THE TRANSCENDENTAL PART OF THE REGULATOR MAP FOR K_1 ON A MIRROR FAMILY OF $K3$-SURFACES

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Abstract

We compute the transcendental part of the normal function corresponding to the Deligne class of a cycle in K_1 of a mirror family of quartic $K3$ surfaces. The resulting multivalued function does not satisfy the hypergeometric differential equation of the periods, and we conclude that the cycle is indecomposable for most points in the mirror family. The occurring inhomogenous Picard-Fuchs equations are related to Painlevé VI–type differential equations.

1. The regulator map and Picard-Fuchs equations

In this paper we study the first nonclassical higher K-group $K_1(X)$ for a smooth complex projective surface X. It was conjectured by H. Esnault around 1995 that certain elements in this group can be detected in the transcendental part of the Deligne cohomology group $H^3_{\mathscr{D}}(X, \mathbb{Z}(2))$ via the regulator (Chern class) map. The transcendental part of the regulator map is defined as an Abel-Jacobi-type integral of holomorphic 2-forms over nonclosed real 2-dimensional chains in X associated to these elements. At that time it was only known that one could detect such classes in the complementary $(1, 1)$-part of Deligne cohomology (see, e.g., [16]). The goal of our paper is to show that Esnault's conjecture is true by looking at the differential equations that are satisfied by the normal functions arising from such classes in a family of surfaces. It turns out that the resulting equations for Abel-Jacobi-type integrals with parameters are strongly connected to a generalization of Painlevé VI–type differential equations.

The higher K-groups $K_1(X), K_2(X), \ldots$ of an algebraic variety X were defined around 1970 by D. Quillen [19]. Later S. Bloch [2] showed that on smooth quasi-projective varieties all their graded pieces with respect to the γ-filtration may be computed as

$$\mathrm{gr}^p_\gamma K_n(X)_{\mathbb{Q}} \cong \mathrm{CH}^p(X, n)_{\mathbb{Q}},$$

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where $\mathrm{CH}^p(X,n)$ are Bloch's higher Chow groups in [2]. This isomorphism gives an explicit presentation of higher K-groups modulo torsion via algebraic cycles.

Let us look more closely at the particular case of $K_1(X)$ for a smooth complex projective surface X. There it is known that $\mathrm{CH}^1(X,1)=\mathbb{C}^\times$ and $\mathrm{CH}^p(X,1)=0$ for $p\geq 4$. The remaining interesting parts of K_1 are therefore $\mathrm{CH}^2(X,1)$ and $\mathrm{CH}^3(X,1)$. The last group consists of zero cycles on $X\times\mathbb{P}^1$ in good position, and therefore the map

$$\tau:\quad \mathrm{CH}^2(X)\otimes_Z\mathbb{C}^\times\to\mathrm{CH}^3(X,1),\qquad x\otimes a\mapsto(x,a)$$

is surjective. Therefore the complexity of $\mathrm{CH}^3(X,1)$ is governed by the complexity of $\mathrm{CH}^2(X)$, which is fairly understood by Mumford's theorem, respectively, Bloch's conjecture. We say that $\mathrm{CH}^3(X,1)$ is decomposable. For $\mathrm{CH}^2(X,1)$ the situation is quite different and the complex geometry of X plays an essential role in the understanding of it. The natural map

$$\tau:\quad \mathrm{CH}^1(X)\otimes_Z\mathbb{C}^\times\to\mathrm{CH}^2(X,1),\qquad D\otimes a\mapsto D\times\{a\}$$

is neither surjective nor injective in general. In the literature there are several examples where the cokernel of τ is nontrivial modulo torsion and even infinite-dimensional (see [4], [11], [16], [24]). The kernel of τ is related but not equal to $\mathrm{Pic}^0(X)\otimes\mathbb{C}^\times$ even modulo torsion by [20, Th. 5.2]. Note that the cokernel of τ is a *birational invariant* (by localization) and hence vanishes for rational surfaces and, in fact, for all surfaces with geometric genus $p_g(X)=0$ and Kodaira dimension less than or equal to 1. Bloch's conjecture would imply that it also vanishes for all surfaces of general type which satisfy $p_g(X)=0$. One way to study $\mathrm{CH}^2(X,1)$ is to look at the Chern class maps

$$c_{2,1}:\quad \mathrm{CH}^2(X,1)\to H^3_{\mathscr{D}}\big(X,\mathbb{Z}(2)\big)=\frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z})+F^2H^2(X,\mathbb{C})}. \tag{1.1}$$

The decomposable cycles (the image of τ) are mapped to the subgroup

$$\mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{C}^\times\subseteq\frac{H^2(X,\mathbb{C})}{H^2(X,\mathbb{Z})+F^2H^2(X,\mathbb{C})}$$

generated by the *Néron-Severi group* $\mathrm{NS}(X)\subset H^2(X,\mathbb{Z})$ of all divisors in X. It is known (see [16]) that the image of $c_{2,1}$ is at most countable modulo this subgroup, so that the image of $\mathrm{coker}(\tau)$ in Deligne cohomology modulo $\mathrm{NS}(X)\otimes_{\mathbb{Z}}\mathbb{C}^\times$ is at most countable. One conjectures that even $\mathrm{coker}(\tau)$ itself is countable.

