

# PERIODIC POINTS ON VEECH SURFACES AND THE MORDELL-WEIL GROUP OVER A TEICHMÜLLER CURVE

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ABSTRACT. Periodic points are points on Veech surfaces, whose orbit under the group of affine diffeomorphisms is finite. We characterise those points as being torsion points if the Veech surfaces is suitably mapped to its Jacobian or an appropriate factor thereof. For a primitive Veech surface in genus two we show that the only periodic points are the Weierstraß points and the singularities.

Our main tool is the Hodge-theoretic characterisation of Teichmüller curves. We deduce from it a finiteness result for the Mordell-Weil group of the family of Jacobians over a Teichmüller curve.

## INTRODUCTION

Let  $\Omega M_g$  denote the tautological bundle over the moduli space of curves of genus  $g$ . Its points consist of pairs  $(X^0, \omega^0)$  of a Riemann surface plus a holomorphic one-form on  $X^0$ . There is a natural  $\mathrm{GL}_2^+(\mathbb{R})$ -action on  $\Omega M_g$  (see [Ve89] or [McM03]). In the rare cases where the projection of  $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X^0, \omega^0)$  to  $M_g$  is an algebraic curve  $C$ , this curve is called a *Teichmüller curve* and  $(X^0, \omega^0)$  a *Veech surface*. These surfaces are also characterised by the following property:

Let  $\mathrm{Aff}(X^0, \omega^0)$  denote the group of diffeomorphisms on  $X^0$  that are affine outside the zeros of  $\omega^0$  with respect to the charts determined by integrating  $\omega^0$ . Then  $(X^0, \omega^0)$  is a Veech surface if and only if  $\mathrm{Aff}(X^0, \omega^0)$  is 'as big as possible', i.e. the matrix parts of these diffeomorphisms form a lattice  $\Gamma$  in  $\mathrm{SL}_2(\mathbb{R})$ .

A *periodic point* on a Veech surface is a point whose orbit under  $\mathrm{Aff}(X^0, \omega^0)$  is finite. Examples of periodic points are the zeroes of  $\omega^0$  and Weierstraß points if  $g = 2$  (see [GuHuSc03]).

The aim of this paper is to show:

**Theorem 3.3:** *The difference of two periodic points is torsion on an  $r$ -dimensional quotient of  $\mathrm{Jac}(X^0)$ , where  $r = [\mathbb{Q}(\mathrm{tr}(\Gamma)) : \mathbb{Q}]$ .*

We also show a converse to this statement in terms of torsion sections of the family of Veech surfaces over an unramified cover of the Teichmüller curve. We say for short that periodic points form a *torsion packet* ([Co85]) on  $X^0$ .

As a consequence we also obtain a different proof of the finiteness of the number of periodic points on a Teichmüller curve that does not arise via a torus covering. This was proved by Gutkin, Hubert and Schmidt (see [GuHuSc03]) using the flat geometry induced by  $\omega^0$ .

We apply the characterisation of periodic points via torsion points to show:

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**Theorem 5.1/5.2:** *The only periodic points on a Veech surface in genus 2, which does not arise from a torus cover, are the Weierstraß points and the zeroes of  $\omega^0$ .*

Since 'large' torsion packets are 'rare' (see e.g. [BoGr00]), we expect that for large  $g$  the strata of  $\Omega M_g$  with 'many' (in particular  $2g - 2$  simple) zeroes should contain 'few' Teichmüller curves.

In fact for  $g = 2$  McMullen shows ([McM04b] and [McM04c]), using the above characterisation of periodic points, that in  $\Omega M_2(1, 1)$  there is only one Teichmüller curve not coming from genus 1. It is generated by the decagon. In contrast to that  $\Omega M_2(2)$  contains infinitely many Teichmüller curves that are not obtained via torus coverings (see [McM04a]).

### Contents.

In Section 1 we start by recalling the language of translation surfaces in which periodic points were first studied in [GuHuSc03]. We translate this into the following setting:

To a Veech surface we may associate (via its  $\mathrm{GL}_2^+(\mathbb{R})$ -orbit) a Teichmüller curve  $C \rightarrow M_g$ . A Teichmüller curve defines (by pulling back the universal family over a finite cover of  $M_g$ ) a family of curves (or a *fibred surface*)  $f : X_1 \rightarrow C_1$ , after passing to some finite unramified covering  $C_1$  of  $C$ . Finally each fibre of  $f$  is a Veech surface  $X^0$  and the 1-form  $\omega^0$  can be recovered from the variation of Hodge structures of  $f$ .

We show that periodic points correspond via this dictionary to sections of  $f$ .

In Section 2 we recall some material from [Mo04] in order to define the  $r$ -dimensional factor  $A_1/C_1$  of  $\mathrm{Jac}(X_1/C_1)$  we referred to above.

We will call a Veech surface *geometrically primitive* if it does not arise via coverings from smaller genus. We will call a Veech surface *algebraically primitive* if the relative Jacobian of an associated fibred surface is irreducible. The indefinite article here and whenever we talk of fibred surfaces refers only to the possibility of passing to an unramified covering.

We give an example that (unlike in genus two) these notions do not coincide in higher genus. In Theorem 2.6 we show that nevertheless a Veech surface stems from a unique geometrically primitive Veech surface.

The *Mordell-Weil group*  $\mathrm{MW}(A/C)$  of a family of abelian varieties  $g : A \rightarrow C$  is the group of rational sections of  $g$  or equivalently the group of  $\mathbb{C}(C)$ -valued points of  $A$ . In Section 3 we study the Mordell-Weil group for the factor  $A_1/C_1$  of  $\mathrm{Jac}(X_1/C_1)$  for a Teichmüller curve  $C \rightarrow M_g$ :

**Theorem 3.1:** *For each Teichmüller curve and any given unramified covering  $C_1 \rightarrow C$  the group  $\mathrm{MW}(A_1/C_1)$  is finite.*

We apply this to the characterisation of periodic points via torsion sections (Thm. 3.3 and the converse Prop. 3.5).

The last two sections contain a degeneration argument similar to the one in [McM04c] for the explicit analysis of periodic points in genus two. While in loc. cit. ratios of sines appear, we need for our purposes that only finitely many ratios  $\tan(a)/\tan(b)$  for  $a, b \in \pi\mathbb{Q}$  lie in a quadratic number field. We give an explicit list of them, that is used in the last section.

### Some notation.

Riemann surfaces are usually denoted by  $X^0, Y^0$ , etc., while  $X, X_1, Y, Y_1$ , etc. will be used for families of Riemann surfaces over base curves  $C, C_1$ , etc., whose fibres over  $0 \in C(\mathbb{C})$  are  $X^0, Y^0$ , etc.

Overlines denote completions of curves or fibred surfaces.

