ON THE GALOIS COHOMOLOGY OF UNIPOTENT ALGEBRAIC GROUPS OVER LOCAL FIELDS

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ABSTRACT. In this paper, we give a necessary and sufficient condition for the finiteness of the Galois cohomology of unipotent groups over local fields of positive characteristic.

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INTRODUCTION

Let k be a field, G a linear k-algebraic group. We denote by k_s the separable closure of k in an algebraic closure $\bar{k}$, by $\mathrm{H}^1(k,G) := \mathrm{H}^1(\mathrm{Gal}(k_s/k), G(k_s))$ the usual first Galois cohomology set. It is important to know the finiteness of the Galois cohomology set of algebraic groups over certain arithmetic fields such as local or global fields. A well-known result of Borel-Serre states that over local fields of zero characteristic the Galois cohomology set of algebraic groups are always finite. A similar result also holds for local fields of positive characteristic with restriction to connected reductive groups, see [Se, Chapter III, Section 4]. Joseph Oesterlé also gives examples of unipotent groups with infinite Galois cohomology over local fields of positive characteristic (see [Oe, page 45]). And it is needed to give sufficient and necessary conditions for the finiteness of the Galois cohomology of unipotent groups over local function fields.

In Section 1, we first present several technical lemmas concerning images of additive polynomials in local fields which are needed in the sequel, and we then give the definition of unipontent groups of Rosenlicht type. Then, in Section 2, we apply results in Section 1 to give a necessary and sufficient condition for the finiteness of the Galois cohomology of an arbitrary unipotent group over local function fields, see Theorem 10. It says roughly that a unipotent group over a local function field has the finite Galois cohomology set if and only if it has a decomposition series such that each factor is a unipotent group of Rosenlicht type. Section 3 deals with some calculations on unipotent groups of Rosenlicht type.

We recall after Tits that a unipotent k-group G is called *k-wound* if every k-homomorphism (or even, k-morphism as in [KMT]) $\mathbb{G}_a \to G$ is constant. A polynomial $P := P(x_1, \ldots, x_n)$ in n variables $x_1, \ldots, x_n$ with coefficients in k is said to be *additive* if $P(x+y) = P(x) + P(y)$, for any two elements $x \in k^n, y \in k^n$. If this is the case, P is the so-called p-polynomial, i.e, a k-linear combination of $x_i^{p^j}$. Denote by p^{m_i} the highest degree of x_i appearing in P with coefficient $c_i \in k \setminus \{0\}$. Then the sum $\sum_{i=1}^{n} c_i x_i^{p^{m_i}}$ is called the principal part of the p-polynomial P and denoted by P_{princ}.

1. IMAGES OF ADDITIVE POLYNOMIALS IN LOCAL FIELDS

Let us first recall the notion of valuation independence (see [DK] or [Ku]). Let (K, v) be a valued field, L a subfield of K, and $(b_i)_{i\in I}$ a system of non-zero elements in K, with $I \neq \emptyset$. This system is called *L-valuation indepedent* if for every choice of elements $a_i \in L$ such that $a_i \neq 0$ for only finitely many $i \in I$, we have

$$v(\sum_{i\in I} a_i b_i) = \min_{i\in I} v(a_i b_i).$$

If V is an L-subvector space of K, then this system is called a *valuation basis* of V if it is a basis of V and L-valuation independent.

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Lemma 1. *Let k be a field of characteristic $p > 0$, v a discrete valuation on k,*

$$P(T) = \sum_{i=1}^{r} c_i T_i^{p^m} + \sum_{i=1}^{r} \sum_{j=1}^{m-1} c_{ij} T_i^{p^{m-j}} + T_1$$

a p-polynomial with coefficients in k, where $c_1, \ldots, c_r$ is a k^{p^m}-valuation basis of k. Take $a \in k$ and assume that the set $\{v(a-y) \mid y \in \operatorname{im} P\}$ admits a maximum. Then there exist two constants $\alpha \leq \beta$ depending only on P (not depending on a) such that this maximum is belong to the segment $[\alpha, \beta]$ or ∞.

Proof. Assume that $y_0 \in k$ such that $v(a-y_0)$ is the maximum of $\{v(a-y) \mid y \in P(k \times \cdots \times k)\}$. After replacing a by $a - y_0$ we can assume that $y_0 = 0$.

