A CONSTRUCTION OF A QUOTIENT TENSOR CATEGORY

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Abstract. Let $f : G \to A$ be a surjective homomorphism of transitive groupoid schemes and let L denote the kernel of f. The exact sequence of groupoid schemes $1 \to L \to G \to A \to 1$ induces a sequence of functors between the categories of finite representations of these groupoid schemes $\mathsf{Rep}_f(A) \to \mathsf{Rep}_f(G) \to \mathsf{Rep}_f(L)$. We show that the category $\mathsf{Rep}_f(L)$ is a quotient category of $\mathsf{Rep}_f(G)$ by $\mathsf{Rep}_f(A)$ in an appropriate sense. We also generalize this setting to the framework where the tensor categories are not necessarily Tannaka categories (i.e. not of the form $\mathsf{Rep}_f(G)$ for some groupoid scheme G), where we show under certain assumption the uniqueness of the quotient tensor category.

Introduction

Let $\mathcal{T}$ be a Tannaka category over k with fiber functor ω to vect_K, where $K \supset k$ and let $\mathcal{S}$ be a full tensor subcategory of $\mathcal{T}$ which is closed under taking sub- and quotient objects. The natural inclusion $\mathcal{S} \to \mathcal{T}$ induces a surjective homomorphism of k-groupoids acting upon K

$$G(\mathcal{T}) \to G(\mathcal{S}) \tag{0.1}$$

In [4] it is shown that the kernel of this homomorphism is a discrete K-groupoid which can therefore be identified with a K-group (scheme). Let L denote this K-group, $\mathcal{Q}$ the category of its (finite dimensional) representation and $\mathfrak{q} : \mathcal{T} \to \mathcal{Q}$ the restriction functor. For the sequence of functors

$$\mathcal{S} \to \mathcal{T} \xrightarrow{\mathfrak{q}} \mathcal{Q} \tag{0.2}$$

it is shown that

- (i) An object of $\mathcal{T}$ is isomorphic (in $\mathcal{T}$) to an object from $\mathcal{S}$ iff its image under $\mathfrak{q}$ is trivial (i.e. isomorphic to the direct sum of copies of the unit object) in $\mathcal{Q}$,
- (ii) Each object in $\mathcal{Q}$ is isomorphic (in $\mathcal{Q}$) to a subobject of the image under $\mathfrak{q}$ of an object from $\mathcal{T}$.

The problem we want to address in this work is to give an abstract description of the quotient category $\mathcal{Q}$. This question was posed by P. Deligne in connection with our description of the representation category of L given in [4]. While considering this problem we realize that, with some "technical assumptions", one can assume $\mathcal{T}$ merely to be a rigid tensor category. On the other hand, as already noticed by Milne in [5] for the existence of a quotient $\mathcal{Q}$

Date: February 19, 2006.

the category $\mathcal{S}$ is necessarily a Tannaka category. Let us start by the definition of a (normal) quotient category of a rigid tensor category $\mathcal{T}$.

A (normal) K-*quotient* of $\mathcal{T}$ is by definition a pair $(\mathcal{Q}, \mathfrak{q} : \mathcal{T} \to \mathcal{Q})$ consisting of a K-linear rigid tensor category $\mathcal{Q}$ and k-linear exact tensor functor $\mathfrak{q}$ (the k-linear structure over $\mathcal{Q}$ is induced from the inclusion $k \subset K$), such that:

(i) for an object $X \in \mathcal{T}$ the largest trivial subobject of $\mathfrak{q}(X)$ is isomorphic to the image under $\mathfrak{q}$ of a subobject of X;

(ii) each object of $\mathcal{Q}$ is isomorphic to a subobject of $\mathfrak{q}(X)$ with $X \in \mathcal{T}$.

Given a K-quotient $(\mathcal{Q}, \mathfrak{q})$ of $\mathcal{T}$, let $\mathcal{S}$ denote the full subcategory of $\mathcal{T}$ consisting of those objects of $\mathcal{T}$ whose images in $\mathcal{Q}$ are trivial (i.e. isomorphic to a direct sum of the unit object). Thus $\mathcal{S}$ is a tensor subcategory and is closed under taking sub- and quotient objects. We shall call $\mathcal{S}$ the *invariant subcategory* with respect to the quotient $(\mathcal{Q}, \mathfrak{q})$. $\mathcal{S}$ is a Tannaka category with fiber functor to vect_K (Lemma 3.3).