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## 1. A DICTIONARY

### Translation surfaces.

A *translation surface* is a compact Riemann surface  $X^0$  with an atlas  $\{U_i, i \in I\}$ , covering  $X^0$  except for finitely many points (called *singularities*  $\text{Sing}(X^0)$ ) such that the transition functions between the charts are translations. A holomorphic one-form  $\omega^0$  on a Riemann surface  $X_0$  induces the structure of a translation surface where  $\text{Sing}(X^0)$  can be any finite set containing the zeroes of  $\omega^0$ . A *translation covering* between two translation surfaces  $(X^0, \text{Sing}(X^0))$  and  $(Y^0, \text{Sing}(Y^0))$  is a covering  $\varphi : X^0 \rightarrow Y^0$  such that the charts of  $X^0$  are pulled back from charts of  $Y^0$  and such that  $\varphi^{-1}(\text{Sing}(Y^0)) = \text{Sing}(X^0)$ . These coverings are sometimes called *balanced* translation coverings. We deal exclusively with them.

If the translation structure on  $Y^0$  is induced from  $\omega_Y^0$  and  $\varphi$  is a translation covering then the translation structure on  $X^0$  is induced from  $\omega^0 = \varphi^* \omega_Y^0$ .

Finally note that a (possibly ramified) covering  $\varphi : X^0 \rightarrow Y^0$  plus differentials on  $X^0$  and  $Y^0$  satisfying  $\omega^0 = \varphi^* \omega_Y^0$  induces a translation covering with  $\text{Sing}(Y^0)$  the zeros of  $\omega_Y^0$  and  $\text{Sing}(X^0)$  their preimages. This set will in general properly contain the zeroes of  $\omega^0$ . We will abbreviate a translation covering by  $\pi : (X^0, \omega_X^0) \rightarrow (Y^0, \omega_Y^0)$  or  $\pi : (X^0, \text{Sing}(X^0)) \rightarrow (Y^0, \text{Sing}(Y^0))$ .

### Affine diffeomorphisms, affine group.

On a translation surface we may consider diffeomorphisms that are orientation-preserving, affine with respect to the coordinate charts and that permute the singularities. They form the *group of affine diffeomorphisms* denoted by  $\text{Aff}(X^0, \text{Sing}(X^0))$ . The matrix part of such an affine diffeomorphism is well-defined and this yields a map  $D$  to a subgroup  $\Gamma$  in  $\text{SL}_2(\mathbb{R})$ . We call  $\Gamma$  the *affine group* of  $(X^0, \text{Sing}(X^0))$ . If  $\text{Sing}(X^0)$  consists of the zeroes of  $\omega^0$  we sometimes write  $\text{SL}(X^0, \omega^0)$  for  $\Gamma$ . The kernel of  $D$  consists of conformal automorphisms of  $X^0$  preserving  $\omega^0$  and is hence finite.

**Definition 1.1.** *A point  $P$  on a translation surface is called periodic if its orbit under the group  $\text{Aff}(X^0, \text{Sing}(X^0))$  is finite.*

### Veech surfaces, Teichmüller curves.

If  $\Gamma \subset \text{SL}_2(\mathbb{R})$  is a lattice, the translation surface  $(X^0, \text{Sing}(X^0))$  is called a *Veech surface*. If  $\text{Sing}(X^0)$  consists of the zeroes of  $\omega^0$ , this is the case if and only if the  $\text{GL}_2^+(\mathbb{R})$ -orbit of  $(X^0, \omega^0)$  in  $\Omega M_g$  projects to an algebraic curve  $\mathbb{H}/\Gamma = C \rightarrow M_g$ , which is called a *Teichmüller curve*. The map  $C \rightarrow M_g$  is injective up to finitely many normal crossings. By abuse of notation we will also call Teichmüller curve a map  $C_1 \rightarrow M_g$ , which is the composition of an unramified cover  $C_1 \rightarrow C$  composed by a Teichmüller curve  $C \rightarrow M_g$  in the above sense.

If we restrict to the quotient  $C_1 = \mathbb{H}/\Gamma_1$  for a sufficiently small subgroup  $\Gamma_1 \subset \Gamma$  of finite index,  $C_1$  will map to a finite cover of  $M_g$  over which the universal family exists. If we pull back this universal family to  $C_1$  we obtain a *fibred surface*  $f : X_1 \rightarrow C_1$  associated with the Teichmüller curve. We will also need a smooth semistable model  $f : \overline{X}_1 \rightarrow \overline{C}_1$  over the completion of  $C_1$ .

A Veech surface  $(X^0, \text{Sing}(X^0))$  is called *square-tiled* (or *arises as torus cover*, or *origami*) if it admits a translation covering to a torus with one singular point.

*From now on we will exclusively deal with Teichmüller curves and translation surfaces that are Veech surfaces.*

Note that 'fibred surface' refers to an object of complex dimension two. It contains as one of its fibres the translation surface  $X^0$ , an object of real dimension two.

**Lemma 1.2.** *A point  $P$  on a Veech surface  $(X^0, \omega^0)$  is periodic if and only if there is an (algebraic) section of some fibred surface  $f : X_1 \rightarrow C_1$  associated to the Teichmüller curve, which passes through  $P$  on the fibre  $X^0$  of  $f$ .*

**Proof:** A section  $s$  of  $f$  over  $C_1 = \mathbb{H}/\Gamma_1$  hits  $X^0$  in one point  $P$ . The  $\Gamma$ -orbit of  $P$  consists of at most  $[\Gamma : \Gamma_1]$  points, hence is finite by the choice of  $\Gamma_1$ .

Conversely given a periodic point  $P$  on  $X^0$  we may take a double cover  $\pi : Y^0 \rightarrow X^0$  branched at  $P$  and some zero of  $\omega^0$ . The translation surface  $(Y^0, \omega_Y^0 := \pi^*\omega^0)$  is still a Veech surface: Indeed let  $\text{Sing}(X^0) = Z(\omega^0) \cup \{P\}$ . The affine group of  $(X^0, \text{Sing}(X^0))$ , is of finite index in  $\Gamma$  by the definition of a periodic point. Now  $\pi$  defines a translation covering  $(Y^0, \pi^{-1}(\text{Sing}(X^0))) \rightarrow (X^0, \text{Sing}(X^0))$  and one can apply [GuJu00] Thm. 4.9 to show the Veech property of  $(Y^0, \omega_Y^0)$ .

Over some  $C_1 = \mathbb{H}/\Gamma_1$  for a subgroup  $\Gamma_1$  of finite index in  $\Gamma$  we have a covering  $\pi : Y_1 \rightarrow X_1$  of fibred surfaces over  $C_1$ , such that the original  $\pi$  is the fibre over some point  $0 \in C_1(\mathbb{C})$ . By construction of  $Y_1$  as  $\text{SL}_2(\mathbb{R})$ -orbit of  $(Y^0, \omega_Y^0)$  the differential  $\omega_Y^0$  extends to a section  $\omega_{Y_1}$  of the relative canonical sheaf  $\omega_{Y_1/C_1}$ . Again by definition of the  $\text{SL}_2(\mathbb{R})$ -action the multiplicities of the zeros of  $\omega_{Y_1}$  remain constant over  $C_1$ . Hence passing to a subgroup of finite index in  $\Gamma_1$  (we nevertheless keep the notation) we may assume that the zeros of  $\omega_{Y_1}$  define sections  $s_i$  of  $f_Y : Y_1 \rightarrow C_1$ .

