

# REDUCIBILITY OF THE INTERSECTIONS OF COMPONENTS OF A SPRINGER FIBER

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ABSTRACT. The description of the intersections of components of a Springer fiber is a very complex problem. Up to now only two cases were described completely. The complete picture for the hook case was obtained by N. Spaltenstein and J.A. Vargas, and for two row case it was obtained by F.Y.C. Fung. They showed in particular that the intersection of a pair of components of a Springer fiber is either irreducible or empty. In both cases all the components are non-singular and this irreducibility strongly related to the non-singularity. Here we use a bijection between orbital varieties and components of corresponding Springer fiber in  $GL_n$  which extends to the bijection between the irreducible components of the intersections of orbital varieties and the irreducible components of the intersections of components of Springer fiber preserving their codimensions, shown in [8], to compute the intersections of the irreducible components of Springer fibers for two column case. In this case in general the components are singular. As we show the intersection of two components is non-empty. The main result of the paper is a purely combinatorial necessary and sufficient condition for the intersection of two components of the Springer fiber to be irreducible in two column case; as an application of this characterization, we give the first examples of pairs of components which intersection is reducible and not of pure dimension.

## 1. INTRODUCTION

1.1. Let  $G$  denote the complex linear algebraic group  $GL_n$  with Lie algebra  $\mathfrak{g} = \mathfrak{gl}_n$  on which  $G$  acts by the adjoint action. For  $g \in G$  and  $u \in \mathfrak{g}$  we denote this action by  $g.u := gug^{-1}$ .

We fix the standard triangular decomposition  $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$  where  $\mathfrak{n}$  is the subalgebra of strictly upper-triangular matrices,  $\mathfrak{n}^-$  is the subalgebra of strictly lower triangular matrices and  $\mathfrak{h}$  is the subalgebra of diagonal matrices of  $\mathfrak{g}$ . Let  $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$  be the standard Borel subalgebra (w.r.t.  $\Pi$ ) so that  $\mathfrak{n}$  is its nilpotent radical. Let  $B$  be the (Borel) subgroup of invertible upper-triangular matrices in  $G$  and  $\mathfrak{b} := \text{Lie}(B)$ .

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The associated Weyl group is identified with the symmetric group  $\mathbf{S}_n = \langle s_i \rangle_{i=1}^{n-1}$  by taking  $s_i$  to be the elementary permutation interchanging  $i$  and  $i + 1$ .

Let  $\mathcal{F} := G/B$  denote the flag manifold. Let  $G \times^B \mathfrak{n}$  be the space obtained as the quotient of  $G \times \mathfrak{n}$  by the right action of  $B$  given by  $(g, x).b := (gb, b^{-1}.x)$  with  $g \in G$ ,  $x \in \mathfrak{n}$  and  $b \in B$ . By the Killing form we identify the space  $G \times^B \mathfrak{n}$  with the cotangent bundle of the flag manifold  $T^*(G/B)$ . Let  $g * x$  denote the class of  $(g, x)$ . The map  $G \times^B \mathfrak{n} \rightarrow \mathcal{F} \times \mathfrak{g}$ ,  $g * x \mapsto (gB, g.x)$  is an embedding which identifies  $G \times^B \mathfrak{n}$  with the following closed subvariety of  $\mathcal{F} \times \mathfrak{g}$  (see. [10, p. 19]):

$$\mathcal{Y} := \{(gB, x) \mid x \in g.\mathfrak{n}\}$$

The map  $f : G \times^B \mathfrak{n} \rightarrow \mathfrak{g}$ ,  $g * x \mapsto g.x$  is called the **Springer resolution**. It satisfies the following commutative diagram:

$$\begin{array}{ccc} G \times^B \mathfrak{n} & \xrightarrow{\simeq} & \mathcal{Y} \\ & \searrow f & \swarrow pr_2 \\ & & \mathfrak{g} \end{array}$$

where  $pr_2 : \mathcal{F} \times \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $(gB, x) \mapsto x$  is the natural projection. The map  $f$  is proper (because  $G/B$  is complete) and its image is exactly  $G.\mathfrak{n} = \mathcal{N}$ , the **nilpotent variety** of  $\mathfrak{g}$  (cf. [13]).

Let  $x$  be a nilpotent element in  $\mathfrak{n}$ . By the diagram above we have :

$$\mathcal{F}_x := f^{-1}(x) = \{gB \in \mathcal{F} \mid x \in g.\mathfrak{n}\} = \{gB \in \mathcal{F} \mid g^{-1}.x \in \mathfrak{n}\}. \quad (*)$$

The variety  $\mathcal{F}_x$  is called the **Springer fiber** above  $x$  and has been studied by many authors. They arise as fibers of Springer's resolution of singularities of the nilpotent cone in [10, 13]. In the course of these investigations, Springer defined  $\mathcal{W}$ -module structures on the rational homology groups  $H_*(\mathcal{F}_x, \mathbb{Q})$  on which also the finite group  $A(x) = Z_G(x)/Z_G^0(x)$  (where  $Z_G(x)$  is a stabilizer of  $x$  and  $Z_G^0(x)$  is its identity component) acts compatibly. Set  $d = \dim(\mathcal{F}_x)$ , the  $A(x)$ -fixed subspace  $H_{2d}(\mathcal{F}_x, \mathbb{Q})^{A(x)}$  of the top homology is known to be irreducible [14].

In [4], D. Kazhdan and G. Lusztig tried to understand Springer's work connecting nilpotent classes and representations of Weyl groups. Among problems they have posed, Conjecture 6.3 in [4] has stimulated the research of the relation between the Kazhdan-Lusztig basis and Springer fibers.

Let  $x \in \mathfrak{n}$  be some nilpotent element and let  $\mathcal{O}_x = G.x$  be its orbit. Consider  $\mathcal{O}_x \cap \mathfrak{n}$ . Its irreducible components are called **orbital varieties** associated to  $\mathcal{O}_x$ . By Spaltenstein's construction [12] there is a tight connection between  $\mathcal{F}_x$  and  $\mathcal{O}_x \cap \mathfrak{n}$  (see Section 2.1).

1.2. For  $x \in \mathfrak{n}$  its Jordan form is completely defined by  $\lambda = (\lambda_1, \dots, \lambda_k)$  a partition of  $n$  where  $\lambda_i$  is the length of  $i$ -th Jordan block. Arrange the numbers in a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  in the decreasing order (that is  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 1$ ) and write  $J(x) = \lambda$ . Note that the nilpotent orbit  $\mathcal{O}_x$  is completely defined by  $J(x)$ . We set  $\mathcal{O}_{J(x)} := \mathcal{O}_x$  and  $\text{sh}(\mathcal{O}_x) := J(x)$ .

In turn an ordered partition can be presented as a Young diagram  $D_\lambda$  – an array with  $k$  rows of boxes starting on the left with the  $i$ -th row containing  $\lambda_i$  boxes. In such a way there is a bijection between Springer fibers (resp. nilpotent orbits) and Young diagrams.

Fill the boxes of Young diagram  $D_\lambda$  with  $n$  distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom we call such an array a Young tableau or simply a tableau of shape  $\lambda$ . Let  $\mathbf{Tab}_\lambda$  be the set of all Young tableaux of shape  $\lambda$ . For  $T \in \mathbf{Tab}_\lambda$  we put  $\text{sh}(T) := \lambda$ .