Let $c_{10} := c_1$ and I the set of indexes j, $0 \leq j \leq m-1$, such that $c_{1j} \neq 0$. We set

$$\beta := \max_{j \in I} \left\{ \frac{-v(c_{1j})}{p^{m-j}-1} \right\}.$$

Any monomial of $P(T_1, \ldots, T_r) - P_{\text{princ}}(T_1, \ldots, T_r)$ is of the form $\lambda T_j^{p^{m-s}}$, $\lambda \in k^*, 1 \leq j \leq r, s \geq 1$, and for such a monomial, we set

$$a(\lambda T_j^{p^{m-s}}) = a_{\lambda,s,j} = \frac{v(\lambda) - v(c_j)}{p^m - p^{m-s}},$$

and set

$$\alpha := \min_{\lambda,j,s} \{v(\lambda) + p^{m-s} a_{\lambda,s,j}\}.$$

It is trivial that $0 \in I$, hence $\beta \geq \frac{-v(c_1)}{p^m-1}$. On the other hand, for the polynomial T_1 in $P - P_{\text{princ}}$, the constant $a(T_1) = a_{1,m,1} = \frac{v(1)-v(c_1)}{p^m-1} = \frac{-v(c_1)}{p^m-1}$, hence $\alpha \leq \frac{-v(c_1)}{p^m-1} \leq \beta$.

We shall show that $\alpha \leq v(a) \leq \beta$ or $v(a) = \infty$

a) Suppose that $\infty > v(a) > \beta$. Then for all $j \in I$, we have

$$v(a) > \beta \geq \frac{-v(c_{1j})}{p^{m-j}-1},$$

and hence,

$$v(c_{1j} a^{p^{m-j}}) = v(c_{1j}) + p^{m-j} v(a) > v(a).$$

Now we set

$$y := \sum_{j \in I} c_{1j} a^{p^{m-j}} + a = P(a, 0, \cdots, 0) \in \operatorname{im} P.$$

Then we get

$$v(a-y) = v(\sum_{j \in I} c_{1j} a^{p^{m-j}}) \geq \min_{j \in I} v(c_{1j} a^{p^{m-j}}) > v(a),$$

a contradiction.

b) Now suppose that $v(a) < \alpha$. Since $c_1, \ldots, c_r$ is a k^{p^m}-valuation basis, we can write

$$a = c_1 a_1^{p^m} + \cdots + c_r a_r^{p^m},$$

where $a_i \in k$ for all i and

$$v(a) = \min_i v(c_i a_i^{p^m}).$$

For any monomial $\lambda T_j^{p^{m-s}}$ of $P - P_{\text{princ}}$, appearing in $P(T_1, ..., T_r)$, $\lambda \in k, 1 \leq j \leq r, s \geq 1$, if $v(a_j) < a_{\lambda,j,s}$ then by the definition of $a_{\lambda,s,j}$, we have

$$v(\lambda a_j^{p^{m-s}}) = v(\lambda) + p^{m-s} v(a_j) > v(c_j) + p^m v(a_j) \geq v(a).$$

Also, if $v(a_j) \geq a_{\lambda,j,s}$ then by the definition of α, we have

$$v(\lambda a_j^{p^{m-s}}) \geq v(\lambda) + p^{m-s} a_{\lambda,j,s} \geq \alpha > v(a).$$

Thus, for all j, we always have

$$v(\lambda a_j^{p^{m_j-s}}) > v(a).$$

Hence

$$v(a - P(a_1, \dots, a_r)) = v(P(a_1, ..., a_r) - P_{\mathrm{princ}}(a_1, ..., a_r)) > v(a),$$

a contradiction. □

Lemma 2. *Let k be a local field of characteristic $p > 0$ and let $P(T_1, \dots, T_r)$ be a separable p-polynomial with coefficients in k with the principal part $P_{\mathrm{princ}}(T_1, \dots, T_r) = \sum_{i=1}^{r} c_i T_i^{p^m}$. Assume that $c_1, \dots, c_r$ is a k^{p^m}-basis of k and $v(c_1), \dots, v(c_r)$ are pairwise distinct modulo p^m. Then the quotient group $k/\operatorname{im} P$ is finite.*

Proof. We write

$$P(T_1, \dots, T_r) = \sum_{i=1}^{r} c_i T_i^{p^m} + (\cdots) + a_1 T_1 + \cdots + a_r T_r.$$

