ON THE HOMOTOPY EXACT SEQUENCE FOR NORI'S FUNDAMENTAL GROUP

HÉLÈNE ESNAULT, PHÙNG HÔ HAI, AND ECKART VIEHWEG

ABSTRACT. Unlike Grothendieck's étale fundamental group, Nori's fundamental group does not fulfill the homotopy exact sequence in general. We give necessary and sufficient conditions which force exactness of the sequence.

1. INTRODUCTION

Let $f : X \to S$ be a proper separable morphism of connected schemes, with Y locally noetherian. Let $x \to X$ be geometric point, with image $s \to S$. Grothendieck shows in [13, Exposé X, Corollaire 1.4] that one has a homotopy exact sequence

$$\pi_1(\bar{X}_s, x) \longrightarrow \pi_1(X, x) \longrightarrow \pi_1(S, s) \longrightarrow 1$$

where $\pi_1(X, x)$ is his étale fundamental group. From this he deduces [13, Exposé X, Corollaire 1.7] that if S is a k scheme, where k is an algebraically closed field, and $X = Y \times_k S$, with Y proper, taking f to be the projection and $x = (y, s)$, one has Künneth formula

$$\pi_1(Y \times_k S, x) = \pi_1(Y, y) \times \pi_1(S, s).$$

Then he shows [13, Exposé X, Corollaire 1.8] that this implies base change

$$\pi_1(X, x) = \pi_1(X \otimes K, x \otimes K)$$

if X is proper connected over an algebraically closed field k and $K \supset k$ is an algebraically closed extension.

For a reduced and geometrically connected scheme X over a perfect field k, endowed with a rational point $x \in X(k)$, Nori in [8] defined his fundamental group $\pi^N(X, x)$, a k-proalgebraic group, as the projective system of k-finite flat algebraic groups G for which there is a flat G-torsor $h : Y \to X$ with trivialization $h^{-1}(x) = G$. If X is proper, he shows that $\pi^N(X, x)$ is the Tannaka group of the category of essentially finite bundles, themselves being subquotients of finite bundles in the category of bundles, semistable of degree zero on all curves. Finite bundles are those which satisfy a nontrivial polynomial equation which coefficients in $\mathbb{Z}$. The rational point x yields the neutralization of the category. Mehta-Subramanian [6] show the Künneth formula if $X = Y \times_k S$ with Y, S

Date: August 4, 2009.

Partially supported by the DFG Leibniz Preis, the SFB/TR45, the ERC Advanced Grant 226257 and by NAFOSTED Vietnam.

proper and geometrically connected, and show that base change need not hold true (see [5, Theorem 4] and the references given there for further developments).

In this article, we show that the homotopy exact sequence need not hold true for Nori's fundamental group, even if $f : X \to S$ is projective smooth and S is projective smooth as well.

We further investigate conditions which force exactness. Let X be a reduced and geometrically connected scheme X over a perfect field k, endowed with a rational point $x \in X(k)$. Let us call a finite bundle V *G-saturated* with respect to x if $V = h_*\mathcal{O}_Y$ where $h : Y \to X$ is a principal bundle under a finite k-group scheme G such that $h^{-1}(x) = G$. The notion in fact does not depend on the choice of the rational point x if one allows a finite field extension (see Proposition 3.3). Our main theorem says

Theorem 1.1. *Let k be a perfect field of characteristic $p \geq 0$, let S be an irreducible reduced scheme over k, such that $H^0(S, \mathcal{O}_S) = k$. Let $f : X \to S$ be a proper separable morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ (thus $H^0(X, \mathcal{O}_X) = k$). Let $x \in X(k)$, with image $s = f(x) \in S(k)$. Then the following two conditions are equivalent:*

I. *For all finite k-group schemes G, and for all G-saturated finite bundles V on X with respect to x, f_*V satisfies base change in a neighbourhood $U(V)$ of s.*

II. *The complex of Nori's fundamental k-group schemes*

$$\pi^N(X_s, x) \longrightarrow \pi^N(X, x) \longrightarrow \pi^N(S, s) \longrightarrow 1$$

is exact.

Here the notion of a *separable morphism* is the one defined by Grothendieck in [13, Exposé X, Définition 1.1]. It implies in particular that f is flat, and that for all closed points $t \in S$, the fiber X_t is reduced. Thus, together with the assumption $f_*\mathcal{O}_X = \mathcal{O}_S$, for any closed point $t \in S$ and any point $x_t \in X$, rational over the residue field $k(t)$ of t, and mapping to it, $\pi_1^N(X_t, x_t)$ is well defined as a $k(t)$-progroup scheme.

The theorem has the immediate corollary:

Corollary 1.2. *Let the assumptions be as in the Theorem. Then for all essentially finite bundles V on X, f_*V is an essentially finite bundle on S.*

Acknowledgments: We thank Michel Raynaud who mentioned to us his theorem written in [10, Théorème 5], which is an important ingredient of the construction of the counter-example. The first author thanks Jakob Stix and Olivier Wittenberg for enlightening discussions on the homotopy exact sequence for Grothendieck's fundamental group. We also thank Alexander Beilinson who pointed out an example of a $\mathbb{P}^1/k$-finite flat group scheme, which is a subgroup scheme of a finite flat constant k-group scheme, yet is not constant.

