X}, \mathbb{Z}_\ell(1)\big) \to H^3\big(\tilde{X}, \mathbb{Z}_\ell(1)\big).$$

By purity ([8], Theorem 2.1.1), the restriction map $H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1))$ is injective, and $H^2_{D_{\mathrm{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) = \bigoplus_i \mathbb{Z}_\ell \cdot [D_i]$. By base change, $H^i(\tilde{X}, \mathbb{Z}_\ell(1)) = H^i(D_{\mathrm{red}}, \mathbb{Z}_\ell(1))$. Thus this group is 0 for $i \geqq 3$, equal to $\bigoplus_i \mathbb{Z}_\ell \cdot [D_i]$ for $i = 2$, and equal to $\mathrm{Pic}(D_{\mathrm{red}})[\ell]$ for $i = 1$. In fact, since $H^2(D_{\mathrm{red}}, \mathbb{Z}_\ell(1))$ is torsion free, one has $\mathrm{Pic}(D_{\mathrm{red}})[\ell] = \mathrm{Pic}^0(D_{\mathrm{red}})[\ell]$, where ^{0} means of degree 0 on each component D_i. Furthermore, by definition, the map $\bigoplus_i \mathbb{Z}_\ell \cdot [D_i] \to \bigoplus_i \mathbb{Z}_\ell \cdot [D_i]$ is defined by $[D_i] \mapsto \bigoplus_j \deg \mathcal{O}_{D_j}(D_i)$. Since the intersection matrix is definite, the map is injective, with finite torsion cokernel $\mathcal{T}$. (This cokernel is 0 if and only if the intersection matrix is unimodular.) Again by purity,

$$H^3_{D_{\mathrm{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \subset \bigoplus_i H^1(D_i^0, \mathbb{Z}_\ell) \quad \text{where } D_i^0 = D_i \setminus \bigcup_{j \neq i} D_i \cap D_j.$$

In particular, $H^3_{D_{\mathrm{red}}}(\tilde{X}, \mathbb{Z}_\ell(1))$ is torsion free. So we extract from (4.1) for any surface singularity the relations

$$(4.2) \qquad H^1(\tilde{X}, \mathbb{Z}_\ell(1)) \to H^1(U, \mathbb{Z}_\ell(1)) = \mathrm{Pic}(D_{\mathrm{red}})[\ell] = \mathrm{Pic}^0(D_{\mathrm{red}})[\ell]$$

and an exact sequence

$$(4.3) \qquad 0 \to \mathcal{T} \to H^2(U, \mathbb{Z}_\ell(1)) \to H^3_{D_{\mathrm{red}}}(\tilde{X}, \mathbb{Z}_\ell(1)) \to 0$$

with finite $\mathcal{T}$ and torsion free $H^3_{D_{\mathrm{red}}}(\tilde{X}, \mathbb{Z}_\ell(1))$. As $\mathrm{Pic}^0(D_{\mathrm{red}})$ is a semiabelian variety, we see that (4.2) implies that (1), (2) and (7) are equivalent conditions.

From the exact sequence

$$(4.4) \qquad 1 \to \mathcal{O}^\times_{D_{\mathrm{red}}} \to \bigoplus_i \mathcal{O}^\times_{D_i} \to \bigoplus_{i<j} k^\times_{D_i \cap D_j} \to 1$$

one has that (6) and (7) are equivalent. Furthermore, from the structure of $\mathrm{Pic}(\tilde{X})$ explained in section 3, one has that (5) is equivalent to (7).

We show that (2) is equivalent to (3). The condition (2) implies that $H^1(U, \mu_{\ell^n}) \subset \mathcal{T}$ for all $n \geqq 0$, thus there are finitely many μ_{ℓ^n} torsors on U. This shows (2) implies (3). On the other hand, if $\mathrm{Pic}^0(D_{\mathrm{red}})$ is not trivial, then $\mathrm{Pic}(D_{\mathrm{red}})[\ell]$ contains $\mathbb{Z}_\ell$. Thus $H^1(U, \mathbb{Z}_\ell(1))$ contains $\mathbb{Z}_\ell$ as well by (4.2). Thus (3) implies (2).

Since obviously (4) implies (3), it remains to see that (3) implies (4). We assume (3). For any commutative finite k-group-scheme G, with Cartier dual $G' = \mathrm{Hom}(G, \mathbb{G}_m)$, one has the exact sequence

$$(4.5) \qquad 0 \to H^1(X, G') \to H^1(U, G') \to \mathrm{Hom}(G, \mathrm{Pic}(U)) \to 0.$$

(See [5], III, Théorème 4.1, and [5], III, Corollaire 4.9, for the 0 on the right, which we will use only on the proof of Theorem 4.2, as $k = \bar{k}$.) We apply it for $G = \mathbb{Z}/p^n$ for some $n \in \mathbb{N} \setminus \{0, 1\}$. Since $\mathrm{Pic}(U)$ is an extension of a discrete (étale) group by $\mathrm{Pic}^0(U)$ which is a smooth, connected, commutative unipotent algebraic group-scheme over k by (5), one has $\mathrm{Hom}(\mu_{p^n}, \mathrm{Pic}(U)) = 0$. On the other hand, $A \xrightarrow{x \mapsto (x^{p^n} - x)} A$ is surjective, as A is complete.