Let $I := \{i \mid a_i \neq 0\}$. Then I is a non-empty set since P is separable. Since $v(c_1), \dots, v(c_r)$ are pairwise distinct modulo p^m, there exists the unique index $i_0 \in I$ such that

$$v(c_{i_0} a_{i_0}^{-p^m}) = \max_{i \in I} v(c_i a_i^{-p^m}).$$

We may and shall assume that $i_0 = 1$. Let us change the variables $X_1 := a_1 T_1 + \cdots + a_r T_r$, and $X_i := T_i$ for all $i > 1$. Then the polynomial P becomes

$$Q(X_1, \dots, X_r) = \sum_{i=1}^{r} b_i X_i^{p^m} + (\cdots) + X_1,$$

where $b_1 = a_1^{-p^m} c_1$, $b_i = c_i - c_1 a_1^{-p^m} a_i^{p^m}$, for all $i > 1$. Then $\operatorname{im} P = \operatorname{im} Q$. We also have $v(b_1) = v(c_1) - p^m v(a_1)$, and $v(b_i) = v(c_i)$ for all $i \geq 2$. So $v(b_1), \dots, v(b_r)$ are still pairwise distinct modulo p^m. It is then clear that $b_1, \dots, b_r$ is a k^{p^m}-valuation basis of k.

By [DK], we know that the images of additive polynomials in $(\mathbb{F}_q((t)), v_t)$ have the optimal approximation property, that is, for every $z \in K$ and every additive polynomial $F(X_1, \dots, X_n)$ with coefficients in $\mathbb{F}_q((t))$ then the set $\{v(z - y) \mid y \in \operatorname{im} F\}$ admits a maximum.

Let α, β be the constants depending only on Q as in Lemma 1. Take a in K and suppose that $a \notin \operatorname{im} Q$. Then by Lemma 1, there is $y_1 \in \operatorname{im} Q$ such that $v(a - y_1) = m_1$, where $\alpha \leq m_1 \leq \beta$. There is some $j_1 \in \mathbb{F}_q$ such that

$$v(a - y_1 - j_1 t^{m_1}) = m_2,$$

where $m_2 > m_1$. Again by Lemma 1 applying for $a - j_1 t^{m_1}$, we have $m_2 \leq \beta$, or $m_2 = \infty$. If $m_2 \neq \infty$ then there is some $j_2 \in \mathbb{F}_q$ such that

$$v(a - y_1 - j_1 t^{m_1} - j_2 t^{m_2}) = m_3,$$

where $m_3 > m_2$. Denote by $[x]$ the biggest integer not exceeding x then since the segment $[\alpha, \beta]$ has at most $u := [\beta - \alpha] + 1$ natural numbers, there exist $j_1, \dots, j_s \in \mathbb{F}_q$, where $s \leq u$, and $m_1 < \cdots < m_s$, with $m_i \in [\alpha, \beta] \cap \mathbb{N}$, such that

$$v(a - y_1 - j_1 t^{m_1} - j_2 t^{m_2} - \cdots - j_s t^{m_s}) = m_{s+1} > \beta.$$

Again by Lemma 1, we have $m_{s+1} = \infty$ and $a - y_1 - j_1 t^{m_1} - j_2 t^{m_2} - \cdots - j_s t^{m_s} \in \operatorname{im} Q$, hence

$$a \in \sum_{j \in \mathbb{F}_q} \sum_{m \in [\alpha, \beta] \cap \mathbb{N}} j t^m + \operatorname{im} Q.$$

Therefore, $k/\operatorname{im} P = k/\operatorname{im} Q$ has at most $q([\beta - \alpha] + 1)$ elements. □

Lemma 3 ([DK], Lemma 4)**.** *Let k be a local function field of characteristic $p > 0$, $P = f_1(T_1) + \cdots + f_r(T_r)$ an additive (i.e. p-) polynomial with coefficients in k in r variables, with its principal part not vanishing over $k^r \setminus \{0\}$. Let $S = \operatorname{im}(P) = f_1(k) + \cdots + f_r(k)$. Then there are additive polynomials $g_1, \dots, g_s \in k[X]$ in one variable X such that*
a) $S = g_1(k) + \cdots + g_s(k)$;
b) all polynomials g_i have the same degree $d = p^\nu$, for some non-negative integer ν;
c) the leading coefficients $b_1, \dots, b_s$ of $g_1, \dots, g_s$ are such that $v(b_1), \dots, v(b_s)$ are distinct elements of $\{0, 1, \dots, d-1\}$.