The Chern class maps $c_{2,1}$ are defined as follows: let $Z=\sum a_jZ_j\in\mathrm{CH}^2(X,1)$ be a cycle. Each Z_j is an integral curve and inherits a rational map $f_j:Z_j\to\mathbb{P}^1$ from the projection map $X\times\mathbb{P}^1\to\mathbb{P}^1$. Let γ_0 be a path on $\mathbb{P}^1$ connecting 0 with

∞ along the real axis; then $\gamma := \bigcup \gamma_j := \bigcup f_j^{-1}(\gamma_0)$ is a closed homological 1-cycle, Poincaré dual to a cohomology class in $F^2H^3(X, \mathbb{Z})$ and therefore torsion (see [16]). If we assume that $\gamma = 0$ (e.g., if $b_1(X) = 0$), then we write $\gamma = \partial\Gamma$ for a real piecewise smooth 2-chain Γ. As a current, that is, as a linear functional on differentiable complex-valued 2-forms on X, the defining property of $c_{2,1}(Z)$ is

$$c_{2,1}(\alpha) = \frac{1}{2\pi i} \sum_j \int_{Z_j - \gamma_j} \log(f_j)\alpha + \int_\Gamma \alpha. \tag{1.2}$$

Now let X be a projective $K3$-surface. Then $p_g(X) = 1$, $b_2(X) = 22$, and $b_1(X) = 0$. The intersection form on $H^2(X, \mathbb{Z})$ is known to be the unimodular form $2E_8 \oplus 3H$, where H is the 2-dimensional standard hyperbolic form.

The Néron-Severi lattice $\mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$ has an orthogonal complement $T(X) \subset H^2(X, \mathbb{Z})$. In particular, there is a well-defined morphism

$$\mathrm{Coker}(\tau) \to \frac{T(X) \otimes \mathbb{C}^\times}{F^2}.$$

If we have an arbitrary smooth family $f : X \to B$ of complex algebraic surfaces over a quasiprojective complex variety B, and an algebraic family of cycles $Z_b \in \mathrm{CH}^2(X_b, 1)$ for all $b \in B$, that is, $Z_b = \mathscr{Z}|X_b$ for some given cycle $\mathscr{Z} \in \mathrm{CH}^2(X, 1)$, then we may define the *normal function*

$$\nu(b) := c_{2,1}(Z_b) \in \frac{T(X_b) \otimes \mathbb{C}^\times}{F^2}.$$

One can easily show that ν is a holomorphic section of the corresponding family of generalized tori $T(X_b) \otimes \mathbb{C}^\times / F^2$. Coming back to the case of $K3$-surfaces, there the canonical bundle ω_X is trivial; hence the group $H^{0,2}(X) = H^0(X, \Omega_X^2)^* = \mathbb{C}$ is 1-dimensional and generated by the dual of ω_X. In a smooth algebraic family X_b of $K3$-surfaces, the composition of the regulator with the projection onto

$$\frac{H^{0,2}(X_b)}{\mathrm{Im}\, H_2(X_b, \mathbb{Z})}$$

produces a multivalued holomorphic function on B, denoted by $\bar{\nu}(b)$, which has poles at all b where the family degenerates (see proof of Lem. 3.1). It is given by the formula

$$\bar{\nu}(b) = \int_{\Gamma_b} \omega_{X_b}$$

since the integral of ω_X over any effective divisor vanishes. If $\mathscr{D}_{\mathrm{PF}}$ denotes the Picard-Fuchs differential operator of the Gauss-Manin connection associated to the family X_b of $K3$-surfaces, then $\mathscr{D}_{\mathrm{PF}}$ annihilates all periods of the family. Therefore we obtain the following result.

LEMMA 1.1
Let $B \subset \bar{B}$ be a smooth compactification of B. Then with the notation above, $\mathscr{D}_{\mathrm{PF}}(\bar{\nu})$ extends to a single-valued meromorphic function on $\bar{B}$ with poles only along degeneracies of X_b, and therefore it satisfies a differential equation

$$\mathscr{D}_{\mathrm{PF}}\big(\bar{\nu}(b)\big) = g(b), \tag{1.3}$$

where g is a rational function in $b \in \bar{B}$.

The proof is given in the appendix. Altogether, we have obtained a map

$$\big\{\text{families of cycles in } \mathrm{CH}^2(X_b, 1)\big\} \longrightarrow \{\text{differential equations/rational functions}\}. \tag{1.4}$$

For each such family of $K3$-surfaces, it sends a family of cycles to the equation $\mathscr{D}_{\mathrm{PF}}\bar{\nu} = g$, respectively, the rational function g, which is the same information on a given family. One should view the resulting solutions $\bar{\nu}(b)$ as *new transcendental functions* arising from the family of K-theoretic cycles in $\mathrm{CH}^2(X_b, 1)$. If g is a nontrivial function, then $\bar{\nu}$, and hence ν, is a nonflat section of the family of Deligne cohomology groups of X_b. In [16] the relationship between the infinitesimal behavior of such normal functions and the mixed Hodge structure of the total space X has already been investigated.

This situation is very reminiscent of a method developed by R. Fuchs [10] in the case of the Legendre family $y^2 = x(x-1)(x-t)$ of elliptic curves and investigated further in the work of Yu. Manin [14, p. 134]. In particular, there is a strong connection with differential equations of a generalized form of type Painlevé VI (see [14]).