Thus $H^1(U,\mathbb{Z}/p^n) = H^1(X,\mathbb{Z}/p^n) = 0$. This shows that (3) implies (4) and finishes the proof of the theorem. □

Theorem 4.2. *Let (X,x) be a surface singularity over an algebraically closed field k.*

(1) *If $\pi^N_{\mathrm{loc}}(U,X,x)$ is a finite group-scheme, (X,x) is a rational singularity, in particular the dualizing sheaf ω_U has finite order.*

(2) *If in addition, the order of ω_U is prime to p, then there is*

$$\big(h : V \to U, \pi^N(U,x), y\big) \in \mathscr{C}_{\mathrm{loc}}(U,x)$$

such that the surface singularity (Y,y) of the integral closure $\tilde{h} : Y \to X$ is a rational double point.

(3) *If $\pi^N_{\mathrm{loc}}(U,X,x) = 0$, then (X,x) is a rational double point.*

Proof. We show (1). If $\pi^N_{\mathrm{loc}}(U,X,x)$ is a finite group-scheme, then, by Lemma 2.5, the condition (3) of Theorem 4.1 is fulfilled, thus $\mathrm{Pic}^0(\tilde{X}) = \mathrm{Pic}^0(U)$ is a smooth, connected commutative unipotent algebraic group-scheme over k. We apply (4.5) to $G = \mathbb{Z}/p^n$. If $\mathrm{Pic}^0(U)$ is not trivial, then $\mathrm{Hom}\big(\mathbb{Z}/p^n, \mathrm{Pic}(U)\big) \neq 0$ for all $n \geqq 0$. Thus U admits nontrivial μ_{p^n}-torsors for all $n \geqq 1$, which do not come from X. This contradicts the finiteness of $\pi^N_{\mathrm{loc}}(U,X,x)$. Thus $\mathrm{Pic}^0(U) = \mathrm{Pic}^0(\tilde{X}) = 0$. We apply Lemma 3.1 to finish concluding that (X,x) is a rational singularity. Again by Lemma 3.1, all line bundles on U, in particular the dualizing sheaf ω_U of U, is torsion. This proves (1).

We show (2). So there is an $M \in \mathbb{N}\backslash\{0\}$ such that $\omega_U^M \cong \mathcal{O}_U$. Choosing such a trivialization yields an $\mathcal{O}_U$-algebra structure on $\mathscr{A} = \bigoplus_0^{M-1} \omega_U^i$ and thus a flat nontrivial μ_M-torsor $h : V = \mathrm{Spec}_{\mathcal{O}_U} \mathscr{A} \to U$. Since $(M,p) = 1$, h is étale, thus (Y,y) is normal. In fact one has $Y = \mathrm{Spec}_{\mathcal{O}_X} \mathscr{B}$ where $\mathscr{B}$ is the $\mathcal{O}_X$-algebra $j_*\mathscr{A}$, $j : U \subset X$. By duality theory, $h_*\omega_Y = \mathscr{H}om_{\mathcal{O}_X}(h_*\mathcal{O}_Y, \omega_X) \cong_{\mathcal{O}_X} h_*\mathcal{O}_Y$. Let $y \in Y$ be the closed point of Y. Thus (Y,y) is a Gorenstein normal surface singularity. On the other hand, since h is a μ_M-torsor, one has $\pi^N(V,y) \subset \pi^N(U,x)$, thus $\pi^N_{\mathrm{loc}}(V,Y,y) \subset \pi^N_{\mathrm{loc}}(U,X,x)$, and therefore is a finite k-group-scheme. Thus by (1) it is a rational singularity. Thus (Y,y) is a Gorenstein rational singularity, thus is a rational double point ([6]).

Now (3) follows directly from (2) as ω_U has then order 1. □

We now refer to [3], Section 3, for the notation, and we go to Artin's list [3], Section 4/5, to conclude using Theorem 4.2 (3):

Corollary 4.3. *If $\pi^N_{\mathrm{loc}}(U,X,x) = 0$, then X admits a finite morphism $f : \hat{\mathbb{A}}^2 \to X$. The morphism f is the identity (i.e. (X,x) is smooth) except possibly in the cases:*

(1) $\mathrm{char}(k) = 2$, E_8^1, E_8^3,

(2) $\mathrm{char}(k) = 3$, E_8^1.

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Remark 4.4. Aside of Artin's classification used in Corollary 4.3, the only place where Nori's fundamental group is used in a non-commutative way is the proof of Theorem 4.2 (2).

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