Remark. As in the proof of this lemma, we may and we shall choose $d=\max\limits_i p^{m_i}$ and $s=\sum\limits_{i=1}^r d\cdot p^{-m_i}$, where $p^{m_i}=\deg f_i$.

Lemma 4. *Let k be a local function field of characteristic $p>0$. Let $P(T_1,\cdots,T_r)$ be a separable p-polynomial in r variables with coefficients in k with the principal part*

$$P_{\mathrm{princ}}=\sum_{i=1}^{r_1}c_iT_i^{p^M}+\sum_{i=r_1+1}^{r_1+r_2}c_iT_i^{p^{M-1}}+\cdots+\sum_{i=r_1+\cdots+r_{M-1}+1}^{r_1+\cdots+r_M}c_iT_i^p$$

vanishing nowhere on $k^r\setminus\{0\}$. Then we always have the following

$$r_1+pr_2+\cdots+p^{M-1}r_M\le p^M,$$

and the equality holds if and only if the quotient group $k/\operatorname{im}P$ is finite.

Proof. We write

$$P=f_1(T_1)+\cdots+f_r(T_r),$$

where each f_i is a p-polynomial in one variable T_i with coefficients in k and of degree p^{m_i}. Let

$$S=\operatorname{im}(P)=f_1(k)+\cdots+f_r(k).$$

Choose $g_1,\dots,g_s$ with leading coefficients $b_1,\dots,b_s$, with $d:=\max\limits_i p^{m_i}$, $s:=\sum\limits_{i=1}^r d\cdot p^{-m_i}$ as in Lemma 3. Then

$$d=p^M \text{ and } s=r_1+pr_2+\cdots+p^{M-1}r_M.$$

Let

$$Q(T_1,\dots,T_s)=g_1(T_1)+\cdots+g_s(T_s).$$

Since $v(b_1),\dots,v(b_s)$ are distinct elements of $\{0,1,\dots,d-1\}$, the principal part of Q vanishes nowhere over $k^s\setminus\{0\}$ and $b_1,\dots,b_s$ is a k^{p^M}-linearly independent subset of k. Hence

$$s=r_1+pr_2+\cdots+p^{M-1}r_M\le p^M.$$

Assume that $s<p^M$. Then by [TT, Proposition 4.5], $k/\operatorname{im}P=k/\operatorname{im}Q$ is infinite. Now assume that $s=p^M$. Then by Lemma 2, $k/\operatorname{im}P$ is finite. Therefore $k/\operatorname{im}P$ is finite if and only if $s=p^M$. □

The above lemma motivates the following definition.

Definition 5. Let k be a non-perfect field of characteristic $p>0$, $P(T_1,\dots,T_{d+1})$ a separable p-polynomial with coefficients in k. The polynomial P is called of *Rosenlicht type* if its principal part (after reindexing variables)

$$P_{\mathrm{princ}}=\sum_{i=1}^{r_1}c_iT_i^{p^M}+\sum_{i=r_1+1}^{r_1+r_2}c_iT_i^{p^{M-1}}+\cdots+\sum_{i=r_1+\cdots+r_{M-1}+1}^{r_1+\cdots+r_M}c_iT_i^p$$

vanishes nowhere on $k^{d+1}\setminus\{0\}$ and

$$\begin{cases}r_1+r_2+\cdots+r_M=d+1\\ r_1+pr_2+\cdots+p^{M-1}r_M=p^M\\ r_1\ge 1, r_i\ge 0\end{cases}$$

A unipotent k-group G is called of *Rosenlicht type* if G is k-isomorphic to a k-subgroup of $\mathbb{G}_a^{d+1}$, where $d=\dim G$, which is defined as the kernel of a separable p-polynomial $P(T_1,\dots,T_{d+1})\in k[T_1,\dots,T_{d+1}]$ of Rosenlicht type.

The two following corollaries are derived immediately from Lemma 4.

