Seshadri fibrations and the Nagata-Biran conjecture on algebraic surfaces

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Introduction

Recently several authors studied the question of how submaximal Seshadri constants determine the geometry of the underlying variety [5], [3], [7]. The results say that if the Seshadri constant of an ample line bundle at a general (and hence at every) point of a variety is small relatively to the maximal possible bound (which depends on L), then the variety is fibered by subvarieties responsible for the Seshadri constants being small. The question was studied in arbitrary dimension in [3] and on surfaces in [5] and [7], where an optimal bound for the existence of the fibration was given. Here we pass to the multiple point Seshadri constants and show that the picture is not only similar to the one point situation but somewhat surprisingly the the result is stronger, it verifies, at least asymptotically, the Nagata-Biran conjecture on a big class of surfaces. For very ample divisors this was observed by different methods by Harbourne [2, Theorem I.1].

Definitions and preliminaries

Let X be a smooth projective variety of dimension n and let L be an ample line bundle on X. Demailly [1] introduced invariants measuring in effect the positivity of L along a subvariety. We recall the definition in the set up relevant for us.

Definition.

Let X be a smooth projective variety, let L be an ample line bundle on X and let $P_1, \ldots, P_r \in X$ be mutually distinct points. The r-tuple Seshadri constant of L at

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$P_1, \dots, P_r$ is the real number

$$\varepsilon(L; P_1, \dots, P_r) = \inf_{C \cap \{P_1, \dots, P_r\} \neq \emptyset} \frac{L.C}{\sum \operatorname{mult}_{P_i} C} \ ,$$

where the infimum is taken over all (irreducible) curves passing through at least one of the points $P_1, \dots, P_r$.
We say that a curve C is a Seshadri curve if $\varepsilon(L; P_1, \dots, P_r) = \frac{L.C}{\sum \operatorname{mult}_{P_i} C}$.

As a function on X^r the Seshadri constant $\varepsilon(L; \cdot, \dots, \cdot)$ is semi-continuous and attains its maximal value at a very general point of X^r i.e. on the subset of X^r being the complement of a union of at most countably many Zariski closed subsets. This maximal value we will abbreviate by $\varepsilon(L; r)$. It is conjectured that for r sufficiently large $\varepsilon(L; r)$ has the maximal possible value which is $\varepsilon_{\max}(L; r) := \sqrt[n]{\frac{1}{r} L^n}$.

Conjecture. (Nagata-Biran) Let X be a smooth projective variety and L an ample line bundle on X. Then there exists a number r_0 (depending on X and L) such that for all $r \geq r_0$

$$\varepsilon(L; r) = \varepsilon_{\max}(L; r).$$

For an effective statement, background and equivalent formulations see [6].

The result

The main result of this paper is the following

Theorem. *Let X be a smooth projective surface, L an ample line bundle on X and $r \geq 2$ a fixed integer. If*

$$\varepsilon(L; r) < \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; r) \tag{1}$$

then there exists a fibration $f : X \longrightarrow D$ over a curve D such that given $P_1, \dots, P_r \in X$ very general, the fiber $f^{-1}(f(P_i))$ computes $\varepsilon(L; P_1, \dots, P_r)$ for arbitrary $i = 1, \dots, r$. Moreover the factor $\sqrt{\frac{r-1}{r}}$ is optimal for every r.

Remark. For $r = 1$ the above Theorem says nothing. It was proved in [7] that $\varepsilon(L; 1) < \sqrt{\frac{3}{4}} \cdot \varepsilon_{\max}(L; 1)$ implies the existence of the fibration by computing curves.

Corollary. *If the surface X admits no fibration over a curve (e.g. general surface of general type), then*

$$\varepsilon(L; r) \geq \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; r).$$

In particular the Nagata-Biran conjecture holds asymptotically.

The proof

For the proof we need several additional results. The first Lemma was proved by Xu [8].

Lemma 1 *Let X be a smooth projective surface, let $(C_t, (P_1)_t, \ldots, (P_r)_t)$ be a one parameter family of pointed curves on X and let m_i be integers such that $\operatorname{mult}_{(P_i)_t} C_t \geq m_i$ for all $i = 1, \ldots, r$, then*

$$C_t^2 \geq \sum_{i=1}^{r} m_i^2 - \min\{m_1, \ldots, m_r\}.$$

The second Lemma was obtained by Küchle in [4].

Lemma 2 *Let $r \geq 2$ and $m_1, \ldots, m_r \in \mathbb{Z}$ be integers with $m_1 \geq \ldots \geq m_r \geq 1$ and $m_1 \geq 2$. Then we have*

$$(r+1)\sum_{i=1}^{r} m_i^2 > \left(\sum_{i=1}^{r} m_i\right)^2 + m_r(r+1).$$

Some versions of the next statement were obtained in [6]. As this version is more general and the proof is way easier, we include them both for the sake of completeness.