For each object $X \in \mathcal{T}$ let $X_{\mathcal{S}}$ denote the maximal subobject of X which is isomorphic to an object of $\mathcal{S}$. We refer to 4.11, 5.3, 5.7 for the condition that $\mathcal{T}$ is flat over $\mathcal{S}$ and over K. Our main results are:

Let $\mathcal{T}$ be a rigid tensor category over k and $\mathcal{S}$ be a full tensor subcategory which is closed under taking sub- and quotient objects.

(i) *Assume that $\mathcal{S}$ is a Tannaka category with fiber functor ω to vect_K, and $\mathcal{T}$ is flat over K and over $\mathcal{S}$. Then if a K-quotient $(\mathcal{Q}, \mathfrak{q})$ of $\mathcal{T}$ by $\mathcal{S}$ exists, it is equivalent to the category, whose objects are triples $(X, Y, f \in \omega(X^\vee \otimes Y)_{\mathcal{S}})$, and morphisms are appropriately defined. Consequently $(\mathcal{Q}, \mathfrak{q})$ is uniquely determined, up to a tensor equivalence.*

(ii) *If $\mathcal{T}$ is a Tannaka category, then it is flat over any full tensor subcategory which is closed under sub- and quotient objects, and the quotient of $\mathcal{T}$ with respect to such a subcategory exists.*

The work is organized as follows. We first recall some basic fact about an exact sequence of algebraic group schemes $1 \to L \to G \to A \to 1$. The construction recalled here will be generalized in the subsequent sections. In section 2 we first define the kernel L of a morphism of transitive groupoid schemes $f : G \to A$ and provide some basic properties of L, for instance, the transitivity. Then using the kernel we describe the base changes of a groupoid scheme. After that we provide a description of the representation category of L generalize the one mentioned in section 1 for group schemes. In section 3 we introduce the notion of quotient tensor category of a rigid tensor category $\mathcal{T}$ by a Tannaka subcategory $\mathcal{S}$ which is closed under taking sub- and quotient objects. In sections 4 and 5 we give a description of this category. Section 4 is devoted to the case when $\mathcal{S}$ is a neutral Tannaka category and section 5 is devoted to the general case. Results of section 2 show in particular that $\mathsf{Rep}_f(L)$ (the category of finite dimensional representation of L) is the quotient of $\mathsf{Rep}_f(G)$ by $\mathsf{Rep}_f(A)$. A consequence of this result which might be useful is a criterion 5.11 for a sequence of groupoid schemes to be exact. Unfortunately our description of the quotient category depends on an assumption about the flatness (of $\mathcal{T}$ over $\mathcal{S}$ and K), which we cannot check when $\mathcal{T}$ is not Tannaka category. Nevertheless we believe that the assumptions have the potential to hold true. To this end some open questions are mentioned in 5.12.

Acknowlegment

The author thanks Prof. H. Esnault for communicating with him the question of Prof. P. Deligne and for many stimulating and enlightening discussions. This work results from these discussions. The author thanks Prof. P. Deligne for his interest in the work and useful comments. The work is partially supported by a Heisenberg-Stipendium of the DFG (Grant Nr. PH-155/1-1), the Leibniz-Preis of Professors H. Esnault and E. Viehweg and the NFBS of Vietnam.

1. Preliminaries

Consider a homomorphism $f : G \to A$ of affine group schemes (not necessarily of finite type) over a field k, which we shall call groups for short. Let L denote the kernel of f, which is then a normal subgroup of G. We collect here some known information on the relationship between these groups.

1.1. We shall use the notation $\mathcal{O}(G)$ to denote the function algebra over G, which is a Hopf k-algebra. The reader is referred to [6] for details on the structure of $\mathcal{O}(G)$. We shall use the following notations for defining the structure maps on $\mathcal{O}(G)$:

- $m : \mathcal{O}(G) \otimes \mathcal{O}(G) \to \mathcal{O}(G)$ for the multiplication;
- $u : k \to \mathcal{O}(G)$ for the unit element map;
- $\epsilon : \mathcal{O}(G) \to k$ for the counit map (which is induced from the unit element e of G);
- $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ for the coproduct (which is induced from the product on G);
- ι for the antipode map (which is induced from the inverse element map on G).