Corollary 6. *Let k be a local field of characteristic $p>0$, G be a commutative unipotent k-wound group of exponent p. Then $\mathrm{H}^1(k,G)$ is finite if and only if G is of Rosenlicht type.*

Corollary 7. *Let k be a local function field of characteristic $p>0$ and G a k-unipotent group of Rosenlicht type of dimension d. Assume that G is k-isomorphic to a k-subgroup of $\mathbb{G}_a^{d+1}$, which is defined as the kernel of a separable p-polynomial $P(T_1,\dots,T_{d+1})\in k[T_1,\dots,T_{d+1}]$ with the principal part vanishing nowhere over $k^{d+1}\setminus\{0\}$. Then the polynomial P is of Rosenlicht type.*

2. Main theorem

We first recall some basic results of the theory of unipotent groups over an arbitrary field (see [Ti], [Oe]). A unipotent algebraic group G over a field k of characteristic $p > 0$ is called k-wound if any k-morphism of affine groups $\mathbb{G}_a \to G$ is constant. Let G be a unipotent group over k. Then there exists a maximal central connected k-subgroup K of G which is killed by p. This group is called *cckp-kernel* of G and denoted by *cckp*(G) or $\kappa(G)$. Here $\dim(\kappa(G)) > 0$ if G is not finite.

The following statements are equivalent:
(i) G is wound over k,
(ii) $\kappa(G)$ is wound over k.

If the two equivalences are satisfied then $G/\kappa(G)$ is also wound over k.

We set

$$\begin{aligned} \kappa^1(G) &:= \kappa(G), \kappa_1(G) := G/\kappa^1(G) \\ \kappa^2(G) &:= \kappa(\kappa_1(G)), \kappa_2(G) := \kappa_1(G)/\kappa^2(G) \\ &\dots \\ \kappa^{n+1}(G) &:= \kappa(\kappa_n(G)), \kappa_{n+1}(G) := \kappa_n(G)/\kappa^{n+1}(G) \end{aligned}$$

We call $\kappa^1(G), \kappa^2(G), \dots, \kappa^n(G), \dots$ the *cckp-kernel series* of G. Then there exists n such that $\kappa^n(G) = 0$, and the least such number will be called the *cckp-kernel length* of G and denoted by *lcckp*(G)

We now recall the following result of Oesterlé (see [Oe, Chap. IV, 2.2]), which is used frequently in the sequel.

Lemma 8. *Let G be a linear algebraic group defined over a field k and U a normal unipotent algebraic subgroups of G defined over k. Then the canonical map*

$$\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, G/U)$$

is surjective.

Proposition 9. *Let G be a smooth connected unipotent group which is defined and wound over a local function field k of characteristics $p > 0$. Let $\kappa^1(G), \kappa^2(G), \dots, \kappa^n(G) = 1$, $n =$ lcckp(G), be its cckp-kernel series. Then $\mathrm{H}^1(k, G)$ is finite if and only if $\mathrm{H}^1(k, \kappa^i(G))$ is finite for all i.*

Proof. Suppose that $\mathrm{H}^1(k, G)$ is finite. From the exact sequence

$$1 \to \kappa^1(G) \to G \to \kappa_1(G) \to 1,$$

we derive the following exact sequence

$$\kappa_1(G)(k) \xrightarrow{\delta} \mathrm{H}^1(k, \kappa^1(G)) \to \mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \kappa_1(G)).$$

We endow $(\kappa_1(G))(k)$ and $\mathrm{H}^1(k, \kappa^1(G))$ with the topology induced from the natural topology (the valuation topology) on k, then the map δ is continuous. Since $\kappa_1(G)$ is k-wound, $(\kappa_1(G))(k)$ is compact by [Oe, Chap. V, Sec. 1]. On the other hand, since $\kappa^1(G)$ is commutative, k-wound and of exponent p, $\kappa^1(G)$ is k-isomorphic to a k-subgroup of $\mathbb{G}^{d+1}$ which is given as the kernel of a separable p-polynomial F in $d+1$ variables, where $d = \dim(\kappa^1(G))$ and F is considered as a homomorphism $F : \mathbb{G}_a^{d+1} \to \mathbb{G}_a$. One thus find that $\mathrm{H}^1(k, \kappa^1(G)) \simeq k/F(k^{d+1})$. One checks that the subgroup $F(k^{d+1}) \subset k$ is open, since F is a separable morphism and we may use the implicit function theorem (see [Se2]) in this case. Hence the topology on $\mathrm{H}^1(k, \kappa^1(G))$ is discrete. Since the map δ is continuous, its image $\mathrm{im}(\delta)$ is compact in the discrete topological group $\mathrm{H}^1(k, \kappa^1(G))$. Therefore $\mathrm{im}(\delta)$ is finite and by twisting argument (see [Se1, Chap. I, Sec. 5.4, Cor. 3]), and the finiteness assumption of $\mathrm{H}^1(k, G)$, we get $\mathrm{H}^1(k, \kappa^1(G))$ is finite. We also know that the natural map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \kappa_1(G))$ is surjective, so $\mathrm{H}^1(k, \kappa_1(G))$ is also finite.