Given  $x \in \mathfrak{n}$  such that  $J(x) = \lambda$  by Spaltenstein ([11]) and Steinberg ([16]) there is a bijection between the set of irreducible components of  $\mathcal{F}_x$  (resp. orbital varieties associated to  $\mathcal{O}_\lambda$ ) and  $\mathbf{Tab}_\lambda$  (cf. Section 2.2). For  $T \in \mathbf{Tab}_\lambda$ , set  $\mathcal{F}_T$  to be the corresponding component of  $\mathcal{F}_x$ . Respectively set  $\mathcal{V}_T$  to be the corresponding orbital variety associated to  $\mathcal{O}_\lambda$ . Moreover as it was established in [8] (cf. Section 2.1) the number of irreducible components and their codimensions in  $\mathcal{F}_T \cap \mathcal{F}_{T'}$  is equal to the number of irreducible components and their codimensions in  $\mathcal{V}_T \cap \mathcal{V}_{T'}$ . Thus, the intersections of orbital varieties are equivalent to the intersections of the irreducible components of the Springer fiber.

The conjecture of Kazhdan and Lusztig mentioned above is equivalent to the irreducibility of certain characteristic varieties [1, Conjecture 4]. They were shown to be reducible in general by Kashiwara and Saito [3]. Nevertheless, the description of pairwise intersections of the irreducible components of the Springer fibers is still open.

The complete picture of the intersections of the components were described by J.A. Vargas for hook case in [18] and by F.Y.C. Fung for two row case in [2]. Both in hook and two-row cases, all the components are non-singular, all the intersections are irreducible or empty.

In this paper we consider the intersections of a pair of components for two column case (that is  $\lambda = (2, 2, \dots)$ ). The two column case and the hook case are two extreme cases in the following sense: For all the nilpotent orbits of the given rank  $k$  the orbit  $\lambda = (k, 1, 1, \dots)$  is the most nondegenerate and the orbit  $\lambda = (2, 2, \dots)$  (with dual partition  $\lambda^* = (n - k, k)$ ) is the most degenerate, in the following sense  $\overline{\mathcal{O}}_{(k, 1, \dots)} \supset \overline{\mathcal{O}}_\mu \supset \mathcal{O}_{(2, \dots, 2, 1, \dots)}$ . However, it seems that the general picture must be

more close to the two column case than to the hook case, which is too simple and beautiful.

1.3. In general we have only Steinberg's construction for orbital varieties and it is not much more satisfactory from geometric point of view than the construction of irreducible components of  $\mathcal{F}_x$ . However there is a nice exclusion of orbital varieties in  $\mathfrak{gl}_n$  associated to nilpotent orbits of nilpotent order 2 (that is corresponding to Young diagrams with two columns). Here each orbital variety is union of finite number of  $B$ -orbits and we can apply [7] to get the full picture of intersections of orbital varieties. In [7] the special so called rank matrix is attached to a  $B$ -orbit. It defines the corresponding  $B$ -orbit completely. Here we use the technique of these matrices to determine the intersection of two orbital varieties of nilpotent order two. In particular we show that the intersection of two orbital varieties associated to an orbit of nilpotent order 2 is not empty (see Proposition 3.15). We give the purely combinatorial and easy to compute necessary and sufficient condition for the irreducibility of the intersection of two orbital varieties of nilpotent order 2 and provide some examples showing that in general such intersections are reducible and not necessary equidimensional (see examples in Section 3.8).

In the subsequent paper (cf. [9]), we show that the intersections of codimension 1 in two column case are irreducible. This together with computations in low rank cases permits us to conjecture

**Conjecture 1.1.** Given  $S, T \in \mathbf{Tab}_\lambda$ . If  $\text{codim}_{\mathcal{F}_S} \mathcal{F}_T \cap \mathcal{F}_S = 1$  then  $\mathcal{F}_T \cap \mathcal{F}_S$  is irreducible.

Let us now give a brief outline of the contents of the paper.

- To make the paper as self content as possible we explain in short in Section 2 Spaltenstein's and Steinberg's constructions and quote the connected results essential in further analysis.
- In Section 3 we provide the main result of this paper namely a purely combinatorial necessary and sufficient condition for the intersection of two components of the Springer fiber to be irreducible in two column case; as an application of this characterization, we give the first examples for which the intersections of two components of the Springer fiber are reducible and are not of pure dimension. This is the most technical part of the paper.
- In Section 4 give some other counter-examples related to the construction of orbital varieties and to a geometric property relating their intersections in codimension one.

## 2. GENERAL CONSTRUCTION

2.1. Given  $x \in \mathfrak{n}$  denote  $G_x = \{g \in G \mid g^{-1}xg \in \mathfrak{n}\}$ . Set  $f_1 : G_x \rightarrow \mathcal{O}_x \cap \mathfrak{n}$  by  $f_1(g) = g.x$  and  $f_2 : G_x \rightarrow \mathcal{F}_x$  by  $f_2(g) = gB$ . Define  $\pi : \mathcal{F}_x \rightarrow \mathcal{O}_x \cap \mathfrak{n}$ ,  $gB \mapsto \pi(gB) := f_1(f_2^{-1}(gB))$ . Then Spaltenstein showed that  $\pi$  induces a surjection  $\hat{\pi}$  from the set of irreducible components of  $\mathcal{F}_x$  to the set of irreducible components of  $\mathcal{O}_x \cap \mathfrak{n}$ , moreover the fiber of this surjective map is exactly an orbit under the action of the component group  $A(x) := Z_G(x)/Z_G^o(x)$  (cf. [12]). He showed also that  $\mathcal{F}_x$  and  $\mathcal{O}_x \cap \mathfrak{n}$  are equidimensional and got the following useful relations:

$$(2.1) \quad \dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(Z_G(x)) = \dim(\mathcal{F}_x) + \dim(B)$$

$$(2.2) \quad \dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(\mathcal{F}_x) = \dim(\mathfrak{n})$$

$$(2.3) \quad \dim(\mathcal{O}_x \cap \mathfrak{n}) = \frac{1}{2} \dim(\mathcal{O}_x).$$

In our setting, for the case  $G = \mathrm{GL}_n$ , the component is always trivial, so  $\hat{\pi}$  is actually a bijection. As an extension of his work, we established in [8] the following result

**Proposition 2.1.** *Let  $x \in \mathfrak{n}$  and let  $\mathcal{F}_1, \mathcal{F}_2$  be two irreducible components of  $\mathcal{F}_x$  and  $\{\mathcal{E}_l\}_{l=1}^t$  the set of irreducible components of  $\mathcal{F}_1 \cap \mathcal{F}_2$ . Then  $\{\pi(\mathcal{E}_l)\}_{l=1}^t$  is exactly the set of irreducible components of  $\mathcal{V}_1 \cap \mathcal{V}_2$  and  $\mathrm{codim}_{\mathcal{F}_1}(\mathcal{E}_l) = \mathrm{codim}_{\mathcal{V}_1}(\pi(\mathcal{E}_l))$ .*

This simple proposition shows that in the case of  $\mathrm{GL}_n$ , orbital varieties associated to  $\mathcal{O}_x$  are equivalent to the irreducible components of  $\mathcal{F}_x$ .

2.2. The parametrization of the irreducible components of  $\mathcal{F}_x$  in  $\mathrm{GL}_n$  by standard Young tableaux is as follows.

In this case  $\mathcal{F}$  is identified with the set of complete flags  $\xi = (V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$  and  $\mathcal{F}_x \cong \{\xi = (V_i) \in \mathcal{F} \mid x(V_i) \subset V_{i-1}\}$ .