There exists a *formula* to derive g: there is a so-called inhomogeneous Picard-Fuchs equation

$$\mathscr{D}_{\mathrm{PF}}\omega_X = d_{\mathrm{rel}}\beta \tag{1.5}$$

before integration over Γ, where β is a section of the vector bundle of (meromorphic) 1-forms in the fibers of the family $f : X \to B$. We say that Γ *does not depend on* b if it can be defined as a real semialgebraic subset via *flat coordinates*, that is, via coordinate functions that are horizontal with respect to the Gauss-Manin connection, and such that the defining inequalities of Γ are polynomials not depending on b. This shows on the one hand that for closed Γ the periods satisfy the Picard-Fuchs equation, and on the other hand that for nonclosed Γ (not depending on b) with $\partial\Gamma = \gamma$, we get

$$g(b) = \mathscr{D}_{\mathrm{PF}} \int_\Gamma \omega_X = \int_\Gamma d_{\mathrm{rel}}\beta = \int_\gamma \beta. \tag{1.6}$$

The last equality uses a version of Stokes's theorem for currents since some of the differential forms involved, in general, have integrable singularities. Hence Stokes's theorem for currents (see [12, Chap. 3]) also implies that β is integrable over γ. In general, Γ depends also on b, and then there is an additional contribution from the derivatives of the boundaries of the integral. In the case of the Legendre family $y^2 = x(x-1)(x-t)$ of elliptic curves, β is a meromorphic function (zero-form) after a double covering

$$\frac{y}{2(x-t)^2}$$

by [10, p. 310] and [14, p. 76]. Manin has put these equations into a more formal context (so-called μ-equations), so that one can understand the sections and operators in a coordinate-free way in terms of certain locally free sheaves on B. This also plays a role in his work on the functional Mordell conjecture. Furthermore, after uniformizing the elliptic curves, the inhomogeneous Picard-Fuchs equation is equivalent to a version involving the Weierstrass $\mathfrak{p}$-function (see [14, p. 137]):

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \mathfrak{p}_z\Big(z + \frac{T_j}{2}, \tau\Big),$$

where α_j are constants parametrizing the family of differential equations and where $(T_0, \dots, T_3) = (0, 1, \tau, 1+\tau)$ are the vertices of the fundamental parallelogram. In this way the transcendental aspect of the solutions and also the connection to integrable systems become apparent (see [14, p. 139]). In the future we hope to investigate further the transcendental properties of our solutions (using again uniformization) and to study the attached integrable systems.

The rest of this article is devoted to a particular solution of the inhomogeneous Picard-Fuchs equation for a certain family of $K3$-surfaces introduced in Section 2. In Section 3 we deduce Esnault's conjecture from the nonvanishing of the $\mathscr{D}_{\mathrm{PF}}(\bar{\nu})$ in the special case of $b = \sqrt{-1}$. In Section 4 we study a certain Shioda-Inose model of X_b which has isomorphic transcendental cohomology. This leads to an explicit computation of β in this case.

2. An example: A mirror family of $K3$-surfaces

We study the 1-dimensional family of $K3$-surfaces given by the quartic equations

$$X_b := \big\{(x, y, z, w) \in \mathbb{P}^3 \mid f(x, y, z, w) = xyz(x + y + z + bw) + w^4 = 0\big\} \quad (2.1)$$

with $b \in \mathbb{P}^1$. Note that this surface, for general b, is not smooth but has six singular points defining a rational singularity of type A_3. The six points are (see [17, Sec. 4])

$$\begin{aligned} P_1 &= (0, 1, -1, 0), & P_2 &= (1, -1, 0, 0), \\ P_3 &= (1, 0, -1, 0), & P_4 &= (1, 0, 0, 0), \\ P_5 &= (0, 1, 0, 0), & P_6 &= (0, 0, 1, 0). \end{aligned}$$

The minimal resolution of the singularities defines a generically smooth family of $K3$-surfaces. In [17] the following theorem was shown.

THEOREM 2.1 (N. Narumiya and H. Shiga)
The family X_b has the following properties.

(1) *It arises as a mirror family from the dual of the simplest polytope P of dimension three. The dual mirror family is the family of all quartic $K3$-surfaces.*

(2) *The rank of* $\mathrm{Pic}(X)$ *is greater than or equal to* 19 *for all* $b \in \mathbb{P}^1 \setminus \{0, \pm 4, \infty\}$ *and equal to* 19 *for very general b (see [17, Sec. 4]).*

(3) $T(X_b)$ *has signature* $(2, 1)$ *for* $b \in \mathbb{P}^1 \setminus \{0, \pm 4, \infty\}$.

(4) *The periods of X_b satisfy the Picard-Fuchs equation*

$$(1-u)\Theta^3 - \frac{3}{2}u\Theta^2 - \frac{11}{16}u\Theta - \frac{3}{32}u = 0$$

(where $\Theta = u(d/du)$) of the generalized Thomae hypergeometric function (see [22])

$$F_{3,2}\Big(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; u\Big)$$

and where we set $u := (4/b)^4$.

(5) *In other words, the Picard-Fuchs equation is given by*

$$(1-u)u^2\Phi''' + 3u\Big(1 - \frac{3}{2}u\Big)\Phi'' + \Big(1 - \frac{51}{16}u\Big)\Phi' - \frac{3}{32}\Phi = 0. \tag{2.2}$$

(6) *The mirror map of the family X_b is given by the arithmetic Thompson series $T(q)$ of type* 2A *in the classification of J. Conway and S. Norton [5]:*

$$T(q) = \frac{1}{q} + 8 + 4372q + 96256q^2 + 124002q^3 + 10698752q^4 + \dots.$$

Proof
We refer to [17] for more details, but we sketch the proof of (4) and (5) since this is crucial; (1)–(3) follow from the construction. In particular, six A_3-singularities give rise to 18 independent cohomology classes of type (1,1), so that the Picard number is greater than or equal to 19. Since (5) is an easy corollary of (4), we prove (4). In [17] the periods are computed as power series in $1/b$, and the differential equation in (4) follows from [22]. As in [17], we consider the new affine equation

$$f(x, y, z) = xyz(x + y + z + 1) + \frac{1}{b^4} = 0$$

obtained by substituting $w' := bw$ and setting $w' = 1$. The periods are integrals of the form

$$I(b) = \frac{1}{2\pi i}\int_{|x|=|y|=|z|=1/4} \frac{dx\,dy\,dz}{xyz(x+y+z+1) + 1/b^4}.$$