The images of  $s_i$  under  $\pi$  are sections of  $f$ . One of them passes through  $P$ , as  $\pi$  is ramified over  $P$ .  $\square$

## 2. ALGEBRAIC AND GEOMETRIC PRIMITIVITY

Let  $K = \mathbb{Q}(\text{tr}(\Gamma))$  denote the trace field of the affine group of a Veech surface  $(X^0, \omega^0)$ . It remains unchanged if we replace  $\Gamma$  by a subgroup of finite index. Let  $r := [K : \mathbb{Q}]$ . We recall from [Mo04] the decomposition of the variation of Hodge structures (VHS) over a Teichmüller curve generated by  $(X^0, \omega^0)$ :

Let  $\mathbb{V} = R^1 f_* \mathbb{Z}$ . In [Mo04] Prop. 2.3 we have shown that there is a decomposition as polarized VHS

$$\mathbb{V}_{\mathbb{Q}} = \mathbb{W} \oplus \mathbb{M}, \quad \mathbb{W}_K = \bigoplus_{\sigma \in \text{Gal}(K/\mathbb{Q})} \mathbb{L}^{\sigma} \quad (1)$$

Here  $\mathbb{L}^{\sigma}$  are rank two local systems over  $K$  and none of the irreducible factors of  $\mathbb{M}_{\mathbb{C}}$  is isomorphic to any of the  $\mathbb{L}_{\mathbb{C}}^{\sigma}$ . This yields (see loc. cit. Thm. 2.5) a decomposition of the Jacobian

$$\text{Jac}(X_1/C_1) \sim A_1 \times B_1 \quad (2)$$

up to isogeny, where  $A_1$  has dimension  $r$  and real multiplication by  $K$ .

Recall furthermore that the graded quotients of a VHS together with the Gauss-Manin connection form a Higgs bundle  $(\mathcal{E}, \Theta)$ . The summands  $\mathbb{L}^{\sigma}$  of  $\mathbb{V}_K$  give rank-two sub-Higgs bundles  $(\mathcal{L}^{\sigma} \oplus (\mathcal{L}^{\sigma})^{-1}, \tau^{\sigma})$ , where  $S = \overline{C} \setminus C$  and

$$\tau^{\sigma} : \mathcal{L}^{\sigma} \rightarrow (\mathcal{L}^{\sigma})^{-1} \otimes \Omega_{\overline{C}}^1(\log S).$$

The subbundle  $\mathcal{L}^{\text{id}}$  of  $f_*\omega_{\overline{X_1}/\overline{C_1}}$  is distinguished by the property that its restriction to the fibre  $X^0$  gives  $\mathbb{C} \cdot \omega^0$ .

Teichmüller curves are characterised (see [Mo04] Thm. 5.3) by a decomposition of the VHS as above plus the property that  $\mathcal{L}^{\text{id}}$  is *maximal Higgs*, i.e. that  $\tau^{\text{id}}$  is an isomorphism. We need only two properties of this notion here: It is stable under replacing  $C_1$  by a finite unramified cover and the VHS over a Teichmüller curve has precisely one rank-two subbundle that is maximal Higgs.

The last property is stated explicitly in [Mo04] Lemma 3.1 for the  $\mathbb{L}^\sigma$ . But any two maximal Higgs subbundles become isomorphic after replacing  $C_1$  by a finite étale cover and none of the irreducible summands of  $\mathbb{M}_{\mathbb{C}}$  is isomorphic to any of the  $\mathbb{L}_{\mathbb{C}}^\sigma$ .

**Definition 2.1.** *A Teichmüller curve is called algebraically primitive if its relative Jacobian  $\text{Jac}(X_1/C_1)$  is irreducible as abelian variety.*

We will also say that a Veech surface is algebraically primitive, if the corresponding Teichmüller curve is algebraically primitive.

**Lemma 2.2.** *A Teichmüller curve is algebraically primitive if and only if  $r = g$ .*

**Proof:** We only have to show that  $\mathbb{W}$  is irreducible over  $\mathbb{Q}$ . This follows immediately from the irreducibility of the  $\mathbb{L}^\sigma$  and the fact that for  $\sigma \neq \tau$  the local systems  $\mathbb{L}^\sigma$  and  $\mathbb{L}^\tau$  are not isomorphic (see [Mo04] Lemma 2.2).  $\square$

Note that irreducibility of  $\text{Jac}(X_1/C_1)$  does not depend on replacing  $C_1$  by unramified covers.

Further note that irreducibility of  $\text{Jac}(X_1/C_1)$  does not exclude that special fibres of  $f : X \rightarrow C$  may have reducible Jacobians.

There is also a natural geometric notion of primitivity for translation surfaces and one for Teichmüller curves without explicitly referring to any differential. We show that these two geometric definitions coincide.

**Definition 2.3.** *A translation surface  $(X^0, \omega_X^0)$  is geometrically imprimitive if there exists a translation surface  $(Y^0, \omega_Y^0)$  of smaller genus and a covering  $\pi : X^0 \rightarrow Y^0$  such that  $\pi^*\omega_Y^0 = \omega_X^0$ .*

**Definition 2.4.** *A Teichmüller curve  $C \rightarrow M_g$  is geometrically imprimitive if there is an unramified cover  $C_1 \rightarrow C$ , a fibred surface  $f_Y : Y_1 \rightarrow C_1$  coming from a Teichmüller curve  $C_1 \rightarrow M_h$  with  $h < g$  and a (possibly ramified) covering  $\pi : X_1 \rightarrow Y_1$  over  $C_1$ . It is called geometrically primitive otherwise.*

**Proposition 2.5.** *A Teichmüller curve is geometrically primitive if and only if a corresponding Veech surface is geometrically primitive.*

**Proof:** Suppose that  $C \rightarrow M_g$  is geometrically imprimitive and let  $\pi : X_1 \rightarrow Y_1$  be the covering of fibred surfaces associated with the Teichmüller curves. We restrict the covering  $\pi$  to some fibre  $X_1^0 \rightarrow Y_1^0$  over  $0 \in C_1(\mathbb{C})$ . We have to show that the differentials  $\omega_{X_1^0}^0$  on  $X_1^0$  and  $\omega_{Y_1^0}^0$  on  $Y_1^0$  that generate the Teichmüller curves define a translation covering. We let  $\text{Sing}(Y_1^0)$  be the zeroes of  $\omega_{Y_1^0}^0$  and  $\text{Sing}(X_1^0)$  be their preimage via  $\pi$ . It suffices to see that (up to a multiplicative constant) we have  $\pi^*\omega_{Y_1^0}^0 = \omega_{X_1^0}^0$ . By definition  $\omega_{X_1^0}^0$  and  $\omega_{Y_1^0}^0$  are obtained from  $\mathcal{L}_{X_1^0}^{\text{id}}$  resp.  $\mathcal{L}_{Y_1^0}^{\text{id}}$  by restriction to 0. By the properties of maximal Higgs subbundles listed above, we have  $\mathcal{L}_{X_1^0}^{\text{id}} = \pi^*\mathcal{L}_{Y_1^0}^{\text{id}}$ . This completes the 'if'-part.