Similarly, by replacing G by $\kappa_1(G)$ then we can show that $\mathrm{H}^1(k, \kappa^2(G))$ is finite, since $\kappa^2(G) = \kappa(\kappa_1(G))$ by definition. Inductively, we can prove that $\mathrm{H}^1(k, \kappa^i(G))$ is finite for all i.

Conversely, assume that $\mathrm{H}^1(k, \kappa^i(G))$ is finite for all i. Since $1 = \kappa^n(G) = \kappa(\kappa_{n-1}G)$ and $\kappa_{n-1}G$ is connected, $\kappa_{n-1}G$ is trivial. Hence from the exact sequence

$$1 \to \kappa^{n-1}G \to \kappa_{n-2}G \to \kappa_{n-1}G \to 1,$$

and the assumption that $\mathrm{H}^1(k, \kappa^{n-1}(G))$ is finite, we deduce that $\mathrm{H}^1(k, \kappa_{n-2}(G))$ is finite. Inductively, we can prove that $\mathrm{H}^1(k, \kappa_{n-2}(G))$,..., $\mathrm{H}^1(k, \kappa_1(G))$ are finite. Then $\mathrm{H}^1(k, G)$ is also finite. □

We now have the following main result of this paper.

Theorem 10. *Let k be a local field of characteristic $p > 0$, G a smooth unipotent group defined over k. Let G_s be the k-split part of G. Then $\mathrm{H}^1(k, G)$ is finite if and only if G is connected and $\kappa^i(G/G_s)$ is of Rosenlicht type for all i.*

Proof. Assume that $\mathrm{H}^1(k, G)$ is finite. Let G^0 be the connected component of G, then G/G^0 is of dimension 0. Since the natural map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, G/G^0)$ is surjective, $\mathrm{H}^1(k, G/G^0)$ is finite. Then by [TT1, Prop. 4.7], G/G^0 is trivial and G is connected. Also, since the natural map $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, G/G_s)$ is surjective, $\mathrm{H}^1(k, G/G_s)$ is finite. Hence by Proposition 9, $\mathrm{H}^1(k, \kappa^i(G/G_s))$ is finite for all i since G/G_s is connected and wound over k. By Corollary 6, $\kappa^i(G/G_s)$ are of Rosenlicht type since such groups are commutative, k-wound and of exponent p.

Conversely, assume that G is connected and for all i, $\kappa^i(G/G_s)$ is of Rosenlicht type. Then by Corollary 6, $\mathrm{H}^1(k, \kappa^i(G/G_s))$ is finite for all i. By Proposition 9, $\mathrm{H}^1(k, G/G_s)$ is finite. Combining with the fact that $\mathrm{H}^1(k, G_s)$ is trivial since G_s is k-split, we deduce that $\mathrm{H}^1(k, G)$ is finite. ☐

3. Unipotent groups of Rosenlicht type

In this section we shall make some calculations on unipotent groups of Rosenlicht type of small dimension.

First we have the following result concerning the dimension of unipotent groups of Rosenlicht type.

Corollary 11. *Let G be a unipotent algebraic group over a non-perfect field k of characteristic $p > 0$. If G is of Rosenlicht type then $\dim(G)$ is divisible by $p - 1$.*

Proof. Let $\dim G = d$ and G is defined by a separable p-polynomial P as in Definition 5. Then for some integer numbers $M > 0$, $r_1 > 0$, $r_2, \ldots, r_M \geq 0$, we have

$$r_1 + r_2 + \cdots + r_M = d + 1 \tag{3.1}$$

$$r_1 + pr_2 + \cdots + p^{M-1} r_M = p^M \tag{3.2}$$

Take (3.1) minus (3.2), we get $(p-1)r_2 + \cdots + (p^{M-1} - 1)r_M = (p^M - 1) + d$. This yields that $p - 1$ divides d since $p^i - 1$ is divisible by $p - 1$ for all i. ☐