On the other hand, one has a geometric series expansion at $b = \infty$:

$$\frac{1}{xyz(x+y+z+1)+1/b^4} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{-4n}}{(xyz)^{n+1}(x+y+z+1)^{n+1}}.$$

Changing the order of summation and integration, we obtain

$$\begin{aligned} I(b) &= \frac{1}{2\pi i} \int_{|x|=|y|=|z|=1/4} \sum_{n=0}^{\infty} \frac{(-1)^n b^{-4n}\, dx\, dy\, dz}{(xyz)^{n+1}(x+y+z+1)^{n+1}} \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|x|=|y|=|z|=1/4} \frac{(-1)^n b^{-4n}\, dx\, dy\, dz}{(xyz)^{n+1}(x+y+z+1)^{n+1}}. \end{aligned}$$

Now one can apply three times the residue theorem and get

$$I(b) = (2\pi i)^2 \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} b^{-4n}.$$

Observing the identity involving Pochhammer symbols

$$\frac{(4n)!}{(n!)^4} = \frac{(1/4)_n (2/4)_n (3/4)_n}{(1)_n (1)_n (1)_n} (4^4)^n,$$

we have shown that

$$I(b) = (2\pi i)^2 F_{3,2}\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1, 1; \left(\frac{4}{b}\right)^4\right).$$

Substituting $u := (4/b)^4$, one gets a multiple of the functions $F_{3,2}(1/4, 2/4, 3/4, 1, 1; u)$ which satisfies a differential equation of order three precisely of the type described in (4), respectively, (5), by [22]. □

COROLLARY 2.2

In b-coordinates, the Picard-Fuchs equation can be written as

$$\left(\left(\frac{b}{4}\right)^4 - 1\right)\left(\frac{b}{4}\right)^3 \Phi''' + \frac{3}{4}\left(\frac{b}{4}\right)^2 \left(1 + \left(\frac{b}{4}\right)^4\right)\Phi'' + \frac{1}{16}\frac{b}{4}\left(\left(\frac{b}{4}\right)^4 - 6\right)\Phi' + \frac{3}{32}\Phi = 0. \tag{2.3}$$

Proof

Use the chain rule. □

To make the following computations easier, we follow [17] and perform the following birational coordinate change (written in affine coordinates):

$$X = xy, \qquad Y = i\left(\frac{bxy}{2} + \frac{1+xyz}{z}\right), \qquad Z = yz.$$

Then X, Y, Z are affine coordinates and define the family of surfaces S_b in Weierstrass form

$$S_b: \quad Y^2 = X\Big(X^2 + X\Big(Z + \frac{1}{Z} - \frac{b^2}{4}\Big) + 1\Big)$$

as an elliptic fibration over $\mathbb{P}^1$ in the Z-coordinate. The inverse transformation is given by

$$x = -2\frac{iX(1+ZX)}{(-2Y+ibX)Z}, \qquad y = \frac{1}{2}\frac{iZ(-2Y+ibX)}{1+ZX}, \qquad z = -2\frac{i(1+ZX)}{-2Y+ibX}$$

in affine coordinates. The following is taken from [17] (with a slight correction).

LEMMA 2.3

The surfaces S_b are ramified coverings of $\mathbb{P}^1 \times \mathbb{P}^1$ (in X, Z coordinates). In X, Y, Z coordinates, the canonical holomorphic 2-form on S_b is given up to a nonzero constant by

$$\omega = \frac{dX\,dZ}{YZ} = \frac{dX\,dY}{X^2(Z-1/Z)}, \tag{2.4}$$

where X, Z are flat coordinates and where

$$Y = Y(b) = \sqrt{P(X,Z)} = \sqrt{X\Big(X^2 + X\Big(Z + \frac{1}{Z} - \frac{b^2}{4}\Big) + 1\Big)}.$$

Proof

In the x, y, z coordinate system, the holomorphic 2-form is given up to a constant by $(dx\,dy)/f_z$ in affine (x, y, z)-coordinates with $w = 1$. Then, using the coordinate transformations above, one computes that

$$\omega = \sqrt{-1} \cdot \frac{dx\,dy}{f_z} = \frac{dX\,dY}{X^2(Z-1/Z)} = \frac{dX\,dZ}{YZ}$$

since $2Y\,dY = X^2(1-1/Z^2)\,dZ + (\partial P/\partial X)\,dX$. □

Now, if we apply the Picard-Fuchs operator $\mathscr{D}_{\mathrm{PF}}$ from Corollary 2.2 to $(dX\,dZ)/YZ$, we get an expression of the form

$$\mathscr{D}_{\mathrm{PF}}\frac{dXdZ}{YZ} = K(X,Z) \cdot \frac{dXdZ}{Y^7 Z},$$

where $K(X, Z)$ is a polynomial function in X, Z.

3. The normal function and the Picard-Fuchs equation

On any elliptic surface, the easiest way to find cycles in K_1 is to use fibers. However, sometimes configurations coming from Néron fibers (degenerate into a union of $\mathbb{P}^1$'s) do have trivial class in K_1, as has already been observed by A. Beĭlinson in [1]. But one can use one smooth fiber together with a bunch of sections (rational) and rational curves in degenerate fibers. In our example, let us take the following cycles: denote by S_b the surface defined by the equation

$$S_b : \quad ZY^2 = X\left(X^2 Z + \left(Z^2 + 1 - Z\frac{b^2}{4}\right)X + Z\right). \tag{3.1}$$

Let C_b be the smooth elliptic fiber over $Z = 1$ of this surface. Hence its defining equation is

$$C_b : \quad Y^2 = X\left(X^2 + \left(2 - \frac{b^2}{4}\right)X + 1\right).$$

The quadratic term $X^2 + (2 - b^2/4)X + 1$ in the right-hand side has two negative real roots if b is purely imaginary, for example, if $b = \sqrt{-1}$. The points $X = 0$ and $X = \infty$ are rationally equivalent on C_b after taking a multiple of two since they are ramification points. The real line from 0 to ∞ does not hit the other ramification points by this observation. The surface X_b in this birational model has 0 and $X = \infty$ as sections. The fiber over $Z = 0$ on S_b decomposes into three rational curves with multiplicity counted. Hence one can construct a cycle in $\mathrm{CH}^2(S_b, 1)$ for general b by using C_b, the two sections, and the degenerate fibers and appropriate rational functions on all curves. In X, Z coordinates the region Γ is given by the real square $0 \le Z \le 1,\ 0 \le X \le \infty$. For $b = \sqrt{-1}$ we make the following observation.