Given  $x \in \mathfrak{n}$  let  $J(x) = \lambda$ . By a slight abuse of notation we will not distinguish between the partition  $\lambda$  and its Young diagram. By R. Steinberg [17] and N. Spaltenstein [11] we have a parametrization of the irreducible components of  $\mathcal{F}_x$  by the set  $\mathbf{Tab}_\lambda$ : Let  $\xi = (V_i) \in \mathcal{F}_x$ , then we get a saturated chain in the poset of Young diagrams

$$\mathrm{St}(\xi) := (J(x) \supset J(x|_{V_{n-1}}) \supset \cdots \supset J(x|_{V_2}) \supset J(x|_{V_1}))$$

where  $x|_{V_i}$  is the nilpotent endomorphism induced by  $x$  by restriction to the subspace  $V_i$  and  $J(x|_{V_{i+1}})$  differs from  $J(x|_{V_i})$  by one corner box. It is easy to see that the data of such a saturated chain is equivalent to give a standard Young tableau. So we get a map  $\mathrm{St} : \mathcal{F}_x \rightarrow \mathbf{Tab}_\lambda$ . Then the collection  $\{\mathrm{St}^{-1}(T)\}_{T \in \mathbf{Tab}_\lambda}$  is a partition of  $\mathcal{F}_x$  into

smooth irreducible subvarieties of the same dimension and  $\overline{\{\text{St}^{-1}(T)\}}_{T \in \mathbf{Tab}_\lambda}$  are the set of the irreducible components of  $\mathcal{F}_x$  which will be denoted by  $\mathcal{F}_T := \overline{\text{St}^{-1}(T)}$  where  $T \in \mathbf{Tab}_\lambda$ .

On the level of orbital varieties the construction is as follows. For  $1 \leq i < j \leq n$  consider the canonical projections  $\pi_{i,j} : \mathfrak{n}_n \rightarrow \mathfrak{n}_{j-i+1}$  acting on a matrix by deleting the first  $i-1$  columns and rows and the last  $n-j$  columns and rows. For any  $u \in \mathcal{O}_\lambda \cap \mathfrak{n}$  set  $J_n(u) := J(u) = \lambda$  and  $J_{n-i}(u) := J(\pi_{1,n-i}(u))$  for any  $i : 1 \leq i \leq n-1$ . Exactly as in the previous construction we get a standard Young tableau corresponding to the saturated chain  $(J_n(u) \supset \dots \supset J_1(u))$ , therefore we get a map  $\text{St}_1 : \mathcal{O}_\lambda \cap \mathfrak{n} \rightarrow \mathbf{Tab}_\lambda$ . Again the collection  $\{\text{St}_1^{-1}(T)\}_{T \in \mathbf{Tab}_\lambda}$  is a partition of  $\mathcal{O}_\lambda \cap \mathfrak{n}$  into smooth irreducible subvarieties of the same dimensions and  $\overline{\{\text{St}_1^{-1}(T) \cap \mathcal{O}_\lambda\}}_{T \in \mathbf{Tab}_\lambda}$  are orbital varieties associated to  $\mathcal{O}_\lambda$ . Put  $\mathcal{V}_T := \overline{\text{St}_1^{-1}(T) \cap \mathcal{O}_\lambda}$  where  $T \in \mathbf{Tab}_\lambda$ ; in particular,  $\coprod_{\lambda \vdash n} \mathbf{Tab}_\lambda$  parameterizes the set of orbital varieties contained in  $\mathfrak{n}$ .

2.3. A general construction for orbital varieties by R. Steinberg (cf. [16]) is as follows. For  $w \in \mathbf{S}_n$  consider the subspace

$$\mathfrak{n} \cap {}^w \mathfrak{n} := \bigoplus_{\alpha \in \mathcal{R}^+ \cap {}^w \mathcal{R}^+} \mathfrak{g}_\alpha$$

contained in  $\mathfrak{n}$ . Then  $G \cdot (\mathfrak{n} \cap {}^w \mathfrak{n})$  is an irreducible locally closed subvariety of the nilpotent variety  $\mathcal{N}$ . Since  $\mathcal{N}$  is a finite union of nilpotent orbits, it follows that there is a unique nilpotent orbit  $\mathcal{O}$  such that  $\overline{G \cdot (\mathfrak{n} \cap {}^w \mathfrak{n})} = \overline{\mathcal{O}}$ . Moreover  $\overline{B \cdot (\mathfrak{n} \cap {}^w \mathfrak{n})} \cap \mathcal{O}$  is an orbital variety associated to  $\mathcal{O}$  and the fundamental result in Steinberg's work is that every orbital variety can be obtained in this way [16]; in particular there is a surjective map  $\varphi : \mathbf{S}_n \rightarrow \coprod_{\lambda \vdash n} \mathbf{Tab}_\lambda$ . The preimages of this map  $\mathcal{C}_T := \varphi^{-1}(T)$  are called the **geometric** (or **left**) **cells** of  $\mathbf{S}_n$ . The geometric cells are given by Robinson-Schensted correspondence, namely for  $T \in \mathbf{Tab}_\lambda$ , one has  $\mathcal{C}_T = \{\text{RS}(T, S) : S \in \mathbf{Tab}_\lambda\}$ , where RS represents the Robinson-Schensted correspondence.

### 3. TWO COLUMN CASE

3.1. In this section we use intensively the results of [7] and we adopt its notation. Set  $\mathcal{X}_2 := \{x \in \mathfrak{n} \mid x^2 = 0\}$  to be the variety of nilpotent upper-triangular matrices of nilpotent order 2. Denote  $\mathbf{S}_n^2 := \{\sigma \in \mathbf{S}_n \mid \sigma^2 = id\}$  the set of involutions of  $\mathbf{S}_n$ . For every  $\sigma \in \mathbf{S}_n^2$ , set  $N_\sigma$  to be the "strictly upper-triangular part" of its corresponding

permutation matrix, that is

$$(3.1) \quad (N_\sigma)_{i,j} := \begin{cases} 1 & \text{if } i < j \text{ and } \sigma(i) = j; \\ 0 & \text{otherwise.} \end{cases}$$

Let  $\mathbf{Tab}_n^2$  be the set of all Young tableaux of size  $n$  with two columns. For  $T \in \mathbf{Tab}_n^2$ ,

write it as  $T = (T_1, T_2)$ , where  $T_1 = \begin{pmatrix} t_{1,1} \\ \vdots \\ t_{n-k,1} \end{pmatrix}$  is the first column of  $T$  and

$T_2 = \begin{pmatrix} t_{1,2} \\ \vdots \\ t_{k,2} \end{pmatrix}$  is the second column of  $T$ . And define the following involution

$$(3.2) \quad \sigma_T := (i_1, j_1) \dots (i_k, j_k)$$

where  $j_s := t_{s,2}$ ;  $i_1 := t_{1,2} - 1$ , and  $i_s := \max\{d \in T_1 - \{i_1, \dots, i_{s-1}\} \mid d < j_s\}$  for any  $s > 1$ . For example, take

$$T = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$$

Then  $\sigma_T = (3, 4)(2, 5)(6, 7)(1, 8)$ .

**Remark 3.1.** To define  $T \in \mathbf{Tab}_n$  it is enough to know columns  $T_i$  as sets (we denote them by  $\langle T_i \rangle$ ), or equivalently the different column positions  $c_T(i)$  of integers  $i : 1 \leq i \leq n$  since the entries increase from up to down in the columns. Thus given  $\sigma_T$  we can reconstruct  $T$ . Indeed,  $\langle T_2 \rangle = \{j_1, \dots, j_k\}$  and  $\langle T_1 \rangle = \{i\}_{i=1}^n - \langle T_2 \rangle$ ."