For $d = k(p - 1)$, we want to solve the following equations with integer variables $M, r_1 \geq 1$, $r_2, \ldots, r_M \geq 0$:

$$r_1 + r_2 + \cdots + r_M = k(p-1) + 1 \tag{3.3}$$

$$r_1 + pr_2 + \cdots + p^{M-1} r_M = p^M \tag{3.4}$$

Equation (3.4) yields that p divides r_1, and then $r_1 = l_1 p$, for some integer number $l_1 \geq 1$. By substituting $r_1 = l_1 p$ in (3.3), we have $l_1 + r_2 + \cdots + p^{M-2} r_M = p^{M-1}$. Then $l_1 + r_2 = l_2 p$, for some integer numbers$l_2 \geq 1$. Similarly, there are natural numbers $l_3, \ldots, l_M \geq 1$ such that

$$r_1 = l_1 p, l_1 + r_2 = l_2 p, l_2 + r_3 = l_3 p, \ldots, l_{M-1} + r_M = l_M p. \tag{3.5}$$

By substituting (3.5) in (3.3)-(3.4), we get

$$\begin{cases} l_1 + l_2 + \cdots + l_M = k \\ l_M = 1 \end{cases}$$

For example:

If $k = 1$ then $M = 1$, $l_1 = 1$.

If $k = 2$ then $M = 2$, $l_1 = l_2 = 1$.

If $k = 3$ then $M = 2$, $l_1 = 2, l_2 = 1$ or $M = 3$, $l_1 = l_2 = l_3 = 1$.

From the above calculations, we have the following proposition.

Proposition 12. *Let k be a non-perfect field of characteristic $p > 0$.*
(a) Every unipotent k-groups of Rosenlicht type of dimension $p - 1$(resp. $2(p - 1)$) is k-isomorphic to a k-subgroup of $\mathbb{G}_a^p$ (resp. $\mathbb{G}_a^{2p-1}$) defined as the kernel of a separable p-polynomial $P(T_1, \ldots, T_p)$ (resp. $P(T_1, \ldots, T_{2p-1})$) with the principal part of the form

$$P_{\mathrm{princ}} = c_1 T_1^p + \cdots + c_p T_p^p,$$

$$(\textit{resp. } P_{\text{princ}} = c_1T_1^{p^2} + \cdots + c_pT_p^{p^2} + c_{p+1}T_{p+1}^p + \cdots + c_{2p-1}T_{2p-1}^p,)$$

which vanishes nowhere over $k^p \setminus \{0\}$ *(resp.* $k^{2p-1} \setminus \{0\}$*).*

(b) Every unipotent k-groups of Rosenlicht type of dimension $3(p-1)$ *is k-isomorphic to a k-subgroup of* $\mathbb{G}_a^{3p-2}$ *defined as the kernel of a separable p-polynomial* $P(T_1, \dots, T_{3p-2})$ *with the principal part of the form*

$$P_{\text{princ}} = c_1T_1^{p^3} + \cdots + c_{2p}T_{2p}^{p^3} + c_{2p+1}T_{2p+1}^p + \cdots + c_{3p-2}T_{3p-2}^p,$$

or of the form

$$P_{\text{princ}} = c_1T_1^{p^3} + \cdots + c_pT_p^{p^3} + c_{p+1}T_{p+1}^{p^2} + \cdots + c_{2p-1}T_{2p-1}^{p^2} + c_{2p}T_{2p}^p + \cdots + c_{3p-2}T_{3p-2}^p,$$

which vanishes nowhere over $k^{3p-2} \setminus \{0\}$*.*

We now recall Oesterlé's construction associating a torus defined over an arbitrary field of positive characteristic with a smooth unipotent group defined and wound over that field (see [Oe, Chap. VI, 5]). Let k be a field of characteristic $p > 0$. Let T is a k-torus, k' a finite purely inseparable extension of k of degree p^n. Denote by $G = \prod_{k'/k}(T \times_k k')$, the Weil restriction from k' to k of T where T is considered as an algebraic group over k'. Then G is connected and commutative and T is a maximal torus of G. Denote by $U(T, k, k')$ (or simply by U) the quotient group G/T, then U is a k-wound unipotent group. We show that if k is a local function fields then the groups in the cckp-series of U, $\kappa^i(U)$ are of Rosenlicht type.