LEMMA 3.1
For $b = \sqrt{-1}$, all coefficients occurring in $\mathscr{D}_{\mathrm{PF}}((dX\,dZ)/YZ) = K(X, Z) \cdot ((dX\,dZ)/Y^7 Z)$ are positive integers, that is, all coefficients of $K(X, Z)$ and all coefficients of $Y^2 = X(X^2 + X(Z + 1/Z - b^2/4) + 1)$.

Proof
Here the rules of differentiating are $\partial X/\partial b = \partial Z/\partial b = 0$ and $\partial Y/\partial b = (b/4) \cdot (1/Y^3)$. This implies that odd derivatives of $1/Y$ get multiplied by even powers of b. Now if we look at the coefficients of (2.3), we see that the coefficients at Ψ''' and Ψ' become positive since $(b/4)^4 - 1$ and $(b/4)^4 - 6$ are negative rational numbers and get multiplied with $(b/4)^6$, respectively, $(b/4)^2$, which are both also negative rational numbers. The coefficients at Ψ and Ψ'' already involve 4th powers of b and hence are positive. Consequently, all coefficients occurring are positive. Using any computer

algebra program, this can be verified, and indeed one has

$$\begin{aligned}\mathscr{D}_{\mathrm{PF}}\frac{dX\,dZ}{YZ} = \frac{dX\,dZ}{8192} &\times (349951\,X^3\,Z^3 + 85952\,X\,Z^3 + 85952\,X^5\,Z^3 + 171904\,X^4\,Z^4 \\ &\quad + 171904\,X^4\,Z^2 + 85952\,X^3\,Z^5 + 171904\,X^2\,Z^4 + 294912\,X^2\,Z \\ &\quad + 294912\,X\,Z^2 + 98304\,Z^3 + 98304\,X^3 + 294912\,X^4\,Z \\ &\quad + 909352\,X^2\,Z^3 + 909352\,X^3\,Z^2 + 98304\,X^6\,Z^3 \\ &\quad + 294912\,X^5\,Z^4 + 294912\,X^5\,Z^2 + 909352\,X^4\,Z^3 \\ &\quad + 294912\,X^4\,Z^5 + 909352\,X^3\,Z^4 + 294912\,X\,Z^4 \\ &\quad + 98304\,X^3\,Z^6 + 294912\,X^2\,Z^5 + 85952\,X^3\,Z + 171904\,X^2\,Z^2) \\ &/(4\,X^2\,Z + 4\,X\,Z^2 + 4\,X + X\,Z + 4\,Z)^{7/2}\sqrt{XZ}.\end{aligned}$$

This completes the proof. □

In particular, if we integrate over the positive region Γ, we get a positive and nonzero integral. Since the boundary of Γ is defined as the rectangle $0 \le Z \le 1$, $0 \le X \le \infty$ and since X, Z are flat with respect to the connection, we say that Γ does not depend on b (see introduction), and this suffices to show that the normal function is nontrivial. So we have proved Esnault's conjecture (see [16]).

COROLLARY 3.2
The projected normal function $\bar{\nu}(b)$ does not satisfy the Picard-Fuchs equation

$$\left(\left(\frac{b}{4}\right)^4 - 1\right)\left(\frac{b}{4}\right)^3\Phi''' + \frac{3}{4}\left(\frac{b}{4}\right)^2\left(1 + \left(\frac{b}{4}\right)^4\right)\Phi'' + \frac{1}{16}\frac{b}{4}\left(\left(\frac{b}{4}\right)^4 - 6\right)\Phi' + \frac{3}{32}\Phi = 0. \tag{3.2}$$

In particular, it is not a rational multiple of a period for all but a countable number of values b. For those b, the corresponding cycle Z_b has no integer multiple that is decomposable modulo $\mathrm{Pic}(X_b) \otimes \mathbb{C}^*$.

The main open problem remains to find a 1-form β such that $d\beta = \mathscr{D}_{\mathrm{PF}}((dX\,dZ)/YZ)$. We compute such a β for the Kummer-type model of these $K3$-surfaces in the following section.

4. The solution of $\mathscr{D}\omega = d\beta$

In [17] we can find the description of a 2:1 map $\pi : S_b \to S'_b$ onto a Kummer surface S'_b that has a birational model with the equation

$$u^2 = s(s-1)\left(s - \left(\frac{\nu+1}{\nu-1}\right)^2\right)t(t-1)(t-\nu^2), \tag{4.1}$$

where ν and b are related via the algebraic equation

$$b^2 = -4 \cdot \frac{(\nu^2+1)^2}{\nu(\nu^2-1)}.$$

We prefer to use this equation for a computation of the solution of $\mathscr{D}\omega = d\beta$ since it is slightly easier, but we do not lose essential information. In this description we see that the transcendental part of $H^2(S_b)$ also denoted by $T(S_b)$ has an Inose-Shioda structure (in the sense of [15]) and is therefore related to a variation of a family of elliptic curves. In fact, there are two isogenous elliptic curves $E_1(\nu)$ and $E_2(\nu)$ with equations