One has

**Theorem 3.2.** ([6, 2.2], [5, 4.13])

(i) *The variety  $\mathcal{X}_2$  is a finite union of  $B$ -orbits, namely*

$$\mathcal{X}_2 = \coprod_{\sigma \in \mathbf{S}_n^2} B.N_\sigma.$$

(ii) *For any  $T \in \mathbf{Tab}_n^2$ , one has  $\overline{\mathcal{V}}_T = \overline{B.N}_{\sigma_T}$ .*

The finiteness property is particular for  $\mathcal{X}_2$ . The fact that each orbital variety has a dense  $B$ -orbit is also particular for very few types of nilpotent orbits including orbits of nilpotent order 2 (cf. [5]). The first property permits us to compute the

intersections of any two  $B$ -orbit closures in  $\mathcal{X}_2$ . The second one permits us to apply the results to the intersections of orbital varieties of nilpotent order 2.

We begin with the general theory of the intersections of  $\overline{B.N}_\sigma$  for  $\sigma \in \mathbf{S}_n^2$ .

3.2. In this section we prefer to use the dual transposition  $\lambda^*$  instead of  $\lambda$  since it will be more convenient to write it down for nilpotent orbits of nilpotent order 2. Indeed, for  $x \in \mathcal{X}_2$  one has  $J^*(x) = (n - k, k)$  where  $k$  is number of Jordan blocks of length two in  $J(x)$ .

**Remark 3.3.** For every element  $x \in \mathcal{X}_2$ , the integer  $\text{rk}(x)$  is exactly the number of blocks of length 2 in  $J(x)$ , so it defines the  $\text{GL}_n$ -orbit of  $x$ .

Any element  $\sigma \in \mathbf{S}_n^2$  can be written as a product of disjoint cycles of length 2. Order elements in increasing order inside the cycle and order cycles in increasing order according to the first entries. In that way we get a unique writing of every involution. Thus,  $\sigma = (i_1, j_1)(i_2, j_2) \dots (i_k, j_k)$  where  $i_s < j_s$  for any  $1 \leq s \leq k$  and  $i_s < i_{s+1}$  for any  $1 \leq s < k$ . Set  $L(\sigma) := k$  [do not confuse this notation with the length function], and denote  $\mathcal{O}_\sigma$  the  $\text{GL}_n$ -orbit of  $N_\sigma$ . By definition we have  $L(\sigma) = \text{rk}(N_\sigma)$ .

Let us define the following number

$$(3.3) \quad r_s(\sigma) := \text{card}\{i_p < i_s \mid j_p < j_s\} + \text{card}\{j_p \mid j_p < i_s\}.$$

Note that the definition of  $r_s(\sigma)$  is independent of ordering cycles in increasing order according to the first entries. However if it is ordered then  $r_1(\sigma) = 0$  and to compute  $r_s(\sigma)$  it is enough to check only the pairs  $(i_p, j_p)$  where  $p < s$ . For example, take  $\sigma = (1, 6)(3, 4)(5, 7)$ . Then  $L(\sigma) = 3$  and  $r_1(\sigma) = 0$ ,  $r_2(\sigma) = 0$ ,  $r_3(\sigma) = 2 + 1 = 3$ . By [6, 3.1] one has

**Theorem 3.4.** For  $\sigma = (i_1, j_1)(i_2, j_2) \dots (i_k, j_k) \in \mathbf{S}_n^2$  one has

$$\dim(B.N_\sigma) = kn - \sum_{s=1}^k (j_s - i_s) - \sum_{s=2}^k r_s(\sigma).$$

**Remark 3.5.** By Theorem 3.2 (ii), the orbits  $B.N_{\sigma_T}$  (where  $(\text{sh}(T))^* = (n - k, k)$ ) are the only  $B$ -orbits of maximal dimension inside the variety  $\mathcal{O}_{(n-k, k)^*} \cap \mathfrak{n}$  and  $\dim(B.N_{\sigma_T}) = k(n - k)$ : Indeed any orbit  $B.N_\sigma$  is irreducible and therefore lies inside an orbital variety  $\mathcal{V}_T$ , in particular it lies in  $\overline{\mathcal{V}}_T$ , so if  $\dim B.N_\sigma = \dim \mathcal{V}_T$  we get that  $\overline{B.N}_\sigma = \overline{\mathcal{V}}_T$  thus by Theorem 3.2 (ii)  $\overline{B.N}_\sigma = \overline{B.N}_{\sigma_T}$  which provides  $\sigma = \sigma_T$ .

In particular if  $\sigma = (i_1, j_1) \dots (i_k, j_k)$  is such that  $\dim(B.N_\sigma) = k(n - k)$ , then  $\sigma = \sigma_T$  where  $T$  is the tableau obtained by

$$c_T(s) = \begin{cases} 2 & \text{if } s = j_p \text{ for some } 1 \leq p \leq k \\ 1 & \text{otherwise} \end{cases}$$

3.3. In [7] the combinatorial description of  $\overline{B.N}_\sigma$  (with respect to Zariski topology) for  $\sigma \in \mathbf{S}_n^2$  is provided. Let us formulate this result.

Recall from Section 2.2 the notion  $\pi_{i,j} : \mathfrak{n}_n \rightarrow \mathfrak{n}_{j-i+1}$  and define the rank matrix  $R_x$  of  $x \in \mathfrak{n}$  to be

$$(3.4) \quad (R_x)_{i,j} := \begin{cases} 0 & \text{if } i \geq j; \\ \text{rk}(\pi_{i,j}(x)) & \text{otherwise.} \end{cases}$$

Notice that for any element  $b \in B$ ,  $\pi_{i,j}(b)$  is an invertible upper-triangular matrix in  $\text{GL}_{j-i+1}$ . Therefore we can define an action of  $B$  on  $\mathfrak{n}_{j-i+1}$  by:  $b.y := \pi_{i,j}(b).y$  for  $y \in \mathfrak{n}_{j-i+1}$  and  $b \in B$ .

Let us first establish a result

**Lemma 3.6.** (i) *If  $x, y \in \mathfrak{n}$  are in the same  $B$ -orbit, then they have the same rank matrix.*  
 (ii) *The morphism  $\pi_{i,j}$  is  $B$ -invariant.*

*Proof.* Note that for any two upper-triangular matrices  $a, b$  and for any  $i, j : 1 \leq i < j \leq n$  one has  $\pi_{i,j}(ab) = \pi_{i,j}(a)\pi_{i,j}(b)$ . In particular, if  $a \in B$  then  $\pi_{i,j}(a^{-1}) = (\pi_{i,j}(a))^{-1}$ . Applying this to  $x \in \mathfrak{n}$  and  $y$  in its  $B$  orbit that is  $y = b.x$  for some  $b \in B$  we get  $\pi_{i,j}(y) = \pi_{i,j}(b).\pi_{i,j}(x)$  so that the morphism  $\pi_{i,j}$  is  $B$ -invariant and in particular  $\text{rk}(\pi_{i,j}(y)) = \text{rk}(\pi_{i,j}(x))$ . Hence  $R_x = R_y$ .  $\square$

By this lemma we can define  $R_\sigma := R_{N_\sigma}$  as the rank matrix associated to orbit  $B.N_\sigma$ .

**Remark 3.7.** Note that computation of  $(R_{N_\sigma})_{i,j}$  is trivial – this is exactly the number of non-zero entries in submatrix of  $1, \dots, j$  columns and  $i, \dots, n$  rows of  $N_\sigma$  or in other words the number of ones in  $N_\sigma$  to the left-below of position  $(i, j)$  (including position  $(i, j)$ ).