We shall adopt the Sweedler notation for the coproduct:

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

1.2. The category $\mathsf{Rep}(G)$ of k-linear representation of G is equivalent to the category $\mathcal{O}(G)$-Comod of $\mathcal{O}(G)$ comodules, which consists of pairs $(V, \rho_V : V \to V \otimes \mathcal{O}(G))$, where V is a vector space and ρ_V is a k-linear map satisfying the following commutative diagrams:

$$\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[d, "\rho_V"'] & V \otimes \mathcal{O}(G) \arrow[d, "\rho_V\otimes\mathrm{id}"] \\
V \otimes \mathcal{O}(G) \arrow[r, "\mathrm{id}\otimes\Delta"'] & V \otimes V \otimes \mathcal{O}(G)
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[dr, "\mathrm{id}"'] & V \otimes_t \mathcal{O} \arrow[d, "\mathrm{id}\otimes\epsilon"] \\
 & V
\end{tikzcd}$$

The coproduct $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ defines a right coaction (as well as a left coaction) of $\mathcal{O}(G)$ on itself, this coaction corresponds to the right (resp. the left) regular representation of G in $\mathcal{O}(G)$. The terminologies: G-equivariant and $\mathcal{O}(G)$-colinear are equivalent.

A homomorphism $f : G \to A$ is the same as a homomorphism of Hopf algebras $f^* : \mathcal{O}(A) \to \mathcal{O}(G)$. The fundamental theorem of algebraic group theory claims that $\mathcal{O}(G)$ is faithfully flat over its subalgebra $f^*(\mathcal{O}(A))$, cf. [3].

1.3. Let $q : L \to G$ be the kernel of f. We notice that L can be defined as the fiber product over A of G with $\mathrm{Spec}\,(k)$, where the morphism $e : \mathrm{Spec}\,(k) \to A$ is given by the unit element of A. Thus we have

$$\mathcal{O}(L) \cong \mathcal{O}(G) \otimes_{\mathcal{O}(A)} k \tag{1.1}$$

The homomorphism $q^* : \mathcal{O}(G) \to \mathcal{O}(L)$ is just the projection $\mathcal{O}(G) \to \mathcal{O}(G) \otimes_{\mathcal{O}(A)} k$ obtained by tensoring $\mathcal{O}(G)$ with the map $\epsilon : \mathcal{O}(A) \to k$. We shall assume from now on that $\mathcal{O}(A)$ is a Hopf subalgebra of $\mathcal{O}(G)$.

1.4. The homomorphism $q : L \to G$ induces a tensor functor $\mathsf{res}^q : \mathsf{Rep}(G) \to \mathsf{Rep}(L)$, which restricts a representation of G in a vector space V through q to a representation of L. The functor $\mathsf{res}^q : \mathsf{Rep}(G) \to \mathsf{Rep}(L)$ admits a right adjoint, which is the induced representation functor $\mathsf{ind}_q : \mathsf{Rep}(L) \to \mathsf{Rep}(G)$, that is we have a functorial isomorphism

$$\mathrm{Hom}_L(\mathsf{res}^q(V), U) \cong \mathrm{Hom}_G(V, \mathsf{ind}_q(U)), \quad V \in \mathsf{Rep}(G), U \in \mathsf{Rep}(L) \tag{1.2}$$

The functoriality yields a canonical L-linear map

$$\varepsilon_U : \mathsf{res}^q\mathsf{ind}_q(U) \to U$$

The functor ind_q can be explicitly computed in terms of the invariant space functor $(-)^L$. We prefer here the following Hopf algebraic description, which will be exploited in the next sections.

For an L-representation U, denote by $U\square_{\mathcal{O}(L)}\mathcal{O}(G)$ the equalizer of the following maps

$$\begin{tikzcd}
U \otimes \mathcal{O}(G) \arrow[r, "\mathrm{id}\otimes\Delta"'] \arrow[rr, bend left, "\rho_U\otimes\mathrm{id}"] & U \otimes \mathcal{O}(G) \otimes \mathcal{O}(G) \arrow[r, "\mathrm{id}\otimes q^*\otimes\mathrm{id}"'] & U \otimes \mathcal{O}(L) \otimes \mathcal{O}(G)
\end{tikzcd} \tag{1.3}$$