2. The counter-example

Let $\mathcal{M}$ be a moduli scheme of smooth genus $g \geq 2$ curves over $\operatorname{Spec}(\mathbb{Z})$, as constructed in [7, Corollary 7.14], and let $\bar{\mathcal{M}}$ be the compactification obtained by adding semistable curves at the boundary (see [7, Appendix D to Chapter 5]. Adding a level N-structure for some $N \geq 3$ and replacing $\operatorname{Spec}\mathbb{Z}$ by the open subscheme $U = \operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ make $\bar{\mathcal{M}}$ a fine moduli scheme. Moreover, $\mathcal{M}$ is smooth over U (see [9], for example). The universal family $h : \mathcal{X} \to \bar{\mathcal{M}}$ defines the invertible sheaf $\lambda = \det(h_*\omega_{\mathcal{X}/\bar{\mathcal{M}}})$.

Writing $\mathcal{A}_g$ for the moduli scheme of polarized abelian varieties with a level N-structure, consider the Satake compactification $\hat{\mathcal{A}}_{g,\mathbb{C}}$ of $\mathcal{A}_{g,\mathbb{C}} = \mathcal{A}_g \times_U \operatorname{Spec}(\mathbb{C})$, a projective compactification obtained by adding g disjoint strata, isomorphic to $\mathcal{A}_{i,\mathbb{C}}$ for $i = g-1, \ldots, 0$ (see [7, Appendix D to Chapter 5] and the references given there). The Torelli map $\mathcal{M} \to \mathcal{A}_g$ extends over $\operatorname{Spec}(\mathbb{C})$ to a morphism

$$\varphi : \bar{\mathcal{M}}_{\mathbb{C}} = \bar{\mathcal{M}} \times_U \operatorname{Spec}(\mathbb{C}) \longrightarrow \hat{\mathcal{A}}_{g,\mathbb{C}}$$

by mapping the moduli point of a semistable curve to the moduli point of the Jacobian of the normalization.

If Δ_0 denotes the boundary divisor of $\bar{\mathcal{M}}_{\mathbb{C}}$ corresponding generically to curves of genus $(g-1)$ with one node, its image in $\bar{\mathcal{A}}_{g,\mathbb{C}}$ is $3(g-1)-3$ dimensional, hence of codimension 3 in $\varphi(\bar{\mathcal{M}}_{\mathbb{C}})$. The other boundary components Δ_i parametrize generically the union of two curves of genus $(g-i)$ and i, respectively. Here the codimension of the image is again 3, if $(g-i) \geq i > 1$ whereas it is 2 for $(g-i) > i = 1$.

One way to construct $\hat{\mathcal{A}}_{g,\mathbb{C}}$ is by choosing a smooth compactification $\bar{\mathcal{A}}_{g,\mathbb{C}}$ of $\mathcal{A}_{g,\mathbb{C}}$ by a normal crossing divisor and a Deligne extension of the universal weight one variation of Hodge structures. Then the sections of the ν-th power of the determinant of the $(1,0)$ Hodge bundle define a morphism from $\bar{\mathcal{A}}_{g,\mathbb{C}}$ to $\mathbb{P}^M_{\mathbb{C}}$, whose restriction to $\mathcal{A}_{g,\mathbb{C}}$ is an embedding. For ν sufficiently large the compactification $\hat{\mathcal{A}}_{g,\mathbb{C}}$ is the image of the map. The uniqueness of the Deligne extension implies that the induced rational map

$$\varphi : \bar{\mathcal{M}}_{\mathbb{C}} \longrightarrow \hat{\mathcal{A}}_{g,\mathbb{C}} \subset \mathbb{P}^M_{\mathbb{C}}$$

is in fact a morphism, defined by the global sections of the restriction of the invertible sheaf λ^ν to $\bar{\mathcal{M}}_{\mathbb{C}}$.

This morphism extends to a U-rational map $\varphi : \bar{\mathcal{M}} \longrightarrow \mathbb{P}^M_U$, regular over some open dense subscheme V of U. Writing $\hat{\mathcal{M}}$ for the image of φ one finds for $g \geq 3$:

1) $\hat{\mathcal{M}} \to V$ is a projective compactification of $\mathcal{M} \to V$;
2) For all closed points $s \in V$, $\operatorname{codim}_{\hat{\mathcal{M}}_s}(\hat{\mathcal{M}}_s \setminus \mathcal{M}_s) \geq 2$, where ${}_s = \times_S s$.