$$E_1(\nu): \quad u_1^2 = s(s-1)(s-\nu^2), \qquad E_2(\nu): \quad u_2^2 = t(t-1)\left(t-\left(\frac{\nu+1}{\nu-1}\right)^2\right) \tag{4.2}$$

together with a Nikulin involution (see [23]) on the abelian surface $A = E_1 \times E_2$ such that the associated Kummer surface is S'_b, and one has an isomorphism $T(S_b) \cong T(S'_b)$ under π_*. In addition, this explains why the periods of S_b are squares of other hypergeometric functions related to the family $E_1(\nu)$, respectively, $E_2(\nu)$. More details about the birational map can be found in [17]. Further instances where Inose-Shioda structures and modular forms arise can be found in [8]. Let us now compute the Picard-Fuchs equation of the family $E_1(\nu)$. If we let $\lambda = \nu^2$, we have $\partial\nu/\partial\lambda = 1/(2\nu) = 1/(2\sqrt{\lambda})$, and therefore for any function Φ we have the transformation rules

$$\frac{\partial\Phi}{\partial\lambda} = \frac{\partial\Phi}{\partial\nu}\cdot\frac{\partial\nu}{\partial\lambda} = \frac{\partial\Phi}{\partial\nu}\frac{1}{2\sqrt{\lambda}} = \frac{\partial\Phi}{\partial\nu}\frac{1}{2\nu}$$

and, for the second derivative,

$$\frac{\partial^2\Phi}{\partial\lambda^2} = \frac{1}{4\nu^2}\frac{\partial^2\Phi}{\partial\nu^2} - \frac{1}{4\nu^3}\frac{\partial\Phi}{\partial\nu}.$$

Plugging this into the standard hypergeometric Picard-Fuchs equation

$$\lambda(1-\lambda)\Phi''(\lambda) + (1-2\lambda)\Phi'(\lambda) - \frac{1}{4}\Phi(\lambda) = 0,$$

we get the new equation

$$(1-\nu^2)\Phi''(\nu) + \frac{1-3\nu^2}{\nu}\Phi'(\nu) - \Phi(\nu) = 0$$

and the inhomogeneous variant (equality of 1-forms)

$$\begin{aligned}(1-\nu^2)\frac{\partial^2}{\partial\nu^2}\omega(s) + \frac{1-3\nu^2}{\nu}\frac{\partial}{\partial\nu}\omega(s) - \omega(s) &= 2d_{\mathrm{rel}}\frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2}\\ &= 2d_{\mathrm{rel}}\frac{s^2(s-1)^2}{\sqrt{s(s-1)(s-\nu^2)}^3},\end{aligned}$$

where

$$\omega(s) = \frac{ds}{\sqrt{s(s-1)(s-\nu^2)}}.$$

The rational normalized version of this equation is

$$\frac{\partial^2}{\partial\nu^2}\omega(s) + \frac{1-3\nu^2}{\nu(1-\nu^2)}\frac{\partial}{\partial\nu}\omega(s) - \frac{1}{1-\nu^2}\omega(s) = \frac{2}{1-\nu^2} d_{\mathrm{rel}} \frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2}. \tag{4.3}$$

In a similar way, we use the substitution $\lambda = ((\nu+1)/(\nu-1))^2$ and get the formula $\partial\nu/\partial\lambda = -(\nu-1)^3/(4(\nu+1))$ and hence

$$\begin{aligned}
\frac{\partial\Phi}{\partial\lambda} &= \frac{\partial\Phi}{\partial\nu}\cdot\frac{\partial\nu}{\partial\lambda} = -\frac{(\nu-1)^3}{4(\nu+1)}\frac{\partial\Phi}{\partial\nu},\\
\frac{\partial^2\Phi}{\partial\lambda^2} &= \frac{(\nu-1)^6}{16(\nu+1)^2}\frac{\partial^2\Phi}{\partial\nu^2} + \frac{(\nu-1)^5(\nu+2)}{8(\nu+1)^3}\frac{\partial\Phi}{\partial\nu}.
\end{aligned}$$

Combining all this, we get the equation

$$\begin{aligned}
&\frac{\partial^2}{\partial\nu^2}\omega(t) + \frac{\nu^2-2\nu-1}{\nu(\nu^2-1)}\frac{\partial}{\partial\nu}\omega(t) + \frac{1}{\nu(\nu-1)^2}\omega(t)\\
&\qquad = -\frac{2}{\nu(\nu-1)^2} d_{\mathrm{rel}} \frac{\sqrt{t(t-1)(t-((\nu+1)/(\nu-1))^2}}{(t-((\nu+1)/(\nu-1))^2)^2}
\end{aligned} \tag{4.4}$$

for

$$\omega(t) = \frac{dt}{\sqrt{t(t-1)(t-((\nu+1)/(\nu-1))^2)}}.$$

We have to compute a sort of convolution product of these two equations in the following sense: set

$$\omega = \omega(s)\wedge\omega(t) = \frac{ds\,dt}{\sqrt{s(s-1)(s-\nu^2)t(t-1)(t-((\nu+1)/(\nu-1))^2)}},$$

and notice that we have the product formula

$$\begin{aligned}
\frac{\partial^3}{\partial\nu^3}\omega &= \frac{\partial^3}{\partial\nu^3}\omega(s)\wedge\omega(t) + 3\frac{\partial^2}{\partial\nu^2}\omega(s)\wedge\frac{\partial}{\partial\nu}\omega(t)\\
&\quad + 3\frac{\partial}{\partial\nu}\omega(s)\wedge\frac{\partial^2}{\partial\nu^2}\omega(t) + \omega(s)\wedge\frac{\partial^3}{\partial\nu^3}\omega(t).
\end{aligned}$$

Similar formulas hold for lower derivatives. Note that s, t are flat coordinates, so that differentiating a differential form with respect to the coefficients is a well-defined procedure. Such formulas can be used to compute $(\partial^3/\partial\nu^3)\omega$ and to obtain a Picard-Fuchs differential operator $\mathcal{D}$ for ω together with a solution β of $\mathcal{D}\omega = d_{\mathrm{rel}}\beta$.