$U\square_{\mathcal{O}(L)}\mathcal{O}(G)$ is called the cotensor product over $\mathcal{O}(L)$ of U with $\mathcal{O}(G)$. Then we have a functorial isomorphism

$$\mathsf{ind}_q(U) \cong U\square_{\mathcal{O}(L)}\mathcal{O}(G) \tag{1.4}$$

For the cotensor product there is the following key isomorphism first considered by Takeuchi [7]

$$(U\square_{\mathcal{O}(L)}\mathcal{O}(G))\otimes_{\mathcal{O}(A)}\mathcal{O}(G)\cong U\otimes\mathcal{O}(G),\quad u\otimes g\otimes h\mapsto\sum_{(u)}u_{(0)}\otimes u_{(1)}gh, \tag{1.5}$$

which together with the faithful flatness of $\mathcal{O}(G)$ over $\mathcal{O}(A)$ (cf. 1.2) shows in particular that the functor $\mathsf{ind}_q=(-)\square_{\mathcal{O}(L)}\mathcal{O}(G)$ is faithfully exact.

A direct consequence of (1.5) is the following isomorphism

$$k\square_{\mathcal{O}(L)}\mathcal{O}(G)\cong\mathcal{O}(A) \tag{1.6}$$

For a representation V of G, we have the following G-equivariant isomorphism

$$V\otimes\mathcal{O}(G)\to(V)\otimes\mathcal{O}(G)=\mathcal{O}(G)^{\oplus\dim_k V},\quad v\otimes h\mapsto\sum_{(v)}v_{(0)}\otimes v_{(1)}h \tag{1.7}$$

Therefore, considering V as an $\mathcal{O}(L)$-comodule, (1.6) and (1.7) imply a G-equivariant isomorphism (where G acts diagonally on the right object)

$$V\square_{\mathcal{O}(L)}\mathcal{O}(G)\cong V\otimes\mathcal{O}(A) \tag{1.8}$$

In other words we have $\mathsf{ind}_q\mathsf{res}^q(V)\cong V\otimes\mathcal{O}(A)$. Note that $\mathcal{O}(A)$ acts on $V\otimes\mathcal{O}(A)$ through the action on the second component.

1.5. In general, for any L-representation U there exists an $\mathcal{O}(A)$ module structure $\mu_U:\mathcal{O}(A)\otimes\mathsf{ind}_q(U)\to\mathsf{ind}_q(U)$ on $\mathsf{ind}_q(U)$, induced from the inclusion of $\mathcal{O}(A)$ in $\mathcal{O}(G)$. The action μ_U is G-equivariant where G acts diagonally on $\mathcal{O}(A)\otimes\mathsf{ind}_q(U)$. The functor ind_q thus factors through a functor to the category $\mathsf{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}$ of the so-called $(\mathcal{O}(G)$-$\mathcal{O}(A))$-Hopf modules. By definition, an $(\mathcal{O}(G)$-$\mathcal{O}(A))$-Hopf module is a k-vector space M together with a coaction ρ_M of $\mathcal{O}(G)$ and an action μ_M of $\mathcal{O}(A)$ which are compatible in the sense that μ_M is $\mathcal{O}(G)$-colinear where $\mathcal{O}(G)$ coacts diagonally on $M\otimes\mathcal{O}(A)$. The category $\mathsf{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}$ is in fact a tensor category with respect to the tensor product over $\mathcal{O}(A)$. It follows from the various isomorphisms above that ind_q defines an equivalence of tensor categories between $\mathsf{Rep}(L)$ and $\mathsf{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}$. In particular, the isomorphism in (1.8) is $\mathcal{O}(A)$-linear.

The equivalence mentioned above can be reformulated in the following more geometric language: there exists an equivalence between L-representations and G-equivariant vector bundles over A. This was pointed out to the author by P. Deligne.

1.6. A new consequence of the classical facts in 1.1-1.6 is the following, cf. [4]. It follows from the faithful exactness of ind_q that the canonical homomorphism $\varepsilon_U:\mathsf{res}^q\mathsf{ind}_q(U)\to U$ is surjective. Assume that U has finite dimension over k then we can find a finite dimensional G-subrepresentation V of $\mathsf{ind}_q(U)$ which still maps surjectively on U. Thus U is a quotient of the restriction to L of a finite dimensional representation of G. Consequently, U can also be embedded in to the restriction to L of a finite dimensional G-representation. Thus each finite dimensional representation U of L can be put (in a non-canonical way) in to a sequence $\mathsf{res}^q(V) \overset{\pi}{\twoheadrightarrow} U \overset{\iota}{\hookrightarrow} \mathsf{res}^q(W)$, where V, W are finite dimensional G-representations. In other words, U is equivalent as an L-representation to the image of an L-linear map $g : \mathsf{res}^q(V) \to \mathsf{res}^q(W)$. Using the equivalence between $\mathsf{Rep}(L)$ and $\mathsf{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}$ and the isomorphism (1.8) we can show that such g are in a 1-1 correspondence with morphisms $\bar{g} : V \otimes \mathcal{O}(A) \to W \otimes \mathcal{O}(A)$ in $\mathsf{Mod}_{\mathcal{O}(A)}^{\mathcal{O}(G)}$. Since $\bar{g}$ is $\mathcal{O}(A)$ linear, it is uniquely determined by a G-equivariant map $f : V \to W \otimes \mathcal{O}(A)$.