In 2) the residue field $\kappa(s)$ of s is a finite field. We denote by k the algebraic closure of $\kappa(s)$. We set $M := \mathcal{M}_s \otimes_{\kappa(s)} k$, $\hat{M} := \hat{\mathcal{M}}_s \otimes_{\kappa(s)} k$. By [3, Remark 1.6], the subset $M_{\rm ord}(k)$ of $M(k)$ consisting of points representing ordinary curves is a nontrivial Zariski open. Since k is algebraically closed, by the property 2) there is a smooth projective curve $S \subset M$ such that $S(k) \cap M_{\rm ord}(k)$ is Zariski dense in S. As M is a fine moduli scheme, the universal family of curves induces a nonisotrivial smooth projective family $f : X \to S$ of genus ≥ 3 curves over S, with the property that the locus in S on which the fibers X_s are ordinary is a nontrivial dense open $S^0 \subset S$. By Raynaud's theorem [10, Théorème 5], the nonisotriviality of f implies that f has fibers which are not ordinary, that is $S \setminus S^0 \neq \emptyset$. We first base change f to $f_1 : X_1 \to S_1$ by a finite separable morphism $S_1 \to S$, such that S_1 is a smooth geometrically irreducible projective curve, and f_1 has a section $S_1 \to X_1$. We still have the property that $S_1 \setminus S_1^0 \neq \emptyset$ and the ordinary fibers of f_1 are precisely above S_1^0. By Grothendieck's theorem [1, Chapter 8, Proposition 4] the Picard functor of line bundles on X_1 modulo line bundles coming from S_1 is representable by a scheme $P \to S_1$ locally of finite type. Thus $G := {\rm Ker}(P \xrightarrow{p} P)$ is a group scheme, finite over S_1. Over S_1^0, it has an étale quotient $G^{0{\rm et}} \to S_1^0$, which defines an étale local system of $\mathbb{F}_p$-vector spaces of dimension g. If this local system is trivial, we set $T^0 := S_1^0$. In general, we define $T^0 \to S_1^0$ to be finite étale, with T^0 to be geometrically connected, such that the pullback of $G^{0{\rm et}}$ trivializes. We define T to be the smooth compactification of T^0. The étale morphism $T^0 \to S_1^0$ extends to a ramified cover $T \to S_1$. We denote by $f_T : X_T \to T$ the pullback family. So f_T is not isotrivial. Over the dense open subscheme T^0 the fibers are ordinary and over the non-empty closed subscheme $T \setminus T^0$ they are not. In addition, over T^0 there are g sections σ_j of $G^{0{\rm et}}$, which generate the $\mathbb{F}_p$-vector space $G^{0{\rm et}}$, hence form a basis of the restriction $G^{0{\rm et}}$ to any point in T^0. Since the pullback of G is finite over T, the sections σ_j extend to sections of G, hence of the relative Picard group P. For $t \in T \setminus T^0$ given, the p-rank of G drops at t, thus a suitable non-zero linear combination of the σ_j will be the unit in the restriction P_t for some $t \in T \setminus T^0$. We conclude that there exists a line bundle L' on $X_T = X_1 \times_T T$, such that

1) L'^p is the pullback of a line bundle N' on T,
2) the restriction of L' to any closed fiber of X_t of $X_T \to T$ with $t \in T^0$ is not trivial,
3) there exists a point $t \in T \setminus T^0$ such that $L'|_{X_t} = \mathcal{O}_{X_t}$.

Since f_1 had a section, f_T has a section $S \xrightarrow{\cong} \Sigma \subset X_T$ as well. We obtain in particular $L'^p|_\Sigma \cong N'$. From this we deduce that the degree d of N' is divisible by p. Thus

$$\big(L' \otimes_{\mathcal{O}_{X_T}} f_T^*\mathcal{O}_T(-\frac{d}{p} \cdot t)\big)^p = f_T^* N_0,$$

with $N_0 = N \otimes_{\mathcal{O}_T} \mathcal{O}(-d \cdot t) \in {\rm Pic}^0(T)$. On the other hand, the p-power multiplication ${\rm Pic}^0(T) \xrightarrow{p} {\rm Pic}^0(T)$ is surjective on points. We conclude that there is a

$N \in \mathrm{Pic}^0(T)$, such that $N^p = N_0$. So setting

$$L := L' \otimes_{\mathcal{O}_{X_T}} f_T^*\big(\mathcal{O}_T(-\frac{d}{p} \cdot t) \otimes_{\mathcal{O}_T} N^{-1}\big),$$

one has

a) $L^p = \mathcal{O}_{X_T}$,
b) $L|_{X_s} \cong L'|_{X_s} \neq \mathcal{O}_{X_s}, \ \forall s \in T^0$,
c) $\exists t \in T \setminus T^0$ such that $L|_{X_t} \cong L'_{X_t} \cong \mathcal{O}_{X_t}$.

For this example, we study the sequence

$$\pi^N(X_t, x) \longrightarrow \pi^N(X_T, x) \longrightarrow \pi^N(T, t) \longrightarrow 1.$$

It is exact on the right: f_T has a section so surely $\pi^N(X_T, x) \to \pi^N(T, t)$ is surjective. In fact, even if we did not have a section, the condition $(f_T)_*\mathcal{O}_{X_T} = \mathcal{O}_T$ implies surjectivity [8, 2-nd Corollary, p.90]. We study the left exactness.

Theorem 2.1. *Let $f_T : X_T \to T$ be the example constructed in this section. Let $x \in X_T(k)$ be a rational point, with image $t \in (T \setminus T^0)(k)$. Then the complex*

$$\pi^N(X_t, x) \to \pi^N(X_T, x) \to \pi^N(T, t) \to 1$$

is not exact on the left.