Let  $\mathbb{Z}^+$  be the set of non-negative integers. Put  $\mathbf{R}_n^2 := \{R_\sigma \mid \sigma \in \mathbf{S}_n^2\}$ . By [7, 3.1, 3.3] one has

**Proposition 3.8.**  *$R = (R_{i,j}) \in M_{n \times n}(\mathbb{Z}^+)$  belongs to  $\mathbf{R}_n^2$  if and only if it satisfies*

- (i)  $R_{i,j} = 0$  if  $i \geq j$ ;
- (ii) For  $i < j$  one has  $R_{i+1,j} \leq R_{i,j} \leq R_{i+1,j} + 1$  and  $R_{i,j-1} \leq R_{i,j} \leq R_{i,j-1} + 1$ ;

- (iii) If  $R_{i,j} = R_{i+1,j} + 1 = R_{i,j-1} + 1 = R_{i+1,j-1} + 1$  then
- (a)  $R_{i,k} = R_{i+1,k}$  for any  $k < j$  and  $R_{i,k} = R_{i+1,k} + 1$  for any  $k \geq j$ ;
  - (b)  $R_{k,j} = R_{k,j-1}$  for any  $k > i$  and  $R_{k,j} = R_{k,j-1} + 1$  for any  $k \leq i$ ;
  - (c)  $R_{j,k} = R_{j+1,k}$  and  $R_{k,i} = R_{k,i-1}$  for any  $k : 1 \leq k \leq n$ .

Fix  $\sigma \in \mathbf{R}_n^2$ , then the conditions (i) and (ii) are obvious from Remark 3.7, and the conditions (iii) appears exactly for the coordinates  $(i, j)$  in the matrix when  $j = \sigma(i)$ , with  $i < j$ ; we draw the following picture (see Figure 1 below) to help the reader to visualize the constraints (a), (b), (c) of (iii), with the following rule: the integers which are inside a same white polygon, are equal, and the integers in a same gray rectangle differ by one.

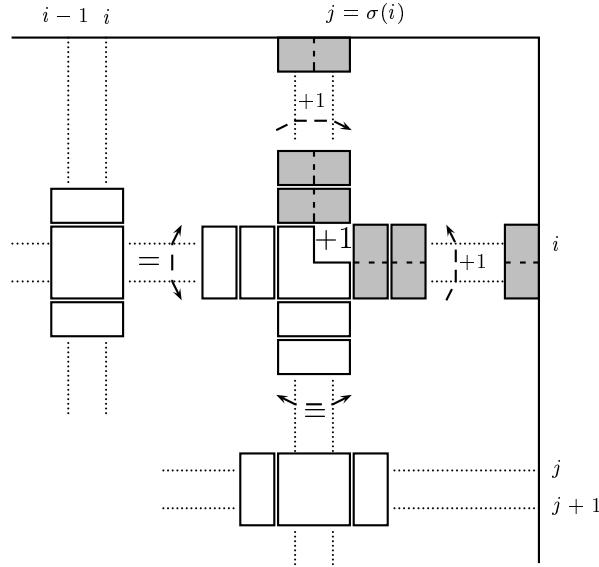


FIGURE 1.

The first part of (c) can be explained in the following: since the integer  $j$  appears already in the second entry of the cycle  $(i, j)$ , so it can not appear again in any other cycle; therefore in the matrix  $N_\sigma$ , the integers of the  $j^{\text{th}}$  row are all 0, and that explains why we should have  $(R_\sigma)_{j,k} = (R_\sigma)_{j+1,k}$  for  $1 \leq k \leq n$ ; the same explanation can also be done for the second part of (c).

When the constrain (iii) appears, let us call the couple  $(i, j)$  a **position** of constrain (iii).

**Remark 3.9.** If two horizontal (resp. vertical) consecutive boxes of a matrix in  $\mathbf{R}_n^2$  differ by one, then it is also the same for any consecutive horizontal (resp. vertical) boxes above (resp. on the right).

As an immediate corollary of Proposition 3.8 we get

**Lemma 3.10.** *Let  $\sigma, \sigma_1$  and  $\sigma_2$  be involutions such that  $\sigma = \sigma_1 \cdot \sigma_2$  and  $L(\sigma) = L(\sigma_1) + L(\sigma_2)$ , then  $R_\sigma = R_{\sigma_1} + R_{\sigma_2}$ ; in particular we have  $\sigma_1, \sigma_2 \preceq \sigma$ .*

*Proof.* The hypothesis  $L(\sigma) = L(\sigma_1) + L(\sigma_2)$  means exactly that any integer appearing a cycle of  $\sigma_1$  does not appear in any cycle of  $\sigma_2$  and conversely [notice that it is also equivalent to say  $\sigma_1 \cdot \sigma_2 = \sigma_2 \cdot \sigma_1 = \sigma$ ]; this means in particular that when the coefficient 1 appears in the matrix  $R_{\sigma_1}$  for the coordinate  $(i, \sigma_1(i))$ , then it can not appear in the  $i^{\text{th}}$  line and in the  $\sigma_1(i)^{\text{th}}$  column of  $R_{\sigma_2}$  and conversely; therefore we get  $N_\sigma = N_{\sigma_1} + N_{\sigma_2}$  and the result follows.  $\square$

**Remark 3.11.** In what follows we identify  $\mathbf{R}_n^2$  with  $\mathbf{S}_n^2$ , and consider  $\mathbf{Tab}_n^2$  as a subset of  $\mathbf{S}_n^2$  via  $\sigma_T$ .

3.4. Define the following partial order on  $M_{n \times n}(\mathbb{Z}^+)$ . For  $A, B \in M_{n \times n}(\mathbb{Z}^+)$  put  $A \preceq B$  if for any  $i, j : 1 \leq i, j \leq n$  one has  $A_{i,j} \leq B_{i,j}$ .

The restriction of this order to  $\mathbf{R}_n^2$  induces a partial order on  $\mathbf{S}_n^2$  by setting  $\sigma' \preceq \sigma$  if  $R_{\sigma'} \preceq R_\sigma$  for  $\sigma, \sigma' \in \mathbf{S}_n^2$ . By [7, 3.5] this partial order describes the closures of  $B.N_\sigma$  for  $\sigma \in \mathbf{S}_n^2$ . Combining [7, 3.5] with Remark 3.5 we get

**Theorem 3.12.** *For any  $\sigma \in \mathbf{S}_n^2$ , one has*

$$\overline{B.N_\sigma} = \coprod_{\sigma' \preceq \sigma} B.N_{\sigma'}.$$

In particular, for  $T \in \mathbf{Tab}_n^2$

$$\mathcal{V}_T = \coprod_{\substack{\sigma' \preceq \sigma_T \\ L(\sigma') = L(\sigma_T)}} B.N_{\sigma'}.$$

3.5. Let  $\pi_{i,j} : \mathfrak{n}_n \rightarrow \mathfrak{n}_{j-i+1}$ . If we denote by  $\hat{\pi}_{s,t} : \mathfrak{n}_{j-i+1} \rightarrow \mathfrak{n}_{t-s+1}$  the same projection, but with the starting-space  $\mathfrak{n}_{j-i+1}$ , then we can easily check the following relation:

$$(3.5) \quad \hat{\pi}_{s,t} \circ \pi_{i,j} = \pi_{s+i-1, t+i-1}.$$

Now if  $R \in \mathbf{R}_n^2$ , it obvious by Remark 3.7 that  $\pi_{i,j}(R)$  fulfill the constraints (i), (ii) and (iii) of Proposition 3.8. Thus, we get