Let $\mathcal{Q}$ be the category, whose objects are triples $(V, W, f : V \to W \otimes \mathcal{O}(A))$, where V, W are finite dimensional representations of G and f is G-equivariant, and whose morphisms are defined in an adequate way. Composing f with the morphism $\mathrm{id} \otimes \epsilon : W \otimes \mathcal{O}(A) \to W \otimes k \cong W$ we obtain a map $f_0 : V \to W$ which is L-linear. We define a functor $\mathcal{Q} \to \mathsf{Rep}(L)$, sending a triple (V, W, f) to the image of f_0. It follows from the discussion of this paragraph that this functor is an equivalence of categories between $\mathcal{Q}$ and the category $\mathsf{Rep}_f(L)$ of finite dimensional representations of L.

2. Groupoids

2.1. Groupoids and their representations. We refer to [2, §1.14] for the definition of an affine groupoid scheme, which we shall call here simply groupoid. Fix a field k and let K be another field containing k. A k-groupoid acting upon $\operatorname{Spec} K$ will usually be denoted like G_k^K. G_k^K is called transitive if it acts transitively upon $\operatorname{Spec} K$, which means that G_k^K is flat over $\operatorname{Spec} K \times_k \operatorname{Spec} K$ with respect to the source and target map $(s, t) : G_k^K \to \operatorname{Spec} K \times_k \operatorname{Spec} K$. The category of K-representation of G_k^K is denoted by $\mathsf{Rep}(G_k^K)$, its subcategory of finite dimensional (over K) representations is denoted by $\mathsf{Rep}_f(G_k^K)$.

2.2. Homomorphisms of groupoids. Assume we are given the following field extensions

$$k_0 \subset k \subset K_0 \subset K \tag{2.1}$$

and transitive groupoids $G = G_k^K$ and $A = A_{k_0}^{K_0}$. A k_0-morphism $f : G \to A$ is called a groupoid homomorphism if for any k-scheme S, f induces a functor

$$f_S : (G(S), K(S)) \to (A(S), K_0(S)) \tag{2.2}$$

of abstract groupoids, where, on objects, f_S is given by the inclusion of fields

$$\begin{tikzcd} & S \arrow[dl, "a"'] \arrow[dr, "f_S(a)"] & \\ \operatorname{Spec} K \arrow[rr] & & \operatorname{Spec} K_0 \end{tikzcd} \tag{2.3}$$

and on morphism f_S is given by

$$\begin{tikzcd} & S \arrow[dl, "\phi"'] \arrow[dr, "f_S(\phi)"] & \\ G \arrow[rr, "f"'] & & A \end{tikzcd} \tag{2.4}$$

That is, we have the following diagram

$$\begin{tikzcd} S \arrow[rr, "f_S(\phi)"] \arrow[d, "{(a,b)}"'] \arrow[drr, "{(f_S(a),f_S(b))}"] & & A \arrow[d] \\ \operatorname{Spec} K \times_k \operatorname{Spec} K \arrow[rr] & & \operatorname{Spec} K_0 \times_{k_0} \operatorname{Spec} K_0 \end{tikzcd} \tag{2.5}$$

Setting $S = G$, $\phi = \mathrm{id}$, we otain the following commutative diagram

$$\begin{tikzcd} G \arrow[rr] \arrow[d] & & A \arrow[d] \\ \operatorname{Spec} K \times_k \operatorname{Spec} K \arrow[rr] & & \operatorname{Spec} K_0 \times_{k_0} \operatorname{Spec} K_0 \end{tikzcd} \tag{2.6}$$

Similarly, f should be compatible with the groupoid structure on G and A which are related to each other by this diagram.