Proof. In [4, Remarks 2.10], we translate exactness in categorial terms. We do not need this precise study here. If the complex was exact on the left, then in particular, any Nori essentially finite bundle on X which restricts to the trivial bundle on X_t would necessarily come from the base. The L constructed is Nori finite by a), it does not come from T by b), yet it trivializes on X_t by c). □

Remark 2.2. Starting from the example $f_T : X_T \to T$ of Theorem 2.1, we could glue to T another smooth projective k-curve C along $t \in (T \setminus T^0)(k)$. We would then glue to X_T the surface $C \times_k X_t$ along X_t. Set $B = T \cup C$, with $T \cap C = \{t\}$, and $X_B = X_T \cap (C \times_k X_t)$ with $X_T \cap (C \times_k X_t) = X_t$. The morphism f_T extends to $f_B : X_B \to B$ by taking the projection to C on $C \times_k X_t$, the bundle L extends to L_B by taking $L|_{C \times_k X_t} = \mathcal{O}_{C \times_k X_t}$. We would thus so obtain an example where for *all* rational points c in the connected component C of B and all rational points y of $X_c = X_t$, the complex

$$\pi^N(X_c, y) \longrightarrow \pi^N(X_B, y) \longrightarrow \pi^N(B, c) \longrightarrow 1$$

is not exact on the left.

3. Conditions which force the homotopy exact sequence

Let X be a proper reduced scheme over a perfect field k with $H^0(X, \mathcal{O}_X) = k$. We denote by $\mathcal{C}^N(X)$ the category of essentially finite bundles defined by Nori in [8]. Recall that the Hom are just those of the coherent category. The *finite* objects are vector bundles V such that two polynomials $f, g \in \mathbb{N}[T]$ exist with

$f \neq g$ and $f(V) \cong g(V)$. The *essentially finite* objects are the bundles which are subquotients of finite bundles in the category of bundles which are semistable of degree 0 on all curves mapping to X.

Lemma 3.1. *Let Z be a proper reduced scheme over k, such that $H^0(Z, \mathcal{O}_Z) = k$.*

1) *Let $K \supset k$ be an extension of perfect fields. Then the map $V \mapsto V \otimes_k K$ defines a tensor functor from $\mathcal{C}^N(X)$ to $\mathcal{C}^N(X \otimes_k K)$.*
2) *Let $\tau : Z \to X$ be a k-morphism. Then the pullback map $V \mapsto \tau^*(V)$ defines a tensor functor $\mathcal{C}^N(X) \to \mathcal{C}^N(Z)$.*

Proof. The first part is contained in [8, p.89]. As for the second part: if V is finite with equation $f(V) \cong g(V)$, then $\tau^*(V)$ is finite with the same equation $f(\tau^*(V)) \cong g(\tau^*(V))$. By [8, Proposition 3.7], a subbundle of a quotient bundle of an essentially finite bundle is essentially finite if and only if it has degree 0 on all curves. So if W is a subquotient of a finite bundle V such that its degree on all curves mapping to X is zero, it is in particular true for the curves which factor over Z. Thus $\tau^*(W)$ is essentially finite. $\square$

We now endow X with a rational point $x \in X(k)$. This neutralizes $\mathcal{C}^N(X)$ by the tensor functor $\omega_x : \mathcal{C}^N(X) \to \mathsf{Vec}_k$, $\omega_x(V) = V|_x$ and defines the Tannaka k-group scheme $\pi^N(X, x) = \mathrm{Aut}^{\otimes}(\omega_x)$. Let $\mathcal{C} \subset \mathcal{C}^N(X)$ be a full finite subcategory. Let $G = \mathrm{Aut}^{\otimes}(\mathcal{C})$ be the Tannaka k-group scheme of $\mathcal{C}$. The group algebra $k[G]$, viewed as a G representation via translation, defines the G-principal bundle $h : Y \to X$ together with the trivialization $h^{-1}(x) = G$. We set $V = h_*\mathcal{O}_Y$. By definition, $V \in \mathcal{C}^N(X)$. Moreover, since h is a torsor, one has

$$V \times_{\mathcal{O}_X} V \cong \mathrm{rank}(V) \cdot V,$$

so in fact V is not only essentially finite, it is finite and fulfills the very simple equation $T^2(V) \cong \mathrm{rank}(V) \cdot 1(V)$ in $\mathbb{N}[T]$.

Definition 3.2. Let G be a finite flat k-group scheme. A finite bundle V on X is called *G-saturated* with respect to $x \in X(k)$ if there is a principal bundle $h : Y \to X$ under G with $H^0(Y, \mathcal{O}_Y) = k$ and with $h^{-1}(x) = G$, such that $V = h_*\mathcal{O}_Y$.

The notion of saturation for a finite bundle depends on the choice of the rational point x. However, if one allows finite extensions of the ground field, it does not.

Proposition 3.3. *Let X be proper reduced scheme over a perfect field k such that $H^0(X, \mathcal{O}_X) = k$. If V be a G-saturated with respect to x, then for any other closed point $y \in X$, there is a finite extension ℓ of the residue field $k(y)$ of y such that $V \otimes_k \ell$ is $G \otimes_k \ell$-saturated with respect to $y \otimes_{k(y)} \ell$.*