Conversely a translation covering defines two Teichmüller curves with commensurable affine groups ([Vo96] Thm. 5.4 or [GuJu00] Thm. 4.9). We pass to a subgroup  $\Gamma_1$  contained in both with finite index and small enough to have universal families. Then the  $\mathrm{SL}_2(\mathbb{R})$ -images of the translation covering patch together to a covering of fibred surfaces over  $\mathbb{H}/\Gamma_1$ .  $\square$

**Examples:** i) The two notions of primitivity coincide for genus 2 and will therefore be abbreviated just by 'primitive': If  $r = 1$  the factor  $A$  of the Jacobian is one-dimensional, hence a family of elliptic curves and the projection onto this factor implies geometric imprimitivity ([GuJu00] Thm. 5.9).

ii) The Riemann surface  $y^2 = x^7 - 1$  with  $\omega_0 = dx/y$  is a Veech surface (see [Ve89]) and  $\Gamma = \Delta(2, 7, \infty)$  has a trace field of degree 3 over  $\mathbb{Q}$ . The corresponding Teichmüller curve is algebraically primitive, hence geometrically primitive.

iii) For genus  $g = 3$  the notions of geometric and algebraic primitivity no longer coincide: The Riemann surface  $y^{12} = x^3(x-1)^4$  with  $\omega^0 = ydx/[x(x-1)]$  is studied in [HuSc01] Thm. 5. It is obtained from unfolding the billiard in the triangle with angles  $3\pi/12$ ,  $4\pi/12$  and  $5\pi/12$ . It is a Veech surface with  $\Gamma = \Delta(6, \infty, \infty)$ . The trace field is of degree only 2 over  $\mathbb{Q}$ . Hence the Teichmüller curve is not algebraically primitive. But it is geometrically primitive, as remarked in loc. cit.: Since  $r = 2$  it cannot arise as a torus cover. If it arose as a genus two cover, this cover would have to be unramified by Riemann-Hurwitz. Hence  $\omega^0$  would have zeros of order at most two. But  $\omega^0$  has indeed a zero of order four.

In [HuSc01] the authors analyse the properties that are preserved if one goes up and down a tree of translation coverings. We show that the situation is simple, if the singularities of the translation surfaces are chosen suitably, i.e. each tree has a root.

**Theorem 2.6.** *A translation surface  $(X^0, \omega^0)$  is obtained as a translation covering from a geometrically primitive translation surface  $(X_{\mathrm{prim}}^0, \omega_{\mathrm{prim}}^0)$ . If the genus  $X_{\mathrm{prim}}^0$  is greater than one, this primitive surface is unique.*

*Moreover, the construction of the primitive surface is equivariant with respect to the  $\mathrm{SL}_2(\mathbb{R})$ -action, if the genus of  $X_{\mathrm{prim}}^0$  is greater than one.*

*If  $(X^0, \omega^0)$  generates a Teichmüller curve, then so does  $(X_{\mathrm{prim}}^0, \omega_{\mathrm{prim}}^0)$ . In this case  $\pi : X^0 \rightarrow X_{\mathrm{prim}}^0$  is branched over periodic points.*

**Proof:** We first prove existence and uniqueness of the primitive surface and drop the superscripts that indicate a special fibre.

Let  $\mathrm{Jac}(X) \rightarrow A_X^{\mathrm{max}}$  be the maximal abelian quotient such that  $\omega_X$  pulls back from  $A_X^{\mathrm{max}}$ . Equivalently  $A_X^{\mathrm{max}}$  is the quotient of  $\mathrm{Jac}(X)$  by all connected abelian subvarieties  $B'$  such that the pullback of  $\omega_X$  to  $B'$  vanishes. If  $\dim A_X^{\mathrm{max}} = 1$  this is the primitive surface, we are done.

Else embed  $X$  into its Jacobian and consider the normalization  $X_A$  of the images of  $X \rightarrow \mathrm{Jac}(X) \rightarrow A_X^{\mathrm{max}} \rightarrow A$  for all isogenies  $A_X^{\mathrm{max}} \rightarrow A$ . Since  $X_A$  generates  $A$  as a group, we have  $g(X_A) \geq \dim A \geq 2$ . The curves  $X_A$  form an inductive system which eventually stabilizes since the genera are bounded below. We let  $X_{\mathrm{prim}}$  be the limit of the  $X_A$  and claim that it has the required properties.

First, by construction of  $A$  and since  $X_{\mathrm{prim}}$  generates  $A$  as a group, there is a differential  $\omega_{\mathrm{prim}}$  on  $X_{\mathrm{prim}}$  such that  $\varphi_X^* \omega_{\mathrm{prim}} = \omega_X$ . Hence  $\varphi_X : X \rightarrow X_{\mathrm{prim}}$  defines a translation cover once the singularities are suitably chosen.

Second,  $(X_{\mathrm{prim}}, \omega_{\mathrm{prim}})$  is indeed primitive: Suppose there is a covering  $X_{\mathrm{prim}} \rightarrow Y$  with  $\pi^* \omega_Y = \omega_{\mathrm{prim}}$  for some differential on  $Y$ . Let  $\mathrm{Jac}(\pi)$  be the induced morphism on the

Jacobians, commuting with  $\pi$  and suitable embeddings of the curves into their Jacobians. Consider the following diagram:

$$\begin{array}{ccccc}
X_{\text{prim}} & \longrightarrow & \text{Jac}(X) & \longrightarrow & A_X^{\text{max}} \\
\downarrow \pi & & \downarrow \cong & \dashrightarrow & \downarrow \text{---} \\
Y & \longrightarrow & Z & & \overline{\text{Jac}(\pi)} \\
& & \downarrow \text{Jac}(\pi) & & \downarrow \exists \\
& & \text{Jac}(Y) & \longrightarrow & A_Y^{\text{max}}
\end{array}$$

By definition  $\omega_{\text{prim}}$  vanishes on  $K := \text{Ker}(\text{Jac}(X) \rightarrow A_X^{\text{max}})$ . Since  $\pi^*\omega_Y = \omega_{\text{prim}}$  the differential  $\omega_Y$  vanishes on  $\text{Jac}(\pi)(K)$ . This means that  $\text{Jac}(\pi)(K)$  is in the kernel of  $\text{Jac}(Y) \rightarrow A(Y)$  and hence  $\text{Jac}(\pi)$  descends to a homomorphism on the quotients  $\overline{\text{Jac}(\pi)} : A_X^{\text{max}} \rightarrow A_Y^{\text{max}}$ . This is an isogeny by construction of  $A_X^{\text{max}}$ . The map  $X_{\text{prim}} \rightarrow Z$ , which is defined as the normalization of the image in  $A_Y^{\text{max}}$ , is an isomorphism by construction and so is  $\pi$ .