**Lemma 3.13.** *If  $R \in \mathbf{R}_n^2$ , then  $\pi_{i,j}(R) \in \mathbf{R}_{j-i+1}^2$  for  $1 \leq i \leq j \leq n$ .*



so that

$$(R_{\sigma_{\sigma(k)}})_{i,j} = \begin{cases} j - i + 1 - (n - k) & \text{if } j - i > n - k - 1 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand by Proposition 3.8 (ii) for any  $\sigma \in \mathbf{S}_n^2$  one has  $(R_\sigma)_{i,j} \geq (R_\sigma)_{i-1,j} - 1 \geq (R_\sigma)_{i-2,j} - 2 \cdots \geq (R_\sigma)_{1,j} - (i - 1)$ . In turn  $(R_\sigma)_{1,j} \geq (R_\sigma)_{1,j+1} - 1 \geq \cdots \geq (R_\sigma)_{1,n} - (n - j)$  so that  $(R_\sigma)_{i,j} \geq (R_\sigma)_{1,n} - (n - j + i - 1)$ . Thus, for any  $\sigma \in \mathbf{S}_n^2(k)$  one has  $(R_\sigma)_{i,j} \geq j - i + 1 - (n - k)$ . As well one has  $(R_\sigma)_{i,j} \geq 0$  so that  $(R_\sigma)_{i,j} \geq \max\{0, j - i + 1 - (n - k)\} = (R_{\sigma_{\sigma(k)}})_{i,j}$ . Thus,  $\sigma \succeq \sigma_{\sigma(k)}$ .

The second part is now a consequence of this result and Theorem 3.12.  $\square$

3.7. Given  $\sigma, \sigma' \in \mathbf{S}_n^2$  we define  $R_{\sigma, \sigma'}$  by

$$(3.8) \quad (R_{\sigma, \sigma'})_{i,j} := \min\{(R_\sigma)_{i,j}, (R_{\sigma'})_{i,j}\}.$$

One has

**Theorem 3.16 (Main Theorem).** *For any  $\sigma, \sigma' \in \mathbf{S}_n^2$  one has*

$$\overline{B \cdot N}_\sigma \cap \overline{B \cdot N}_{\sigma'} = \coprod_{R_\zeta \preceq R_{\sigma, \sigma'}} B \cdot N_\zeta.$$

*This intersection is irreducible if and only if  $R_{\sigma, \sigma'} \in \mathbf{R}_n^2$ .*

*Proof.* To establish this equivalence we need only to prove the “only if” part and to do it we need some preliminary result.

**Lemma 3.17.** *Suppose that  $\overline{B \cdot N}_\sigma \cap \overline{B \cdot N}_{\sigma'}$  is irreducible. Denote  $B'$  the Borel subgroup in  $\mathrm{GL}_{j-i+1}$ . Then  $\overline{B' \cdot N}_{\pi_{i,j}(\sigma)} \cap \overline{B' \cdot N}_{\pi_{i,j}(\sigma')}$  is irreducible.*

*Proof.* Let  $\alpha, \beta$  be two maximal involutions in  $\mathbf{S}_{\langle i,j \rangle}^2$  such  $\alpha, \beta \preceq \pi_{i,j}(\sigma), \pi_{i,j}(\sigma')$ . By Remark 3.14 (ii), we have also  $\alpha, \beta \preceq \sigma, \sigma'$ . By hypothesis we have  $\overline{B \cdot N}_\sigma \cap \overline{B \cdot N}_{\sigma'} = \overline{B \cdot N}_\delta$  for an element  $\delta \in \mathbf{S}_n^2$ . In particular we get  $\alpha, \beta \preceq \delta$ . By Remarks 3.14 (i) and (iii) we get  $\alpha = \pi_{i,j}(\alpha), \beta = \pi_{i,j}(\beta) \preceq \pi_{i,j}(\delta) \preceq \pi_{i,j}(\sigma), \pi_{i,j}(\sigma')$ . Since  $\alpha$  and  $\beta$  are maximal, we get  $\alpha = \beta = \pi_{i,j}(\delta)$ .  $\square$

Note that for  $n = 3$  all the intersections are irreducible so that the claim is trivially true.

Let now  $n$  be minimal such that  $\overline{B \cdot N}_\sigma \cap \overline{B \cdot N}_{\sigma'}$  is irreducible and  $R_{\sigma, \sigma'} \notin \mathbf{R}_n^2$ . Note that constrains (i) and (ii) of Proposition 3.8 are satisfied by any  $R_{\sigma, \sigma'}$ . If  $R_{\sigma, \sigma'} \notin \mathbf{R}_n^2$  then at least one of the conditions (a), (b) and (c) of the constrain (iii) of Proposition 3.8 is not fulfilled. By symmetry around the anti diagonal it is enough to check only Condition (a) and the first part of Condition (c).

As for the first relation in (3.6), we can easily check that

$$(3.9) \quad R_{\pi_{i,j}(\sigma), \pi_{i,j}(\sigma')} = \pi_{i,j}(R_{\sigma, \sigma'}).$$

Let  $B'$  be the Borel subgroup of  $\mathrm{GL}_{n-1}$ . By Lemma 3.17 and Relation (3.9), we deduce that the varieties  $\overline{B' \cdot N}_{\pi_{1,n-1}(\sigma)} \cap \overline{B' \cdot N}_{\pi_{1,n-1}(\sigma')}$ ,  $\overline{B' \cdot N}_{\pi_{2,n}(\sigma)} \cap \overline{B' \cdot N}_{\pi_{2,n}(\sigma')}$  are irreducible so by induction hypothesis

$$(3.10) \quad \pi_{1,n-1}(R_{\sigma, \sigma'}), \pi_{2,n}(R_{\sigma, \sigma'}) \in \mathbf{R}_{n-1}^2,$$

Put  $\zeta \in \mathbf{S}_{n-1}^2$  to be such that  $R_\zeta = \pi_{1,n-1}(R_{\sigma, \sigma'})$  and  $\eta \in \mathbf{S}_{\langle 2, n \rangle}^2$  be such that  $R_\eta = \pi_{2,n}(R_{\sigma, \sigma'})$ .

Suppose that  $R_{\sigma, \sigma'} \notin \mathbf{R}_n^2$ , denote  $(i_o, j_o)$  the position of a constrain (iii) 

|   |     |
|---|-----|
| k | k+1 |
| k | k   |

 which is not satisfied by the matrix  $R_{\sigma, \sigma'}$ .

Condition (a): If the first part of Condition (a) is not satisfied, it means that we can find two horizontal consecutive boxes below of the two boxes 

|   |   |
|---|---|
| k | k |
|---|---|

 which differ by one; but these two boxes and 

|   |   |
|---|---|
| k | k |
|---|---|

 will lies in  $\pi_{2,n}(R_{\sigma, \sigma'}) \in \mathbf{R}_{n-1}^2$ , which is impossible by Remark 3.9.