Proof. Let $\mathcal{C} \subset \mathcal{C}^N(X)$ be the full subcategory spanned by V. The base changed category $\mathcal{C} \otimes_k k(y)$ has two neutral fiber functors $\omega_x \otimes_k k(y)$ and ω_y. By Deligne's theorem [2, Proposition 6.6], $\mathrm{Iso}^{\otimes}(\omega_x \otimes_k k(y), \omega_y)$ is represented by a finite $k(y)$-torsor under both finite $k(y)$-group schemes $\mathrm{Aut}^{\otimes}(\omega_x \otimes_k k(y))$, and $\mathrm{Aut}^{\otimes}(\omega_y)$. By definition $G = \mathrm{Aut}^{\otimes}(\omega_x)$. Thus it splits over a finite extension $\ell \supset k(y)$ of $k(y)$. One has base change for the extensions $\ell \supset k(y) \supset k$ ([8, Proposition 5]). Thus the finite ℓ-group scheme

$$\mathrm{Aut}^{\otimes}(\omega_x \otimes_k \ell) = \mathrm{Aut}^{\otimes}(\omega_x) \otimes_k \ell = G \otimes_k \ell$$

is isomorphic to the finite ℓ-group scheme

$$\mathrm{Aut}^{\otimes}(\omega_y \otimes_{k(y)} \ell) = \mathrm{Aut}^{\otimes}(\omega_y) \otimes_{k(y)} \ell.$$

Thus the regular ℓ-representations $\ell[\mathrm{Aut}^{\otimes}(\omega_x)]$ and $\ell[\mathrm{Aut}^{\otimes}(\omega_y)]$ are isomorphic as well. This shows that $V \otimes_k \ell$ is $G \otimes_k \ell$-saturated with respect to both $x \otimes_k \ell$ and $y \otimes_{k(y)} \ell$. □

Let $f : X \to S$ be a proper separable morphism such that $H^0(X, \mathcal{O}_X) = k$ and $f_*\mathcal{O}_X = \mathcal{O}_S$. Let V be a G-saturated finite bundle with respect to x. Then f_*V is a $\mathcal{O}_S$-coherent sheaf of $\mathcal{O}_S$-algebras. We define $\mu : T = \mathrm{Spec}_S f_*V \to S$. Recall from [13, Exposé X, Définition 1.1] that a k-scheme Z is *separable* if $Z \otimes_k K$ is reduced for all field extensions $K \supset k$, and that a morphism $f : X \to S$ of k-schemes is separable if it is flat, and if for all points $s \in S$, the fiber $X_s := X \times_S s$ is separable. Furthermore, if $f : X \to S$ is separable, and $T \to S$ is any morphism of k-schemes, then the base change morphism $f_T : X_T \to T$ is separable as well.

Theorem 3.4. *Let S be an irreducible reduced scheme over a perfect field k, such that $H^0(S, \mathcal{O}_S) = k$. Let $f : X \to S$ be a proper separable morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $x \in X(k)$, and $s = f(x) \in S(k)$. If, for all finite k-group schemes G and all G-saturated finite bundles W on X with respect to x, f_*W satisfies base change in an open neighbourhood $U(W) \subset S$ of s, then f_*V is a finite K-saturated bundle on S with respect to s, where $K = G(\langle f_*V \rangle)$ is a quotient k-group scheme of G. In other words, μ is a K-principal bundle, $\mu^{-1}(s) = K$ and $H^0(T, \mathcal{O}_T) = k$.*

Proof. By taking the base change $s \to S$ of $Y \to X$, we obtain a G-principal bundle $h_s : Y_s \to X_s$. It may happen that $\dim_k H^0(Y_s, \mathcal{O}_{Y_s}) > 1$. As an extreme example, the principal bundle could completely split. We consider its reduction in the sense of Nori which we explain now. Let H be the Tannaka group of the subcategory $\langle (h_s)_*(\mathcal{O}_{Y_s}) \rangle$ of $\mathcal{C}^N(X_s)$, and let $\beta : R_s \to X_s$ be the principal bundle under H, Tannaka dual to $k[H]$. As in [8, 2.2 Definition], one uses the G-principal bundle h_s to define the tensor functor

$$\mathrm{Rep}_k(G) \longrightarrow \langle (h_s)_*(\mathcal{O}_{Y_s}) \rangle, \ V \longmapsto (V \otimes (h_s)_*(\mathcal{O}_{Y_s}))^G.$$

It induces a homomorphism $H \to G$, together with a factorization

$$\beta : R_s \longrightarrow Y_s \xrightarrow{h_s} X_s.$$

The morphism $R_s \to Y_s$ over X_s is a morphism of principal bundles, under H for R_s and G for Y_s. By construction, $H^0(R_s, \mathcal{O}_{R_s}) = k$. Then (R_s, H, X_s) is called the reduction of (Y_s, G, X_s).

We claim that H is normal in G. By the criterion in [4, Theorem A1], this follows from the fact that each object E in $\langle (h_s)_*(\mathcal{O}_{Y_s})\rangle$ is a subobject of a direct sum of copies of $h_*(\mathcal{O}_Y)|_{X_s} = (h_s)_*\mathcal{O}_{Y_s}$. Indeed, one has

$$E \subset (h_s)_*h_s^*(E) \cong \oplus (h_s)_*\mathcal{O}_{Y_s}.$$

Let $K = G/H$. Then $k[K] \subset k[G]$ and $k[K]$ this the maximal subspace of $k[G]$ on which H acts trivially [11, §16.3]. By Tannaka duality $k[K]$ corresponds to the maximal trivial subobject of $(h_s)_*(\mathcal{O}_{Y_s})$, which is

$$H^0(Y_s, \mathcal{O}_{Y_s}) \otimes_k \mathcal{O}_{X_s} = H^0(X_s, V|_{X_s}) \otimes_k \mathcal{O}_{X_s} \subset (h_s)_*(\mathcal{O}_{Y_s}).$$

Thus H can be characterized as the maximal subgroup of G which acts trivially on $H^0(Y_s, \mathcal{O}_{Y_s})$.