LEMMA 4.1
One has the following inhomogeneous Picard-Fuchs equation involving 2-forms:

$$\frac{\partial^3\omega}{\partial\nu^3} + 3\frac{2\nu+1}{\nu(\nu+1)}\frac{\partial^2\omega}{\partial\nu^2} + \frac{7\nu^4-6\nu^3-4\nu^2+6\nu+1}{(\nu-1)^2(\nu+1)^2\nu^2}\frac{\partial\omega}{\partial\nu} + \frac{\nu^4-2\nu^3-2\nu-1}{(\nu-1)^3(\nu+1)^2\nu^2}\omega = d_{\mathrm{rel}}\beta \quad (4.5)$$

where

$$\begin{aligned}\beta = &-2\frac{s(s-1)(2\nu^4+3\nu^3-\nu^2-3\nu s+s\nu^2-2s)}{\nu(s-\nu^2)^2(\nu^2-1)^2\sqrt{s(s-1)(s-\nu^2)}}\omega(t)\\ &+\frac{6}{1-\nu^2}\frac{\sqrt{s(s-1)(s-\nu^2)}}{(s-\nu^2)^2}\omega'(t)\\ &+2\frac{t(t-1)(2t\nu^4-7t\nu^3+7t\nu^2-t\nu-t-2\nu^4-7\nu^3-7\nu^2-\nu+1)}{\nu^2(\nu-1)^4(t-((\nu+1)/(\nu-1))^2)^2(\nu^2-1)\sqrt{t(t-1)(t-((\nu+1)/(\nu-1))^2)}}\omega(s)\\ &+6\frac{\sqrt{t(t-1)(t-((\nu+1)/(\nu-1))^2)}}{\nu(\nu-1)^2(t-((\nu+1)/(\nu-1))^2)^2}\omega'(s). \end{aligned} \quad (4.6)$$

Proof
We denote the derivative of a function (or a form in flat coordinates) f of ν by f'. Let us carry out the computation in a more general setting. Assume that we have two Picard-Fuchs equations,

$$\omega''(s) - A_s\omega'(s) - B_s\omega(s) = d\beta_s,$$
$$\omega''(t) - A_t\omega'(t) - B_t\omega(t) = d\beta_t,$$

with functions β_s, β_t and A_s, B_s, A_t, B_t depending on ν. Now first note that

$$\omega'''(s) = A_s\omega''(s) + (A_s' + B_s)\omega'(s) + B_s'\omega(s) + \frac{d}{d\nu}\,d\beta_s$$

and furthermore that

$$\frac{d}{d\nu}d\beta_s = d\beta_s'$$

by symmetry of mixed derivatives. A similar relation holds for t. By using the product formulas

$$\omega''' = \omega'''(s)\wedge\omega(t) + 3\omega''(s)\wedge\omega'(t) + 3\omega'(s)\wedge\omega''(t) + \omega(s)\wedge\omega'''(t),$$
$$\omega'' = \omega''(s)\wedge\omega(t) + 2\omega'(s)\wedge\omega'(t) + \omega(s)\wedge\omega''(t),$$

we obtain

$$\begin{aligned}\omega''' - \frac{3}{2}(A_s + A_t)\omega'' = {} & \Big[A_s' + B_s + 3B_t - \frac{1}{2}A_s^2 - \frac{3}{2}A_s A_t\Big]\omega'(s) \wedge \omega(t) \\ & + \Big[A_t' + B_t + 3B_s - \frac{1}{2}A_t^2 - \frac{3}{2}A_s A_t\Big]\omega(s) \wedge \omega'(t) \\ & + \Big[A_s B_s + A_t B_t - \frac{3}{2}(B_s + B_t)(A_s + A_t) + B_s' + B_t'\Big]\omega \\ & + d_{\mathrm{rel}}\bigg[\Big(\beta_s' - \Big(\frac{1}{2}A_s + \frac{3}{2}A_t\Big)\beta_s\Big)\omega(t) + 3\beta_s\omega'(t) \\ & \qquad - \Big(\beta_t' - \Big(\frac{1}{2}A_t + \frac{3}{2}A_s\Big)\beta_t\Big)\omega(s) - 3\beta_t\omega'(s)\bigg]\end{aligned}$$

does not involve any more terms of the form $\omega'(s) \wedge \omega'(t)$. Now let

$$\begin{aligned} A &:= \frac{3}{2}(A_s + A_t), \qquad B := -\frac{1}{2}A_s^2 - \frac{3}{2}A_s A_t + A_s' + B_s + 3B_t, \\ C &:= -\frac{1}{2}A_s B_s - \frac{3}{2}(A_s B_t + A_t B_s) - \frac{1}{2}A_t B_t + B_s' + B_t'. \end{aligned}$$

Then, assuming that we have the equality

$$-\frac{1}{2}A_s^2 - \frac{3}{2}A_s A_t + A_s' + B_s + 3B_t = -\frac{1}{2}A_t^2 - \frac{3}{2}A_s A_t + A_t' + B_t + 3B_s$$