Proposition 13. *Let k be a local field of characteristic* $p > 0$*,* k' *a finite purely inseparable extension of k, T a k -torus. Let U be the unipotent group associated with T as above. Then the groups* $\kappa^i(G)$ *are of Rosenlicht type.*

In particular, if $\dim T = 1$*,* $[k' : k] = p$*, then U is k-isomorphic to a subgroups of* $\mathbb{G}_a^p$ *defined as the kernel of a p-polynomial of the form* $c_1T^p + \cdots + c_pT^p + T_p$*, where* $c_1, \dots, c_p$ *is a* k^p*-basis of k.*

Proof. Let $[k' : k] = p^n$. Since U is wound over k, $\kappa^i(G)$ are all of Rosenlicht type if and only if $\mathrm{H}^1(k, U)$ is finite by Proposition 9, which is in turn equivalent to the fact that both groups $\mathrm{H}^1(k, T)/p^n$ and ${}_{p^n}\mathrm{H}^2(k, T)$ are finite (see [TT1, Prop. 5.1]), where for an abelian A and a natural number n, A/n (respectively ${}_nA$) is the cokernel (respectively kernel) of the natural endomorphism $A \to A, x \mapsto nx$.

Let $X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ be the character group of T. Let $\mathrm{H}^0(k, X(T))^\wedge$ be the completion of the abelian group $\mathrm{H}^0(k, X(T))$ for the topology given by subgroups of finite index. Then by the Tate-Nakayama duality (see [Mi, Chapter I, Corollary 2.4]), there is a duality between the compact group $\mathrm{H}^0(k, X(T))^\wedge$ and the discrete group $\mathrm{H}^2(k, T)$. Furthermore, the group $\mathrm{H}^1(k, T)$ is finite. Since $X(T)$ is a free abelian group of finite rank, we deduce that the group ${}_{p^n}\mathrm{H}^2(k, T)$ is finite. Hence $\mathrm{H}^1(k, U)$ is finite and $\kappa^i(G)$ are all of Rosenlicht type.

Now assume that $[k' : k] = p$ and $\dim T = 1$. Then $\dim U = p-1$ and by Corollary 11, G is itself of Rosenlicht type. By Proposition 12, U is k-isomorphic to a subgroups of $\mathbb{G}_a^p$ defined as the kernel of a p-polynomial of the form $c_1T^p + \cdots + c_pT^p + aT_p$, where $c_1, \dots, c_p$ is a k^p-basis of k, and by changing variable we can take $a = 1$. This completes the proof the proposition. □

Remark.
(1) Let k be a field of characteristic $p > 0$ such that $[k^{1/p} : k] = p$, and t an element in $k - k^p$. Then, with $T = \mathbb{G}_m$ and $k' = k^{1/p}$, J. Oesterlé shows explicitly that $U = U(T, k, k')$ is k-isomorphic to a k-subgroup of $\mathbb{G}_a^p$ defined by the equation $x_0^p + tx_1^p + \cdots + t^{p-1}x_{p-1}^p = x_{p-1}$.
(2) Proposition 13 still holds true if "local field" is replaced by "global field" since the property of being of Rosenlicht type is unchanged under separable extentions.
(3) If k is a global function field and $\dim T > 0$ then the group of k-rational points of unipotent group U constructed as above is infinite (see [Oe, Chap. VI, 5]). Joseph Oesterlé even raised the following question:

Question (Oesterlé)**.** If a wound unipotent group G has an infinite number of rational points over a global function field K, does G have a subgroup defined over K of dimension ≥ 1 such that its underlying variety is K-unirational. Better, does G have a subgroup U of the type as above (associated with the Weil purely inseparable restriction of a torus)?

We conclude the paper by raising three questions concerning unipotent groups of Rosenlicht type, which may give some ideals about answering the above question of Oesterlé.

Question (I)**.** Does any unipotent group of Rosenlicht type arise as a group in the *cckp*-series of a unipotent group constructed by Oesterlé as above (associated with the Weil purely inseparable restriction of a torus)?

Question (II)**.** Let G be a unipotent group of Rosenlicht type over a global function field K of positive characteristic. Is the group of K-rational points $G(K)$ infinite? Is the underlying variety of G is K-unirational?

Question (III)**.** Let K be a global function fields, G a K-wound unipotent K-group. Assume that $G(K)$ is infinite. Is it true that G have a subgroup H of Rosenlicht type?

We note that if one has an affirmative answer for the second part of Oesterlé's question then one also has an affirmative answer for the Question III (see Proposition 13).

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