The exact sequence $1 \to H \to G \to K \to 1$ defines a K-principal bundle $\lambda : Z \to X$ by setting $Z = Y/H$. By definition, $\lambda^{-1}(x) = K$. Let $W = \lambda_*(\mathcal{O}_Z)$. The inclusion $W \subset V = h_*(\mathcal{O}_Y)$ in $\mathcal{C}^N(X)$ is Tannaka dual to the inclusion $\omega_x(W) = k[K] \subset k[G]$. In other words, $W \subset V$ corresponds to the maximal subrepresentation of the representation $k[G]$ of G, on which H acts trivially, that is $k[K]$.

To say that H acts trivially on $\omega_x(W)$ is to say that $W|_{X_s}$ is trivial. One obtains

$$W|_{X_s} = H^0(X_s, V|_{X_s}) \otimes_k \mathcal{O}_{X_s} = H^0(X_s, W|_{X_s}) \otimes_k \mathcal{O}_{X_s}.$$

By assumption, f_*W and f_*V are locally free and satisfy base change in a neighbourhood U of s which is dense in S (as S is irreducible). Thus

$$H^0(X_t, W|_{X_t}) = H^0(X_t, V|_{X_t})$$

has dimension r over the residue field of any point $t \in U$. It follows that one has

$$f_*W = f_*V \text{ and } f^*f_*W = f^*f_*V = W \text{ over } U \subset S.$$

Let $t \in S$ be a closed point in S. Its residue field $k(t)$ is a finite (separable) extension of k. As

$$W \otimes_k k(t) \in \mathcal{C}^N(X \otimes_k k(t)),$$

$W|_{X_t}$ lies in $C^N(X_t)$ by Lemma 3.1. Thus its maximal trivial subobject

$$H^0(X_t, W|_{X_t}) \otimes_k \mathcal{O}_{X_t} \subset W|_{X_t}$$

has rank $\leq r$. On the other hand, by semi-continuity [12, Théorème 7.7.5], $\dim_{k(t)} H^0(X_t, W|_{X_t}) \geq r$, where $k(t)$ is the residue field of t. We conclude that $\dim_{k(t)} H^0(X_t, W|_{X_t}) = r$ for all closed points $t \in S$. This implies (*loc.cit.*) that f_*W is a vector bundle of rank r, fulfills base change on S, and hence satisfies

$$f^*f_*W = W \text{ over } S.$$

On the other hand, since λ is a principal bundle under K, one has the isomorphism $W \otimes_{\mathcal{O}_X} W \cong r \cdot W$ of vector bundles over X. Applying f_* to this relation, together with the assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ and projection formula, we obtain the isomorphism $f_*W \otimes_{\mathcal{O}_S} f_*W \cong r \cdot f_*W$. Thus $\sigma : \mathrm{Spec}_S f_*W \to S$ is a K-principal bundle. By construction it satisfies $\sigma^{-1}(s) = K$. We conclude that f_*W is K-saturated with respect to s

It remains to see that the inclusion $f_*W \subset f_*V$, which is an isomorphism over $U \subset S$, is in fact an isomorphism. Since the inclusion $W \subset V$ comes from Tannaka duality, the quotient is a vector bundle. By [12, Théorème 7.7.6], $f_*(V/W)$ is of the shape $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_S)$ for some coherent sheaf $\mathcal{Q}$. Writing locally $\mathcal{Q}$ as the quotient of two vector bundles, one sees that $f_*(V/W)$ locally lies in a vector bundle. So does f_*V/f_*W as $f_*V/f_*W \subset f_*(V/W)$. But the support of f_*V/f_*W is closed in $S \setminus U$. Since U is dense, this is impossible. This finishes the proof. □

Remarks 3.5.
A. We observe in the theorem, that the bundle f_*W satisfies base change, i.e. that $f_*W \twoheadrightarrow H^0(X_t, W|_{X_t})$ for all closed point $t \in S$. It does not follow from the proof that $f_*V = f_*W$ also has the base change property. Indeed, as we have seen in our counter-example in the previous section, this is not necessarily the case.

B. As we have seen in Remark 2.2, in Theorem 3.4 one really needs the irreducibility of S.

C. Nevertheless one can extend Theorem 3.4 to the case where S is reducible. For each $W \in \mathcal{C}^N(X)$, one has to require that the rank of f_*W is constant on all generic points of S. In addition, one needs rational points s_j in each connected component S_j of S such that the assumption of Theorem 3.4 holds near s_j.

In fact, given the sheaf V on X, one can first restrict everything to an irreducible component of S. The argument used in the proof shows at the end that f_*V is locally free of rank r componentwise on S, and hence f_*V will be locally free of rank r as well.