The *kernel* of f is defined as the fiber product

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} K_0 \arrow[d, "e"] \\ G \arrow[r, "f"'] & A \end{tikzcd} \tag{2.7}$$

where $e : \operatorname{Spec} K_0 \to A$ is given by the unite element of A.

Lemma 2.3. *L is a K_0-groupoid acting transitively upon* $\operatorname{Spec} K$.

Proof. The fiber product of $\operatorname{Spec} K_0$ with $\operatorname{Spec} K \times_k \operatorname{Spec} K$ over $\operatorname{Spec} K_0 \times_{k_0} \operatorname{Spec} K_0$ is $\operatorname{Spec} K \times_{K_0} \operatorname{Spec} K$. Therefore there exists a map $L \to \operatorname{Spec} K \times_{K_0} \operatorname{Spec} K$. We show that L is transitive over $\operatorname{Spec} K \times_{K_0} \operatorname{Spec}$. According to [2, Prop. 3.3], this is equivalent to saying that the associated stack $\mathcal{G}_{K:L}$ is a gerbes. This last fact was shown in [5, 1.2]. □

2.4. Base change. Using the notion of kernel, we define in this section the "lower" base change for groupoids. Let $G = G_k^K$ be a transitive groupoid and $k \subset k_1 \subset K$ an intermediate field. The projection

$$\operatorname{Spec} K \times_k \operatorname{Spec} K \to \operatorname{Spec} k_1 \times_k \operatorname{Spec} k_1 \tag{2.8}$$

induces a homomorphism of groupoids $G_k^K \to \operatorname{Spec} k_1 \times_k \operatorname{Spec} k_1$. The kernel of this homomorphism is called the k_1-*diagonal subgroupoid* of G_k^K and denoted by $G_{k_1}^K$

(2.9)
$$\begin{tikzcd}
G_{k_1}^K \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_k^K \arrow[r] \arrow[d] & \operatorname{Spec} k_1 \times_k \operatorname{Spec} k_1 \arrow[d] \\
\operatorname{Spec} K \times_{k_1} \operatorname{Spec} K \arrow[r] & \operatorname{Spec} K \times_k \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k_1 \times_k \operatorname{Spec} k_1
\end{tikzcd}$$

It follows from definition of $G_{k_1}^K$ that the left square in this diagram is cartesian. In case $k_1 = K$, G_K^K is just the usual diagonal subgroup of G_k^K.

On the other hand, for any extension $K \subset K_1$, Deligne [2] defines a k-groupoid $G_k^{K_1}$ acting (transitively) upon $\operatorname{Spec} K_1$:

(2.10)
$$\begin{tikzcd}
G_k^{K_1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_k^K \arrow[d] \\
\operatorname{Spec} K_1 \times_k \operatorname{Spec} K_1 \arrow[r] & \operatorname{Spec} K \times_k \operatorname{Spec} K
\end{tikzcd}$$

which in our context might be called the "upper" base change. We notice that the category $\mathsf{Rep}(G_k^{K_1})$ is equivalent to the category $\mathsf{Rep}(G_k^K)$ (cf. [2, (3.5.1)]).

Thus, given field extensions $k_0 \subset k \subset K_0 \subset K$ and a homomorphism of transitive groupoids $f : G_k^K \to A_{k_0}^{K_0}$. Then f factors through

$$G_k^K \xrightarrow{f_k} A_k^K \twoheadrightarrow A_k^{K_0} \hookrightarrow A_{k_0}^{K_0} \tag{2.11}$$

Since the last map in (2.11) is an injection, in what follows we shall only consider the situation $k_0 = k$, i.e. field extensions $k \subset K_0 \subset K$.

Lemma 2.5. *Let $q : L_{K_0}^K \to G_k^K$ be the kernel of a homomorphism $f : G_k^K \to A_k^{K_0}$ of transitive groupoids. For any $V \in \mathsf{Rep}(G_k^K)$, the $L_{K_0}^K$-invariant subspace $V^{L_{K_0}^K}$ is a G_k^K-subrepresentation of V, which is the pull-back through f of a representation of $A_k^{K_0}$.*