Now if the second part of Condition (a) is not satisfied, it means that we will find two equal vertical consecutive boxes 

|   |
|---|
| m |
| m |

 on the right of the boxes 

|     |
|-----|
| k+1 |
| k   |

. By Relation (3.10), these four last boxes can not lie inside  $\pi_{1,n-1}(R_{\sigma, \sigma'}), \pi_{2,n}(R_{\sigma, \sigma'})$ ; we deduce in particular that  $i_o = 1$  and that the boxes 

|   |
|---|
| m |
| m |

 belong to the last column. Since  $R_{\sigma, \sigma'}$  satisfies Condition (ii) of Proposition 3.8, the “North-East” corner of  $R_{\sigma, \sigma'}$  must be 

|     |   |
|-----|---|
| m   | m |
| m-1 | m |

. Now if we look at  $\zeta$  (resp.  $\eta$ ) as its own lifting in  $\mathbf{S}_n^2$ , then its configuration in the “North-East” corner will be of the following 

|     |     |
|-----|-----|
| m   | m   |
| m-1 | m-1 |

(resp. 

|     |   |
|-----|---|
| m-1 | m |
| m-1 | m |

). Since the intersection is irreducible, we should find  $\delta \in \mathbf{S}_n^2$  such that  $\delta \succeq \zeta, \eta$  and  $R_\delta \preceq R_{\sigma, \sigma'}$ . Since  $(R_\zeta)_{2,n-1} = (R_{\sigma, \sigma'})_{2,n-1} = m-1$  we get that also  $(R_\delta)_{2,n-1} = m-1$ . Since  $(R_\zeta)_{1,n-1} = (R_{\sigma, \sigma'})_{1,n-1} = m$  we get that also  $(R_\delta)_{1,n-1} = m$ . Since  $(R_\eta)_{2,n} = (R_{\sigma, \sigma'})_{2,n} = m$  we get that also  $(R_\delta)_{2,n} = m$ . But then by Remark 3.9 the “North-East” corner of  $R_\delta$  should be of the following configuration 

|     |     |
|-----|-----|
| m   | m+1 |
| m-1 | m   |

, this is impossible since  $(R_\delta)_{1,n} \leq (R_{\sigma, \sigma'})_{1,n} = m$ .

Condition (c): Suppose that the first part of Condition (c) is not satisfied, it means that we can find two vertical consecutive boxes 

|     |
|-----|
| m+1 |
| m   |

 lying in the  $j_o^{th}$  and  $(j_o + 1)^{th}$



3.8. Let us apply the previous subsection to the elements of the form  $\sigma_T$  to show that in general the intersection  $\mathcal{V}_T \cap \mathcal{V}_{T'}$  is reducible and not equidimensional.

**Example 3.18.** (i) For  $n \leq 4$  all the intersections of  $B$ -orbit closures of nilpotent order 2 are irreducible. The first examples of reducible intersections of  $B$  orbit closures occur in  $n = 5$ . In particular there is the unique example of the reducible intersection of orbital varieties and it is

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \hline \end{array}, \quad R_{\sigma_T} = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T' = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline \end{array}, \quad R_{\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$R_{\sigma_T, \sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since  $(R_{\sigma_T, \sigma_{T'}})_{1,3} = (R_{\sigma_T, \sigma_{T'}})_{1,2} + 1 = (R_{\sigma_T, \sigma_{T'}})_{2,2} + 1 = (R_{\sigma_T, \sigma_{T'}})_{2,3} + 1$  and  $(R_{\sigma_T, \sigma_{T'}})_{3,5} = (R_{\sigma_T, \sigma_{T'}})_{4,5} + 1$  we get that  $R_{\sigma_T, \sigma_{T'}}$  does not satisfy condition (iii)-(c) of Proposition 3.8, therefore  $R_{\sigma_T, \sigma_{T'}} \notin \mathbf{R}_5^2$ . As well  $(R_{\sigma_T, \sigma_{T'}})_{1,4}$ ,  $(R_{\sigma_T, \sigma_{T'}})_{2,5}$  do not satisfy Remark 3.9. Accordingly we find three maximal elements  $R, R', R'' \in \mathbf{R}_5^2$  for which  $R, R', R'' \prec R_{\sigma_T, \sigma_{T'}}$

$$R = R_{(1,3)(2,5)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad R' = R_{(1,4)(3,5)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R'' = R_{(1,5)(2,4)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that  $\dim(B.N_{(1,3)(2,5)}) = \dim(B.N_{(1,4)(3,5)}) = \dim(B.N_{(1,5)(2,4)}) = 4$  so that  $\mathcal{V}_T \cap \mathcal{V}_{T'}$  contains three components of codimension 2.

(ii) The first example of non-equidimensional intersection of orbital varieties occurs in  $n = 6$  and it is

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array}, \quad R_{\sigma_T} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T' = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array}, \quad R_{\sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that

$$R_{\sigma_T, \sigma_{T'}} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since  $(R_{\sigma_T, \sigma_{T'}})_{1,3} = (R_{\sigma_T, \sigma_{T'}})_{1,2} + 1 = (R_{\sigma_T, \sigma_{T'}})_{2,2} + 1 = (R_{\sigma_T, \sigma_{T'}})_{2,3} + 1$  and  $(R_{\sigma_T, \sigma_{T'}})_{1,5} = (R_{\sigma_T, \sigma_{T'}})_{2,5}$  we get that  $R_{\sigma_T, \sigma_{T'}}$  does not satisfy condition (iii) (a) of Proposition 3.8 and  $(R_{\sigma_T, \sigma_{T'}})_{1,5}$ ,  $(R_{\sigma_T, \sigma_{T'}})_{2,6}$  do not satisfy Remark 3.9 so that  $R_{\sigma_T, \sigma_{T'}} \notin \mathbf{R}_6^2$  and the maximal elements  $R, R' \in \mathbf{R}_6^2$  for which  $R, R' \prec R_{\sigma_T, \sigma_{T'}}$  are

$$R = R_{(1,3)(4,6)} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$R' = R_{(1,6)(2,5)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that  $\dim(B.N_{(1,3)(4,6)}) = 6$  and  $\dim(B.N_{(1,6)(2,5)}) = 4$  so that  $\mathcal{V}_T \cap \mathcal{V}_{T'}$  contains one component of codimension 2 and another component of codimension 4.

#### 4. SOME OTHER COUNTER-EXAMPLES

**4.1. Cell graphs.** Let  $T \in \mathbf{Tab}_\lambda$  be a standard tableau and  $\mathcal{C}_T$  its corresponding left cell (cf. Section 2.3). Steinberg's construction provides the way to construct  $\mathcal{V}_T$  with the help of elements of  $\mathcal{C}_T$ . In [8], we got an other geometric interpretation of the  $\mathcal{C}_T$ :

**Theorem 4.1.** ([8]) - Let  $T \in \mathbf{Tab}_\lambda$  and let  $w = \text{RS}(T, T') \in \mathcal{C}_T$ . Then for a  $x \in \mathcal{V}_T \cap B.(\mathfrak{n} \cap^w \mathfrak{n})$  in general position, the unique Schubert cell whose intersection with the irreducible component  $\mathcal{F}_{T'}$  of the Springer fiber is open and dense in  $\mathcal{F}_{T'}$  is indexed by  $w$ .

The cell  $\mathcal{C}_T$  in  $\mathbf{S}_n$  can be visualized as a **cell graph**  $\Gamma_T$  where the vertices are labeled by  $\mathbf{Tab}_\lambda$ , and two vertices  $T'$  and  $T''$  are joined by an edge labeled by  $k$  if  $s_k \text{RS}(T, T') = \text{RS}(T, T'')$ . One can easily see (cf. [8], for example) that if  $T'$  and  $T''$  are joined in  $\Gamma_T$ , then  $\text{codim}_{\mathcal{F}_{T'}} \mathcal{F}_{T'} \cap \mathcal{F}_{T''} = 1$ .