Theorem 3.6. *Let k be a characteristic $p \geq 0$ perfect field, let S be an irreducible reduced scheme over k, such that $H^0(S, \mathcal{O}_S) = k$. Let $f : X \to S$ be a proper separable morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Let $x \in X(k)$, with image $s = f(x) \in S(k)$. If, for all finite k-group schemes G, and for all G-saturated finite bundles V on X with respect to x, f_*V satisfies base change in a neighbourhood $U(V)$ of s, then the complex of Nori's fundamental k-group schemes*

$$\pi^N(X_s, x) \longrightarrow \pi^N(X, x) \longrightarrow \pi^N(S, s) \longrightarrow 1$$

is exact.

Proof. By [8, 2-nd Corollary, p.90], the complex is exact on the right. This amounts to saying that the subcategory $\mathcal{S} := f^*\mathcal{C}^N(S)$ of $\mathcal{C}^N(X)$ is full and closed under taking subquotients in $\mathcal{C}^N(X)$. The functor $f^* : \mathcal{C}^N(S) \to \mathcal{S}$ is an equivalence of categories. Let $\mathcal{C} \subset \mathcal{C}^N(X)$ be a finite subcategory, with $G = \mathrm{Aut}^{\otimes}(\mathcal{C})$ its k-Tannaka group scheme, which is a quotient of $\pi^N(X, x)$. We overtake the notations of the proof of Theorem 3.4. Then f^* identifies $\langle f_*V\rangle$ with $\langle W\rangle$, and $K = \mathrm{Aut}^{\otimes}(\langle f_*V\rangle)$. Thus one has a commutative diagram

$$\begin{tikzcd}
\pi_1(X_s, x) \arrow[r] \arrow[d] & \pi^N(X, x) \arrow[r, "\text{surj.}"] \arrow[d, "\text{surj.}"'] & \pi^N(S, s) \arrow[d, "\text{surj.}"] \\
H \arrow[r] & G \arrow[r, "\text{surj.}"'] & K
\end{tikzcd}$$

where H was defined as $\mathrm{Ker}(G \to K)$.

According to the proof of Theorem 3.4, H is realized as the Tannaka group of $\langle h_*(\mathcal{O}_{Y_s})\rangle \subset \mathcal{C}^N(X_s)$. Thus the induced homomorphism $\pi_1(X_s, x) \to H$ is surjective. This finishes the proof. □

Corollary 3.7. *Let the assumptions be as in Theorem 3.6. Then for all essentially finite bundles V on X, f_*V is an essentially finite bundle on S and satisfies base change in a neighbourhood of s.*

Proof. Let V be an essentially finite bundle on X. Let

$$H^0(X_s, V|_{X_s}) \otimes_k \mathcal{O}_{X_s} \subset V|_{X_s}$$

be the maximal trivial subobject of $V|_{X_s} \in \mathcal{C}(X_s)$. By [4, Remarks 2.10], there is a $W_0 \in \mathcal{C}^N(S)$, such that $W = f^*W_0 \subset V$ and such that

$$W|_{X_s} = H^0(X_s, V|_{X_s}) \otimes_k \mathcal{O}_{X_s}.$$

Thus $W_0 = f_*f^*W_0 \subset f_*V$ is an isomorphism in a neighbourhood of s. We apply the same argument as in the proof of Theorem 3.6 to conclude that $f_*W = W_0 = f_*V$. This shows that f_*V is an essentially finite bundle on S. On the other hand, one has

$$f_*W|_s = W_0|_s = H^0(X_s, V|_{X_s}) = f_*V|_s.$$

Thus f_*V satisfies base change in a neighbourhood of s. This finishes the proof. □

Proof of Theorem 1.1. Theorem 3.6 tells us that I) implies II) and Corollary 3.7 the converse. □

4. Remarks

4.1. **Non-constant finite flat group schemes.** Let $f : X \to S$ satisfy the assumptions of Theorem 3.4 and assume in addition that $k = \bar{k}$. Then we know [4, Remark 2.10] that the pro-étale quotient of $\pi^N(X, x)$ is identified with Grothendieck's fundamental group $\pi_1(X, x)$.

Let G be a finite quotient of $\pi^N(X, x)$ and V be a G-saturated finite bundle with respect to x. By Proposition 3.3, V is G-saturated with respect to all closed points of X. On the other hand, there is a non-empty open $U \subset S$ such that

$f_*V = (f \circ h)_*\mathcal{O}_Y$ satisfies base change. Taking a basis point $s \in U(k)$ and $x \in f^{-1}(U)(k)$, we can apply Theorem 3.4. So

$$T = \mathrm{Spec}\,_S(f \circ h)_*\mathcal{O}_Y \longrightarrow S$$

is a principal bundle under the finite k-group K which is a quotient of G, and f_*V is saturated with respect to K. Thus $T \to S$, which is the Stein factorization of $f \circ h$, is in particular finite étale. On the other hand, in the proof of Theorem 3.4, we constructed a principal bundle on $\beta : R_s \to X_s$ under the k-group scheme $H = \mathrm{Ker}(G \to K)$ such that $H^0(R_s, \mathcal{O}_{R_s}) = k$ and $\beta^{-1}(x) = H$. Thus $\pi_1(X_s, x)$ surjects into H and

$$\pi_1(X_s, x) \longrightarrow \pi_1(X, x) \longrightarrow \pi_1(S, s) \longrightarrow 1$$

is an exact sequence.