(this condition is equivalent to the fact that the elliptic curves $E_1(\nu)$ and $E_2(\nu)$ are isogenous), we have the inhomogeneous Picard-Fuchs equation

$$\omega''' - A\omega'' - B\omega' - C\omega = d_{\mathrm{rel}}\beta,$$

where β is the 1-form

$$\begin{aligned}\beta := {} & \Big(\beta_s' - \Big(\frac{1}{2}A_s + \frac{3}{2}A_t\Big)\beta_s\Big)\omega(t) + 3\beta_s\omega'(t) \\ & + \Big(\beta_t' - \Big(\frac{1}{2}A_t + \frac{3}{2}A_s\Big)\beta_t\Big)\omega(s) + 3\beta_t\omega'(s).\end{aligned}$$

In our case, $A_s = -(1 - 3\nu^2)/(\nu(1 - \nu^2))$, $B_s = 1/(1 - \nu^2)$, $A_t = -(\nu^2 - 2\nu - 1)/(\nu(\nu^2 - 1))$, and $B_t = -1/(\nu(\nu - 1)^2)$. Therefore we get

$$\begin{aligned} A &= -3\frac{2\nu + 1}{\nu(\nu + 1)}, \qquad B = -\frac{7\nu^4 - 6\nu^3 - 4\nu^2 + 6\nu + 1}{(\nu - 1)^2(\nu + 1)^2\nu^2}, \\ C &= -\frac{\nu^4 - 2\nu^3 - 2\nu - 1}{(\nu - 1)^3(\nu + 1)^2\nu^2} \end{aligned}$$

and for β the expression above. This finishes the proof. □

Appendix

In this section we give the proof of the following lemma from Section 1.

LEMMA 1.1
In the situation of Section 1, $\mathscr{D}_{\mathrm{PF}}(\bar{\nu})$ is a single-valued meromorphic function on B with poles only along degeneracies of X_b, and therefore it satisfies a differential equation

$$\mathscr{D}_{\mathrm{PF}}(\bar{\nu}) = g,$$

where g is a rational function in $b \in B$.

Proof
Assume that we have a family $f : \bar{X} \to \bar{B}$ of projective surfaces over a compact Riemann surface $\bar{B}$. Let $\Sigma \subseteq \bar{B}$ be the finite subset over which there are singular fibers. Let $h : X \to B$ be the smooth part of f. We may assume that the family is semistable (semistable reduction) and that there is a cycle $Z \in \mathrm{CH}^2(X, 1)$ such that the restriction of Z to all fibers induces the family of cycles in $\mathrm{CH}^2(X_b, 1)$. (Both reductions perhaps require a finite cover of B, which does not however change the assertion.) The cycle Z has a class $c_{3,2}(Z) \in H^3_{\mathscr{D}}(X, \mathbb{Z}(2))$ in Deligne cohomology. By semistability, $\Delta := f^{-1}\Sigma$ is a divisor with strict normal crossings and its Deligne cohomology can be computed via the logarithmic de Rham complex. Let $\mathscr{V}^2$ be the sheaf of transcendental cohomology classes in $R^2h_*\mathbb{C}$, a local system of rank $22 - \rho(X_b)$ for b general. The Deligne class vanishes in $F^3 \cap H^3(X_b, \mathbb{Z})$, since $b_3(X_b) = 0$ in our case, and therefore induces a holomorphic normal function $\nu \in H^0(B, \mathscr{V}^2 \otimes \mathscr{O}_B/F^2)$ over B. However, since the family is semistable, there is a canonical extension of ν to a holomorphic section of the sheaf $R^2 f_*\Omega^*_{\bar{X}/\bar{B}}(\log \Delta)$. This can be seen as follows. Let

$$\mathbb{Z}_{\mathscr{D},X}(2) = \mathrm{Cone}\big(Rj_*\mathbb{Z}(2) \to \Omega^*_{\bar{X}}(\log \Delta)/F^2\big)[-1]$$

be the Beĭlinson-Deligne complex (see [9]) of X using the inclusion $X \overset{j}{\hookrightarrow} \bar{X}$, and let

$$\mathbb{Z}_{\mathscr{D},h}(2) = \mathrm{Cone}\big(Rj_*\mathbb{Z}(2) \to \Omega^*_{\bar{X}/\bar{B}}(\log \Delta)/F^2\big)[-1]$$

be the relative Beĭlinson-Deligne complex of h. There is a natural surjection of complexes $\mathbb{Z}_{\mathscr{D},X}(2) \to \mathbb{Z}_{\mathscr{D},h}(2)$ which induces a morphism

$$H^3_{\mathscr{D}}\big(X, \mathbb{Z}(2)\big) \to H^0\big(\bar{B}, R^3 f_*\mathbb{Z}_{\mathscr{D},f}(2)\big).$$

Since all fibers of h satisfy $b_3(X_b) = 0$, we conclude that the image of this element in $H^0(\bar{B}, R^3 f_*Rj_*\mathbb{Z}(2))$ vanishes. Therefore the image of $c_{3,2}(Z)$ in $H^0(\bar{B}, R^3 f_*\mathbb{Z}_{\mathscr{D},f}(2))$ is coming (at least locally because of monodromy) from a class in $H^0(\bar{B}, R^2 f_*\Omega^*_{\bar{X}/\bar{B}}(\log \Delta)/F^2)$ and is thus an extension of $\nu \in H^0(B, R^2 f_*\Omega^*_{X/B}/F^2)$ to $\bar{B}$. By construction, it is meromorphic along Σ but still multivalued with indeterminacies in the local system of integral cohomology classes. Now we apply the Picard-Fuchs operator. This makes $g(b)$ a single-valued complex function on B. $\mathscr{D}_{\mathrm{PF}}$ has meromorphic (rational) coefficients in b since they are the coefficients of the characteristic polynomial of the Gauss-Manin connection, which has regular singular points along Σ by P. Deligne [6]. Therefore the resulting function $g(b)$ is holomorphic outside Σ, but it can have higher order poles along Σ. By Chow's theorem, any meromorphic function on $\bar{B}$ is rational. □

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