Third, for the uniqueness we have to show that for a translation covering  $\pi : (X, \omega_X) \rightarrow (Y, \omega_Y)$  there is a morphism  $\pi_{\text{prim}} : X_{\text{prim}} \rightarrow Y_{\text{prim}}$ . As in the second step we have an induced map  $\overline{\text{Jac}(\pi)} : A_X^{\text{max}} \rightarrow A_Y^{\text{max}}$ . The curve  $X_{\text{prim}}$  was obtained as the normalization of the image of  $X$  is some quotient  $q_X : A_X^{\text{max}} \rightarrow A_X$ . It hence maps to the normalization  $Z$  of the image of  $Y$  in  $A_X^{\text{max}} / \langle \text{Ker}(q_X), \text{Ker}(\overline{\text{Jac}(\pi)}) \rangle$ . Since  $Z$  maps to  $Y_{\text{prim}}$  by construction we are done.

For the  $\text{SL}_2(\mathbb{R})$ -equivariance let  $(X^1, \omega^1) = A \cdot (X^0, \omega^0)$  for some  $A \in \text{SL}_2(\mathbb{R})$ . Primitivity implies the existence of a translation coverings  $\pi : A \cdot X_{\text{prim}}^0 \rightarrow X_{\text{prim}}^1$  and  $\pi' : A^{-1} \cdot X_{\text{prim}}^1 \rightarrow X_{\text{prim}}^0$ . Hence either both primitive curves have genus 1 or both have bigger genus greater than one and we are done by uniqueness.

In case of a torus cover the statement that  $(X_{\text{prim}}^0, \omega_{\text{prim}}^0)$  generates a Teichmüller curve is trivial. In the other cases the previous argument implies that the affine group of  $(X_{\text{prim}}^0, \omega_{\text{prim}}^0)$  contains the one of  $(X^0, \omega^0)$ .

If both translation surfaces generate Teichmüller curves there is a subgroup of finite index of  $\text{Aff}(X^0, \omega^0)$  that descends to  $X_{\text{prim}}^0$ . This group has to fix branch points and hence the whole group can generate only finite orbits of branch points.  $\square$

### 3. THE MORDELL-WEIL GROUP

**Theorem 3.1.** *Let  $f : X_1 \rightarrow C_1$  be a fibred surface associated with a Teichmüller curve. Then the Mordell-Weil group of  $A_1/C_1$  is finite. Here  $A_1$  is the factor of  $\text{Jac}(X_1/C_1)$  with real multiplication by  $K$ .*

*In particular if the Teichmüller curve is algebraically primitive then  $\text{MW}(\text{Jac}(X_1/C_1))$  is finite.*

The finiteness of the Mordell-Weil group is invariant under isogenies. Thus there is no need to specify  $A_1$  in its isogeny class. In particular we may suppose that the  $\mathbb{Q}$ -local system  $\mathbb{W}$  comes from a  $\mathbb{Z}$ -local system  $\mathbb{W}_{\mathbb{Z}}$ .

Furthermore the statement of the theorem becomes stronger the smaller the subgroup  $\Gamma_1$  with  $C_1 = \mathbb{H}/\Gamma_1$  is. Thus we may replace  $C_1$  by an unramified cover and suppose that we have unipotent monodromies. To simplify notation we will call this cover  $C$ , which should

not be confused with the notation for the original Teichmüller curve.

Let  $\bar{g} : \bar{A} \rightarrow \bar{C}$  be an extension of  $g : A \rightarrow C$  to a semiabelian scheme. A unique such extension exists due to the unipotent monodromies. We denote by  $H^0(C, \mathcal{O}_C(A/C))$  (resp. by  $H^0(\bar{C}, \mathcal{O}_{\bar{C}}^{\text{an}}(\bar{A}))$ ) the group of algebraic sections of  $A/C$  (resp. the group of analytic sections of  $\bar{A}/\bar{C}$ ).

Two remarks: First, the analytic sections of  $\bar{A}/\bar{C}$  are necessarily algebraic. Nevertheless we have to use the analytic category, because we want to use uniformization in the sequel. Second, by properness of  $g$  any rational section of  $g$  extends to the whole curve  $C$ , hence  $\text{MW}(A/C) = H^0(C, \mathcal{O}_C(A/C))$ .

**Lemma 3.2.** *The restriction map*

$$r : H^0(\bar{C}, \mathcal{O}_{\bar{C}}^{\text{an}}(\bar{A})) \rightarrow H^0(C, \mathcal{O}_C(A/C))$$

*is injective with finite cokernel.*

**Proof:** The proof is from Prop. 6.8 in [Sa93]. We reproduce a sketch for convenience of the reader:

There exists a group scheme  $N \rightarrow \bar{C}$  (called the Néron model of  $\bar{A}/\bar{C}$ ) containing  $\bar{A}/\bar{C}$  as a connected component, with the following property: For smooth  $Y \rightarrow \bar{C}$  a rational map  $Y \dashrightarrow N$  over  $\bar{C}$  extends to a morphism  $Y \rightarrow N$ .

We apply this property to sections  $s : C \rightarrow A$ . They extend to rational maps from  $\bar{C}$  a priori to a projective completion of  $N$ , but in fact to a morphism  $\bar{C} \rightarrow N$ . Hence

$$H^0(C, \mathcal{O}_C(A/C)) \rightarrow H^0(\bar{C}, \mathcal{O}_{\bar{C}}(N/\bar{C}))$$

is an isomorphism. This shows that the cokernel of  $r$  consists of sections of the finite group scheme  $N/\bar{A}$  and is hence finite.  $\square$

If the local system  $\mathbb{W}$  on  $C$  carries a polarized VHS of weight  $m$  (in our case  $m = 1$ ) then  $H^i(\bar{C}, j_*\mathbb{W}_{\mathbb{C}})$  is known to carry a Hodge structure of weight  $m + i$  (see [Zu79]). Here  $j : C \rightarrow \bar{C}$  is the inclusion. Indeed let  $\Omega^{\bullet}(\mathbb{W})_{(2)}$  denote the deRham complex with  $L_2$ -growth conditions at the punctures. Zucker (see [Zu79]) shows (extending Deligne's results to the non-compact case) that we may identify  $H^i(\bar{C}, j_*\mathbb{W}_{\mathbb{C}})$  with the hypercohomology groups  $\mathbf{H}^i(\bar{C}, \Omega^{\bullet}(\mathbb{W})_{(2)})$ . Then the Hodge structure comes from the filtration on  $\Omega^{\bullet}(\mathbb{W})_{(2)}$  induced by the Hodge filtration on  $\mathbb{W}$ .