Said differently, Theorem 3.6 yields a different proof of Grothendieck's homotopy exact sequence theorem in the shape: $f : X \to S$ proper separable over $k = \bar{k}$, X and S geometrically connected, $x \in X(k)$ mapping to $s \in S(k)$, then one has the homotopy exact sequence. Here in the formulation we skipped the properness of S, as it is not used for the torsor property of $T \to S$. In fact, one could also say that we consider the group scheme $\mathcal{H} = \mathrm{Ker}\big(G_S \to \mathrm{Aut}(f_*V)\big)$. As G is defined over k and is k-étale, $\mathcal{H}$ has to be a constant group scheme. By the base change property on a non-trivial open, we conclude that $\mathcal{H} = H \times_k S$.

This invites one to wonder whether in general, over a reduced irreducible scheme S, even proper, if $\mathcal{H} \subset G_S$ is a flat subgroup scheme of a constant finite group scheme, $\mathcal{H}$ is constant as well. This is not the case. Alexander Beilinson pointed out the following example. Let k be a perfect field, $S = \mathbb{P}^1/k$. Let $Z \xrightarrow{\sigma} \mathbb{G}_a^2$ be the blow up of the origin $(0,0)$. Then one has a morphism $a : Z \to \mathbb{P}^1$. The $\mathbb{P}^1$-group scheme

$$\mathcal{H} := \sigma^{-1}(G) \subset G_{\mathbb{P}^1}, \quad \text{with} \quad G = \alpha_p \times_k \alpha_p \subset \mathbb{G}_a^2,$$

is a finite flat (in fact even Zariski locally trivial) subgroupscheme of $G_{\mathbb{P}^1}$, which is not constant.

4.2. **Künneth.** The argument for the Künneth formula in [6] is extreme simple, so there is no point in trying to make it better. Yet we wish to mention that the Theorem 3.6 allows to recover the Künneth formula. Let us briefly explain how.

Consider $X = Z \times_k S$ with f being the second projection and $g : X \to Z$ the first one. The rational point x has the shape $x = (y, s)$. The homomorphism

$$\pi^N(Z \times_k S, (z, s)) \longrightarrow \pi^N(Z, z) \times_k \pi^N(S, s)$$

is defined over k. It is an isomorphism if and only if it is after a finite base change $\ell \supset k$. So given V a G-saturated finite bundle with respect to G, we can find a point x' over a finite extension such that f_*V has base change locally at s' and g_*V has base change locally at z'. So we can apply the construction used in the proof of 3.4 for this bundle and the two projections over a finite field extension, and one obtains both the exact sequence $1 \to H \to G \to K \to 1$ and $1 \to K \to G \to H \to 1$. This splits G over this finite extension.

Of course, we assumed in Theorem 3.6 that S is irreducible. However, as in Remark 3.5 one can easily avoid this assumption in Theorem 3.4. This is all one needs for the Künneth formula.

References

[1] Bosch, S., Lütkebohmert, W., Raynaud, M.: *Néron Models*, Ergebnisse der Mathematik **21**, Springer Verlag (1990).

[2] Deligne, P.: *Catégories tannakiennes,* The Grothendieck Festschrift, Vol. II, 111–195, Progress in Math. **87**, Birkhäuser (1990).

[3] Dwork, B., Ogus, A.: *Canonical liftings of Jacobians*, Compositio math. **58** (1986), 111–131.

[4] Esnault, H, Hai P.-H., Sun, X.: *On Nori's Fundamental Group Scheme,* Progress in Mathematics, Vol. **265** (2007), 377–398.

[5] Garuti, M.: *Barsotti-Tate groups and p-adic representations of the fundamental group scheme*, Math. Ann. **341** (2008), 603–622.

[6] Mehta, V. B., Subramanian, S.: *On the fundamental group scheme*, Invent. math. **148** (2002), 143–150.

[7] Mumford, D.: *Geometric Invariant Theory*, (1965), 2-n enlarged edition in collaboration with Fogarty, J., Ergebnisse der Mathematik **34** (1982)

[8] Nori, M.: *The fundamental group scheme*, Proc. Indian Acad. Sci. **91** (1982), 73–122.

[9] Popp, H.: *Universelle Familien von Kurven*, Special issue dedicated to the seventieth birthday of Erich Kähler. Abh. Math. Sem. Univ. Hamburg **47** (1978), 228–235.

[10] Szpiro, L.: *Propriétés numériques du faisceau dualisant relatif, in* Séminaire sur les pinceaux de courbes de genre au moins 2, Astérisque **86** (1981).

[11] Waterhouse, W.C.: *Introduction to affine group schemes*, Graduate Texts in Mathematics **66**, Springer-Verlag (1979).

[12] EGA III: *Étude Cohomologique des Faisceaux Cohérents*, 2-ième partie, Publ. Math. I.H.E.S. **17** (1963), 5–91.

[13] SGA1: *Revêtements étales et groupe fondamental*, Lect. Notes in Mathematics **224** (1970), Springer Verlag.

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: `esnault@uni-due.de`

Institute of Mathematics, P.O. Box 731, Hanoi, Vietnam
E-mail address: `phung@math.ac.vn`

Universität Duisburg-Essen, Mathematik, 45117 Essen, Germany
E-mail address: `viehweg@uni-due.de`