Proposition 3 *Let (Y, L) be a polarized surface with Picard number ρ and let $P_1, \ldots, P_r$ be points in Y such that $\varepsilon := \varepsilon(L; P_1, \ldots, P_r) < \varepsilon_{\max}(L; r)$. Then there are at most $\rho+r-1$ irreducible and reduced Seshadri curves.*

Proof. Let $\pi : X \longrightarrow Y$ be the blowing up of Y at $P_1, \ldots, P_r$ with exceptional divisors $E_1, \ldots, E_r$ and let $H := \pi^* L$. Suppose $C_1, \ldots, C_s$ are irreducible and reduced curves computing ε, $\widetilde{C}_1, \ldots, \widetilde{C}_r$ are their proper transforms. The $\mathbb{Q}$-divisor $M := H - \varepsilon \sum_{i=1}^{r} E_i$ is nef and big and we have $M.(\sum_{i=1}^{s} \lambda_i \widetilde{C}_i) = 0$ for arbitrary $\lambda_i \geq 0$. The Hodge Index Theorem implies that the intersection matrix of $\widetilde{C}_1, \ldots, \widetilde{C}_s$ is negative definite. Since $\rho(X) = \rho + r$ it implies the assertion $s \leq \rho + r - 1$. □

The above Proposition has an interesting consequence in the case $P_1, \ldots, P_r$ are general points. We say that an r–tuple $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ is *almost-homogeneous* if all but at most one of the coordinates are equal. We say that a curve C is almost-homogeneous at $P_1, \ldots, P_r$ if the r–tuple $(\operatorname{mult}_{P_1} C, \ldots, \operatorname{mult}_{P_r} C)$ is almost-homogeneous.

Lemma 4 *Let (X, L) be a polarized surface with Picard number $\rho = \rho(X)$ and let $P_1, \ldots, P_r$ be general points on X. If $\varepsilon(L; P_1, \ldots, P_r) < \varepsilon_{\max}(L; r)$ then any irreducible Seshadri curve is almost-homogeneous.*

Proof. Since the points are general the monodromy group acts as the full symmetric group S_r i.e. if there is an irreducible curve C with multiplicities $(\operatorname{mult}_{P_1} C, \ldots, \operatorname{mult}_{P_r} C)$, then there exists an irreducible curve C_σ with multiplicities $\operatorname{mult}_{P_i} C_\sigma = \operatorname{mult}_{P_{\sigma(i)}} C$ for $i = 1, \ldots, r$ and $\sigma \in S_r$. The only possibility that there are at most $\rho + r - 1$ curves in the set $\{C_\sigma\}_{\sigma \in S_r}$ is that they are almost-homogeneous. □

Now we are in the position to prove our main result.

Proof of Theorem. Let $P_1, \ldots, P_r \in X$ be very general. Since $\varepsilon(L; P_1, \ldots, P_r)$ is not maximal, there exists a computing curve $C_{P_1,\ldots,P_r}$. Moving the points around we obtain a whole family of such curves. Let $m_1 \geq \ldots \geq m_r$ be integers such that $\operatorname{mult}_{P_i} C_{P_1,\ldots,P_r} = m_i$ for the general member $C_{P_1,\ldots,P_r}$ of the family.

Suppose first that $m_1 \geq 2$ and $m_r \geq 1$. Then the Hodge Index Theorem together with Lemma 1 give:

$$\frac{\sum m_i^2 - m_r}{(\sum m_i)^2} \leq \frac{L^2 \cdot C^2}{(\sum m_i)^2} \leq \frac{(L.C)^2}{(\sum m_i)^2} < \frac{r-1}{r^2} L^2.$$

Hence by Lemma 2 we obtain

$$\sum m_i^2 - m_r < \frac{r-1}{r^2} \left(\sum m_i \right)^2 < \frac{(r-1)(r+1)}{r^2} \left(\sum m_i^2 - m_r \right),$$

a contradiction.

What is left is either
a) $m_1 = \ldots = m_r = 1$
or $m_r = 0$ and
b) $(m_1, \ldots, m_r) = (m, \ldots, m, 0)$ or
c) $(m_1, \ldots, m_r) = (m, 0, \ldots, 0)$.
Note that our Lemma 4 is crucial in order to limit the number of possibilities for the $m_i's$.

Case a) is immediately excluded as $C^2 \geq r - 1$ by Lemma 1 and thus

$$\frac{L.C}{\sum m_i} = \frac{L.C}{r} \geq \frac{\sqrt{r-1}}{r} \sqrt{L^2}$$

by Hodge Index Theorem contradicting our condition (1).

In case b) we may assume $r \geq 3$ as the case $r = 2$ is covered by case c) as well. Again, Lemma 1 implies $C^2 \geq (r-1)m^2 - m$ which together with Hodge Index Theorem leads to the inequality:

$$m < \frac{r^2}{2r^2 - 3r + 1}.$$

For $r \geq 3$ this forces $m = 1$ which is absurd.

Thus we are left with case c) i.e. a computing curve passes through just one of the points $P_1, \ldots, P_r$. As $r \geq 2$ we have

$$\frac{L.C}{m} < \sqrt{\frac{r-1}{r^2} L^2} < \sqrt{\frac{3}{4} L^2}$$

which shows that the curve is submaximal for one point Seshadri constant (see Remark), hence we must have $m = 1$ and there is the desired fibration by [7, Theorem].

□

The following example shows that our Theorem is optimal.

Example. Let $X = \mathbb{P}^2$, let $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and let $r = 2$. Then the line through two given points P_1, P_2 computes $\varepsilon(L; P_1, P_2) = \frac{1}{2} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; 2)$ and there is no fibration on $\mathbb{P}^2$.

More generally, let r be given and let X be a rational normal scroll in $\mathbb{P}^r$ and $L = \mathcal{O}_X(1)$. The scroll is of course fibered but the curves in the ruling are not the Seshadri curves. To see this let $P_1, \ldots, P_r \in X$ be points in general position. Then obviously for a fiber F of the ruling passing through the set $P_1, \ldots, P_r$ we have

$$\frac{L.F}{\sum \mathrm{mult}_{P_i}} = 1.$$

On the other hand the points span a hyperlane in P^r i.e. there is a curve $C \in |L|$ passing through all of them with Seshadri quotient

$$\frac{L.C}{\sum \mathrm{mult}_{P_i}} = \frac{r-1}{r} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; r) < 1.$$

So X is not fibered by the Seshadri curves in this case.

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