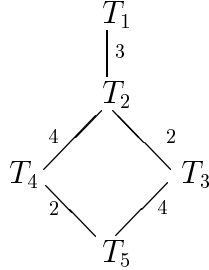
Note that  $T'$  and  $T''$  can be joined by an edge in  $\Gamma_T$  and not joined by an edge in  $\Gamma_S$  for some  $S, T \in \mathbf{Tab}_\lambda$ . Is it true that  $\text{codim}_{\mathcal{F}_{T'}} \mathcal{F}_{T'} \cap \mathcal{F}_{T''} = 1$  if and only if there exists  $T \in \mathbf{Tab}_\lambda$  such that  $T'$  and  $T''$  are joined by an edge in  $\Gamma_T$ ?

The answer is negative as we show by an example in this subsection.

As we show in [9] if  $k \leq 2$  then  $\text{codim}_{\mathcal{V}_T} (\mathcal{V}_T \cap \mathcal{V}_S) = 1$  if and only if there exists  $P \in \mathbf{Tab}_{(n-k, k)^*}$  such that  $T$  and  $S$  are joined by an edge in  $\Gamma_P$  so that the first example occurs in  $n = 6$  for  $\mathbf{Tab}_{(3,3)^*}$ . In that case  $(3,3)^* = (2,2,2)$  and the corresponding orbital varieties are 9-dimensional. Let us put

$$T_1 = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \quad T_3 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 & 6 \\ \hline \end{array}, \quad T_4 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & 6 \\ \hline \end{array}, \quad T_5 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}.$$

One can check that all the cell graphs are the same this graph is

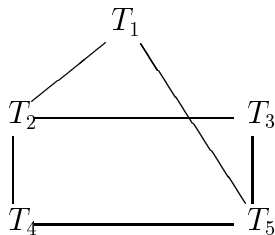


On the other hand one has

$$R_{\sigma_{T_1}, \sigma_{T_5}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = R_{(1,5)(2,6)(3,4)}$$

and  $\dim(B.N_{(1,5)(2,6)(3,4)}) = 8$ , so that  $\text{codim}_{\mathcal{V}_{T_1}} (\mathcal{V}_{T_1} \cap \mathcal{V}_{T_5}) = 1$ . As well the straight computations show that  $\dim(\mathcal{V}_{T_1} \cap \mathcal{V}_{T_4}) = \dim(\mathcal{V}_{T_1} \cap \mathcal{V}_{T_3}) = \dim(\mathcal{V}_{T_2} \cap \mathcal{V}_{T_5}) =$

$\dim(\mathcal{V}_{T_3} \cap \mathcal{V}_{T_4}) = 7$  so that all these intersections are of codimension 2. Further,  $\mathcal{V}_{T_1} \cap \mathcal{V}_{T_4}$ ,  $\mathcal{V}_{T_1} \cap \mathcal{V}_{T_3}$  and  $\mathcal{V}_{T_3} \cap \mathcal{V}_{T_4}$  are irreducible.  $\mathcal{V}_{T_2} \cap \mathcal{V}_{T_5}$  has three components with the following dense  $B$ -orbits:  $B.N_{(1,3)(2,5)(4,6)}$ ,  $B.N_{(1,5)(2,4)(3,6)}$ , and  $B.N_{(1,4)(2,6)(3,5)}$ . Below we draw the graph where two vertices are joined if the corresponding intersection is of codimension 1.



**4.2. Orbital variety's construction.** Let us go back to Steinberg's construction of an orbital variety (see Section 2.3). Given  $T \in \mathbf{Tab}_\lambda$  one has  $\mathcal{V}_T = \overline{B.(n \cap {}^w n)} \cap \mathcal{O}_\lambda$  for any  $w \in \mathcal{C}_T$ . Obviously,

$$\dim(\overline{B.(n \cap {}^w n)} \cap \mathcal{O}_\lambda) = \dim(B.(n \cap {}^w n) \cap \mathcal{O}_\lambda),$$

so that  $\dim(B.(n \cap {}^w n) \cap \mathcal{O}_\lambda) = \dim(\mathcal{O}_\lambda \cap n)$ , therefore  $B.(n \cap {}^w n) \cap \mathcal{O}_\lambda$  is also irreducible in  $\mathcal{O}_\lambda \cap n$ ; in particular  $B.(n \cap {}^w n) \cap \mathcal{O}_\lambda$  is an orbital variety if and only if  $B.(n \cap {}^w n) \cap \mathcal{O}_\lambda$  is closed in  $\mathcal{O}_\lambda \cap n$ . The natural questions connected to the construction are

**Q1:** May be one can always find  $w \in \mathcal{C}_T$  such that  $\mathcal{V}_T = B.(n \cap {}^w n) \cap \mathcal{O}_\lambda$ ?

**Q2:** Or may be  $\mathcal{V}_T = \bigcup_{y \in \mathcal{C}_T} B.(n \cap {}^y n) \cap \mathcal{O}_\lambda$ ?

The answer to both these questions is negative as we show by the following counter-example.

**Example 4.2.** Let  $T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline 4 & \\ \hline \end{array}$ . The corresponding left cell is given by

$$\mathcal{C}_T = \{w_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}, w_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, w_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}\}.$$

We draw here in green the corresponding space  $n \cap {}^w n$ :  
 On the other hand by Theorem 3.12,  $\mathcal{V}_T = B.N_{(2,3)} \cup B.N_{(2,4)} \cup B.N_{(1,3)} \cup B.N_{(1,4)}$ .  
 As one can see from the picture  $N_{(1,4)} \notin B.(n \cap {}^w n)$  for  $w \in \{w_1, w_2, w_3\}$ .

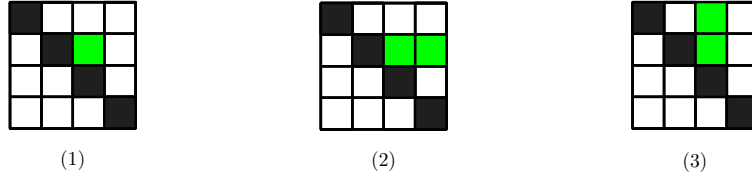


FIGURE 2.

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### INDEX OF NOTATION

Symbols appearing frequently are given below in order of appearance.

- 1.1  $\mathfrak{n}$ ,  $\mathfrak{g}_\alpha$ ,  $\alpha_i$ ,  $\Pi$ ,  $\alpha_{i,j}$ ,  $B$ ,  $\mathbf{S}_n$ ,  $s_i$ ,  $g.u.$ ,  $\mathcal{F}_x$ ,  $\mathcal{O}_x$ ;
- 1.2  $J(x)$ ,  $\mathcal{O}_\lambda$ ,  $\text{sh}(\mathcal{O})$ ,  $\text{sh}(T)$ ,  $\mathbf{Tab}_\lambda$ ,  $\mathcal{F}_T$ ,  $\mathcal{V}_T$ ;
- 2.2  $\pi_{i,j} : \mathfrak{n}_n \rightarrow \mathfrak{n}_{j+1-i}$
- 3.1  $\mathcal{X}_2$ ,  $\mathbf{S}_n^2$ ,  $N_\sigma$ ,  $\mathbf{Tab}_n^2$ ,  $\sigma_T$ ;
- 3.2  $L(\sigma)$ ,  $\mathcal{O}_\sigma$ ;
- 3.3  $R_\sigma$ ,  $\mathbf{R}_n^2$ ;
- 3.5  $\mathbf{S}_{\langle i,j \rangle}^2$ ,  $\pi_{i,j} : \mathbf{S}_n^2 \rightarrow \mathbf{S}_{\langle i,j \rangle}^2$
- 3.6  $\mathbf{S}_n^2(k)$ ,  $\mathbf{Tab}_n^2(k)$

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