But in fact there is another complex more easily accessible and quasi-isomorphic to  $\Omega^{\bullet}(\mathbb{W})_{(2)}$  ([Zu79] Prop. 9.1).

We describe how to calculate the Hodge structure on  $H^i(\bar{C}, j_*\mathbb{W}_{\mathbb{C}})$  in our situation: The Hodge filtration on  $\mathbb{W}$  is

$$0 = \mathcal{F}^2 \subset \mathcal{F}^1 = g_*\omega_{\bar{A}/\bar{C}} \subset \mathcal{F}^0 = (\mathbb{W} \otimes \mathcal{O}_C)_{\text{ext}},$$

where the subscript denotes the Deligne extension to  $\bar{C}$ . As graded pieces we have

$$\mathcal{E}^{1,0} = \mathcal{F}^1/\mathcal{F}^2 = g_*\omega_{\bar{A}/\bar{C}}, \quad \mathcal{E}^{0,1} = \mathcal{F}^0/\mathcal{F}^1 = R^1g_*\mathcal{O}_{\bar{A}}.$$

Combining [Zu79] Thm. 7.13 and Prop. 9.1 (see also the restatement after Lemma 12.14 in loc. cit.) we conclude for  $p \in \{0, 1, 2\}$

$$H^1(\bar{C}, j_*\mathbb{W}_{\mathbb{C}})^{(p, 2-p)} = \mathbf{H}^1(\bar{C}, \mathcal{E}^{p, 1-p} \longrightarrow \mathcal{E}^{p-1, 2-p} \otimes \Omega_{\bar{C}}^1(\log S)) \quad (3)$$

where the mapping in the complex on the right is the graded quotient of the Gauss-Manin connection (equivalently: the Kodaira-Spencer mapping) and  $S = \bar{C} \setminus C$ .

**Proof of the theorem:** The uniformization of  $\overline{A}/\overline{C}$  yields a short exact sequence

$$0 \rightarrow j_* \mathbb{W}_{\mathbb{Z}} \rightarrow \mathcal{E}^{0,1} \rightarrow \mathcal{O}_{\overline{C}}^{\text{an}}(\overline{A}) \rightarrow 0.$$

We take cohomology and note that  $H^0(\overline{C}, \mathcal{E}^{0,1})$  vanishes as  $\overline{A}/\overline{C}$  has no fixed part. Hence

$$\begin{aligned} H^0(\overline{C}, \mathcal{O}_{\overline{C}}^{\text{an}}(\overline{A})) &= \text{Ker}(H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{Z}}) \rightarrow H^1(\overline{C}, \mathcal{E}^{0,1})) \\ &= H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{Z}}) \cap (\text{Ker}(H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{C}}) \rightarrow H^1(\overline{C}, \mathcal{E}^{0,1}))) \\ &= H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{Z}}) \cap H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{C}})^{1,1}. \end{aligned}$$

By (1) and (3) we deduce

$$H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{C}})^{1,1} = \bigoplus_{\sigma_i \in \text{Gal}(K/\mathbb{Q})} \mathbf{H}^1(\overline{C}, \mathcal{L}_i \rightarrow (\mathcal{L}_i)^{-1} \otimes \Omega_{\overline{C}}^1(\log S)),$$

where  $\mathcal{L}_i$  is the  $(1, 0)$ -part of  $(\mathbb{L}^{\sigma_i} \otimes \mathcal{O}_C)_{\text{ext}}$ . As the Kodaira-Spencer map for  $\sigma_1 := \text{id}$  is an isomorphism (this is the definition of 'maximal Higgs'), the first summand vanishes. But the action of  $K$  permutes the summands transitively and hence  $H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{Z}}) \cap H^1(\overline{C}, j_* \mathbb{W}_{\mathbb{C}})^{1,1} = 0$ .  $\square$

**Theorem 3.3.** *Let  $\varphi : X^0 \rightarrow \text{Jac}(X^0) \rightarrow A^0$  be the embedding of a Veech surface into its Jacobian (normalized such that one of the zeros of  $\omega^0$  maps to 0) composed by the projection to the factor  $A^0$ .*

*The periodic points on a Veech surface map via  $\varphi$  to torsion points on  $A^0$ . In particular there is only a finite number of periodic points on a Veech surface if  $r > 1$ , i.e. if the surface is not square-tiled.*

The finiteness result was obtained by Gutkin, Hubert and Schmidt by entirely different methods in [GuHuSc03].

**Proof:** A periodic point of  $(X^0, \omega^0)$  gives a section of some fibred surface  $X_1 \rightarrow C_1$  by Lemma 1.2 and via  $\varphi$  a section of  $A_{C_1} \rightarrow C_1$ . This section has finite order by Thm. 3.1 thus proving the first statement.

By Thm. 5.1 in [Mo04] the family of abelian varieties  $A/C$  and also the section are defined over some number field. We fix some fibre of  $f : X \rightarrow C$  defined over some number field, say our original  $X^0$ . If  $r > 1$  the image  $\varphi(X^0)$  in  $A^0$  is a curve, which generates an abelian variety of dimension  $r$ . Hence it cannot be (a translate of) an abelian subvariety.

In this situation the Manin-Mumford conjecture says that the  $\mathbb{Q}$ -rational points of  $\varphi(X^0)$  have finite intersection with  $A_{\text{tors}}^0$ . We have seen that all periodic points are contained in this intersection.

Proofs of the Manin-Mumford conjecture were obtained in different generality by e.g. Raynaud, Hindry, Vojta, Buium, Hrushovski, McQuillan. For what we need here the proof in [PiRo02] is sufficient and maybe the most easily accessible.  $\square$

**Corollary 3.4.** *If  $(X^0, \omega^0)$  is a Veech surface that generates an algebraically primitive Teichmüller curve, then all periodic points form a torsion packet, i.e. for two periodic points  $P, Q$  the difference  $P - Q$  is torsion (as a divisor class).*

There is a converse to the above theorem, if we look for torsion sections instead of looking fibrewise:

**Proposition 3.5.** *Let  $\varphi_C : X \rightarrow \text{Jac}(X/C) \rightarrow A$  be the family of maps considered in the previous theorem and let  $\varphi_{C_1} : X_1 \rightarrow A_1$  be the map obtained by an unramified base change  $C_1 \rightarrow C$ .*

*Periodic points on  $X$  are precisely the preimages via  $\varphi_{C_1}$  of sections of  $A_1/C_1$  for all unramified coverings  $C_1 \rightarrow C$ .*

**Proof:** We may choose  $A$  as in the proof of Thm. 2.6, as the statement of the proposition is invariant under isogenies. Sections of  $\varphi_{C_1}(X)$  extend to sections of its normalisation, which was called  $f_{\text{prim}} : X_{\text{prim}} \rightarrow C_1$ . Hence they give periodic points on each fibre of  $X_{\text{prim}}$  by the criterion for periodic points given in Lemma 1.2. Finally periodic points remain periodic under passing to a translation cover of Veech surfaces.  $\square$

**Corollary 3.6.** *If  $r > 1$  there is a universal bound depending only on  $g$  for the number of periodic points on a Veech surface of genus  $g$ .*

In fact Buium ([Bu94]) gives a bound on the number of torsion sections of a family of curves in a family of abelian varieties. This bound only depends on the genus of the curve and the dimension of the abelian variety, but it grows very fast with  $g$ .

#### 4. RATIOS OF TANGENTS

For the next section we will need:

**Theorem 4.1.** *For each  $d > 0$ , there are only finitely many pairs of rational numbers  $0 < \alpha < \beta < 1/2$  such that*

$$\mu = \tan(\pi\beta)/\tan(\pi\alpha)$$

*is an algebraic number of degree  $d$  over  $\mathbb{Q}$ .*

**Proof:** This follows from [McM04c] Thm. 2.1 and the addition formula

$$\frac{\tan(\frac{x+y}{2})}{\tan(\frac{x-y}{2})} = \frac{\frac{\sin(x)}{\sin(y)} + 1}{\frac{\sin(x)}{\sin(y)} - 1}.$$

$\square$

For  $d = 2$  a list of these quotients can be obtained easily from table 3 in [McM04c]. For later use we list those quotients, which are non-units. By Galois conjugation in the cyclotomic field containing  $\mu$  we may furthermore suppose  $\alpha = 1/s$  for some  $s \in \mathbb{N}$ .

$\alpha$	$\beta$	$\mu$	Trace	Norm
1/10	1/5	$\sqrt{5}$	0	-5
1/10	2/5	$5 + 2\sqrt{5}$	10	5
1/5	3/10	$1 + 2\sqrt{1/5}$	2	1/5
1/12	1/6	$(3 + 2\sqrt{3})/3$	2	-1/3
1/12	1/3	$3 + 2\sqrt{3}$	6	-3
1/6	1/4	$\sqrt{3}$	0	-3
1/6	5/12	$3 + 2\sqrt{3}$	6	-3
1/4	1/3	$\sqrt{3}$	0	-3
1/3	5/12	$(3 + 2\sqrt{3})/3$	2	-1/3

Table 1: Quadratic ratios of tangents, that are non-units

#### 5. PERIODIC POINTS IN GENUS TWO

The  $\text{GL}_2^+(\mathbb{R})$ -action on  $\Omega M_2$  respects the multiplicities of the zeroes of the differential. We denote the corresponding strata by  $\Omega M_2(2)$  and  $\Omega M_2(1, 1)$ .

5.1. **The stratum  $\Omega M_2(2)$ .** By [McM04a] each Veech surface in the stratum  $\Omega M_2(2)$  contains an  $L$ -shaped surface in its  $\mathrm{GL}_2^+(\mathbb{R})$ -orbit.

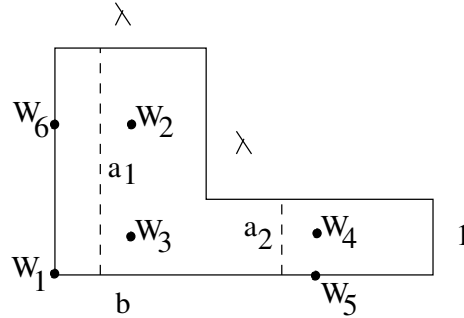


Figure 1: Admissible representative of a primitive Veech surface in  $\Omega M_2(2)$

Here  $\lambda = (e + \sqrt{(e^2 + 4b)})/2$ , where  $e \in \{-1, 0, 1\}$  and  $b \in \mathbb{N}$  with the restriction that  $e + 1 < b$  and if  $e = 1$  then  $b$  is even. A triple  $(b, e, \lambda)$  satisfying these conditions is called *admissible*.

In this section we show:

**Theorem 5.1.** *The only periodic points on a primitive Veech surface in  $\Omega M_2(2)$  are the Weierstraß points.*

**Proof:** Let  $f : X_1 \rightarrow C_1$  be a fibred surface (and  $\overline{X_1} \rightarrow \overline{C_1}$  its stable completion) corresponding to a given primitive Veech surface in  $\Omega M_2(2)$ . Suppose the Veech surface contains a non-Weierstraß periodic point  $P$ . Passing to an unramified cover of  $C_1$  we may suppose by Lemma 1.2 that the Weierstraß points and  $P$  extend to sections  $s_{W_i}$  for  $i = 1, \dots, 6$  and  $s_P$  of  $f$ . We suppose that  $s_{W_1}$  passes through the zero of  $\omega^0$ . By Thm. 3.3 and the primitivity assumption the section  $s_P - s_{W_1}$  is torsion.

We start from an admissible representative  $(X^0, \omega^0)$  and observe what happens if we degenerate in the vertical direction, i.e. along the path

$$(X^t, \omega^t) = \mathrm{diag}(e^t, e^{-t}) \cdot (X^0, \omega^0)$$

for  $t \rightarrow \infty$ , where  $(1 \ 0)^T$  corresponds to the horizontal and  $(0 \ 1)^T$  to the vertical direction. Note that the action of  $\mathrm{diag}(e^t, e^{-t})$  does not change the ratio of the heights of the vertical cylinders.

The curves  $X^t$  are obtained by first cutting  $X^0$  along the centers of the vertical cylinders and then glueing a pair of annuli (of some modulus increasing with  $t$ ) along the cut circles. The limit curve is obtained by 'squeezing' the interior of each vertical cylinder to a point. See [Ma74] for more details.

Hence the stable model of the limit curve  $X^\infty$  is a rational curve with two pairs of points identified. By construction these nodes are fixed points of the hyperelliptic involution.

Normalising suitably, we may suppose that  $X^\infty$  looks as follows: The Weierstraß section  $s_{W_1}$  intersects  $X^\infty$  in  $\infty$ ,  $s_{W_2}$  intersects  $X^\infty$  in zero and the hyperelliptic involution becomes  $z \mapsto -z$ . By a linear transformation we may suppose that  $s_P$  intersects  $X^\infty$  in 1 and that the remaining Weierstraß sections are glued to pairs  $x$  with  $-x$  and  $y$  with  $-y$ . for some  $x, y \in \mathbb{C} \setminus \{0, \pm 1\}$ .

Furthermore  $\omega^0$  comes from a subbundle of  $f_*\omega_{\overline{X_1}/\overline{C_1}}$  and specialises to a section  $\omega^\infty$  of the dualizing sheaf on the singular fibre  $X^\infty$ . Thus the differential  $\omega^\infty$  has to vanish to

the order two at  $\infty$  and it has simple poles at  $x, -x, y$  and  $-y$ . Furthermore we must have

$$\operatorname{Res}_{\omega^\infty}(x) = \operatorname{Res}_{\omega^\infty}(-x) \quad \text{and} \quad \operatorname{Res}_{\omega^\infty}(y) = \operatorname{Res}_{\omega^\infty}(-y).$$

The differential

$$\omega^\infty = \left( \frac{y}{z-x} - \frac{y}{z+x} - \frac{x}{z-y} + \frac{x}{z+y} \right) dz$$

has this property and by Riemann-Roch it is unique up to scalar multiple.

The invariance of the height ratios implies

$$\frac{y}{x} = \frac{\operatorname{Res}_{\omega^\infty}(x)}{\operatorname{Res}_{\omega^\infty}(-y)} = \frac{\int_{a_1} \omega^0}{\int_{a_2} \omega^0} = \frac{\lambda + 1}{1}$$

up to sign and interchanging the roles of  $x$  and  $y$ .

Due to the irreducibility of  $X^\infty$  and [McM04c] Thm. 3.4 the divisor  $s_P - s_{W_1}$  remains torsion on the singular fibre. Thus  $1 - \infty$  is say  $N$ -torsion. In order to have a well-defined map  $g : X^\infty \rightarrow \mathbb{P}^1, z \mapsto (z-1)^N$  we must have both

$$(x-1)^N = (-x-1)^N \quad \text{and} \quad (y-1)^N = (-y-1)^N.$$

This implies that  $x, y \in i\mathbb{R}$ , in fact we must have  $x = i \tan(A\pi/N)$  and  $y = i \tan(B\pi/N)$  for some  $A, B \in \mathbb{Z}$ .

We now use the list of tangent ratios of the previous section: If  $y/x$  is a unit then

$$|\operatorname{Norm}(\lambda + 1)| = |e + 1 - b| = 1.$$

The only admissible triples that satisfy this condition are  $e = 0$  and  $b = 2$  (which gives the octagon) and  $e = -1$  and  $b = 1$  (which gives the pentagon). For these two cases the theorem was proved in [GuHuSc03] Examples 3 and 4. Alternatively one might use that these Teichmüller curves pass through the Riemann surfaces  $y^2 = x(x^4 - 1)$  resp.  $y^2 = x^5 - 1$ , whose torsion points are known ([BoGr00]). Then one can conclude as in the proof of Thm. 5.2. If  $y/x$  is not a unit we can rule out each element of the list of table 1 by the conditions that  $\operatorname{Norm}(\lambda + 1)$  has to be an integer and  $\operatorname{Trace}(\lambda + 1) \in \{1, 2, 3\}$ .  $\square$

**5.2. The stratum  $\Omega M_2(1,1)$ .** This stratum contains only one primitive Teichmüller curve, the one generated by the regular decagon (see [McM04b] and [McM04c]). The decagon corresponds to the Riemann surface  $y^2 = x^5 - 1$  with the differential  $x dx/y$  and its  $\operatorname{GL}_2^+(\mathbb{R})$ -orbit contains the following translation surface:

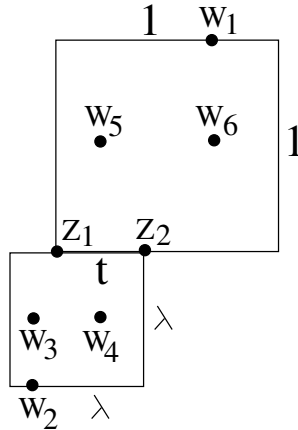


Figure 2: Representative in the  $\mathrm{GL}_2^+(\mathbb{R})$ -orbit of the decagon

Here the side lengths of the squares are 1 and  $\lambda = (-1 + \sqrt{5})/2$  and  $t = \sqrt{5}/5$ . We denote this surface by  $(X^0, \omega^0)$ .

**Theorem 5.2.** *The only periodic points of the decagon are the Weierstraß points and the zeroes of  $\omega^0$ .*

**Proof:** We consider the degeneration of the surface in Figure 2 by squeezing the horizontal direction. The singular fibre  $X^\infty$  in the limit is a  $\mathbb{P}^1$  with two pairs of points identified. We suppose that the sections  $s_{Z_i}$  extending the two zeroes  $Z_1$  and  $Z_2$  of  $\omega^0$  intersect  $X^\infty$  in 0 and  $\infty$  respectively. We may suppose that the hyperelliptic involution acts by  $z \mapsto 1/z$ . This forces the Weierstraß sections  $s_{W_1}$  and  $s_{W_2}$  to intersect in  $X^\infty$  in  $-1$  and  $1$  respectively. Furthermore  $s_{W_3}$  and  $s_{W_4}$  intersect  $X^\infty$  in the pair of identified points  $x$  and  $1/x$  while  $s_{W_5}$  and  $s_{W_6}$  degenerate to the pair of identified points  $y$  and  $1/y$ .

The differential  $\omega^\infty$  has simple zeros in 0 and  $\infty$  and simple poles at  $x$ ,  $1/x$ ,  $y$  and  $1/y$  such that

$$\mathrm{Res}_{\omega^\infty}(x) = \mathrm{Res}_{\omega^\infty}(1/x) \quad \text{and} \quad \mathrm{Res}_{\omega^\infty}(y) = \mathrm{Res}_{\omega^\infty}(1/y).$$

This implies that

$$\omega^\infty = \left( \frac{y - 1/y}{z - x} - \frac{y - 1/y}{z - 1/x} + \frac{1/x - x}{z - y} - \frac{1/x - x}{z - 1/y} \right) dz$$

By Thm. 3.3 the difference  $Z_1 - Z_2$  is torsion. Considering the surface  $y^2 = x^5 - 1$  one notices (see e.g. [BoGr00]) that  $Z_1 - Z_2$  is 5-torsion, that  $Z_1 - W_1$  is 5-torsion and that whenever  $R - W_1$  is torsion with  $R$  a non-Weierstraß point, then  $R - W_1$  is 5-torsion. Hence  $g_1(z) = z^5$  and  $g_2(z) = (z + 1)^5$  are well-defined on  $X^\infty$ . This implies that (up to replacing  $x$  or  $y$  by its inverse or interchanging  $x$  and  $y$ ) we have  $x = \exp(2\pi i/5)$  and  $y = \exp(4\pi i/5)$ . If there was another periodic point  $R$  on the decagon, which becomes  $r$  on  $X^\infty$  then  $g_3(z) = (z - r)^5$  would have to be well-defined on  $X^\infty$ . This implies immediately that  $r$  has to be real. Let  $M_{10}$  be the set of complex numbers with argument a multiple of  $2\pi/10$ . Then  $\{x - r, r \in \mathbb{R}\}$  intersects  $M_{10}$  only for  $r = -1$  or  $r \geq 0$  and  $\{y - r, r \in \mathbb{R}\}$  intersects  $M_{10}$  only for  $r \leq 0$ . Hence there is no possible choice for  $r \notin \{-1, 0, \infty\}$ .  $\square$

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