Proof. By making the lower base change we see that L_K^K is the kernel of $f_K : G_K^K \to A_K^K$, hence is normal in G_K^K as K-groups schemes. Thus $V^{L_K^K} \subset V$ is invariant under the action of G_K^K, but this also implies that $V^{L_K^K}$ is invariant under the action of G_k^K. Since L_K^K can also be considered as the kernel of the morphism $f : G_k^K \to A_k^K$, we see that A_k^K acts on $V^{L_K^K}$. As we noticed after (2.11), $\mathsf{Rep}(A_k^K)$ is equivalent to $\mathsf{Rep}(A_k^{K_0})$, which means that $V^{L_K^K}$ is indeed a representation of G_k^K that comes from a representation of $A_k^{K_0}$ by pulling-back through f. We therefore conclude that $L_{K_0}^K$ acts trivially on this space. Since $V^{L_{K_0}^K} \subset V^{L_K^K}$, these spaces coincide. □

2.6. The function algebra. We refer to [2, 1.14] for the properties of the function algebra $\mathcal{O} := \mathcal{O}(G_k^K)$ of a groupoid G_k^K. See also the appendix to [4]. $\mathcal{O}$ is a k-Hopf algebroid acting on K. The structure maps are denoted as follows:

- $m : \mathcal{O} \otimes_k \mathcal{O} \to \mathcal{O}$, the multiplication;

- $s \otimes_k t : K \otimes_k K \to \mathcal{O}$ – the unit map, induced from the source-target morphism $(s,t) : G_k^K \to \operatorname{Spec} K \times_k \operatorname{Spec}_K$. This map induces two actions of K on $\mathcal{O}$, denoted by s and t;
- $\Delta : \mathcal{O} \to \mathcal{O}_s \otimes_t \mathcal{O}$ – the coproduct map, induced from the product on G_k^K (the tensor product is taken over K with respect to different actions of K induced by the map s and t);
- $\epsilon : \mathcal{O} \to K$ – the counit map, induced from the unit element in G_k^K;
- $\iota : \mathcal{O} \to \mathcal{O}$ – the antipode map, induced from the inverse element map. This map interchanges the actions s and t.

In particular, $(\mathcal{O}, m, s \otimes t)$ is a $K \otimes_k K$-algebra. The k-linear maps $s, t : K \to \mathcal{O}$ induce to structures of K-vector space on $\mathcal{O}$, making it a K-bimodule. We shall assume that the left action of K is given by t and the right one by s. Then $(\mathcal{O}, \Delta, \epsilon)$ is a K-bimodule coalgebra, i.e. $\mathcal{O}$ is considered as K-bimodule, the tensor product is the one for K-bimodules. We adopt Sweedler notation for the coproduct on $\mathcal{O}$:

$$\Delta(h) = \sum_{(h)} h_{(1)s} \otimes_t h_{(2)}.$$

We notice that the K-linearity of Δ reads

$$\Delta(t(\lambda) h s(\mu)) = \sum_{(h)} t(\lambda) h_{(1)s} \otimes_t h_{(2)} s(\mu) \quad h \in \mathcal{O}, \lambda, \mu \in K \tag{2.12}$$

And for ϵ we have $\epsilon(s(\lambda) h t(\mu)) = s(\lambda) \epsilon(h) t(\mu)$.

The category $\mathsf{Rep}(G_k^K)$ of G_k^K representations over K is the same as the category of $\mathcal{O}$-comodules, i.e. of pairs (V, ρ_V) of a K-vector space V and a morphism

$$\rho_V : V \to V \otimes_t \mathcal{O}$$

satisfying

$$\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[d, "\rho_V"'] & V \otimes_t \mathcal{O} \arrow[d, "\rho_V \otimes \mathrm{id}"] \\
V \otimes_t \mathcal{O} \arrow[r, "\mathrm{id} \otimes \Delta"'] & V \otimes V \otimes_t \mathcal{O}
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "\rho_V"] \arrow[dr, "\mathrm{id}"'] & V \otimes_t \mathcal{O} \arrow[d, "\mathrm{id} \otimes \epsilon"] \\
 & V
\end{tikzcd} \tag{2.13}$$

We note that the K-linearity of ρ_V means:

$$\rho_V(\lambda v) = t(\lambda) \rho_V(v).$$

2.7. The induced representation functor. Consider extensions of fields $k \subset K_0 \subset K$ and a homomorphism $f : G_k^K \to A_k^{K_0}$ of groupoids as in 2.2. Let $q : L_{K_0}^K \to G_k^K$ be the kernel of f and $\mathsf{res}^q : \mathsf{Rep}(G_k^K) \to \mathsf{Rep}(L_{K_0}^K)$ denote the restriction functor. By definition $\mathcal{O}(L_{K_0}^K) = \mathcal{O}(G_k^K) \otimes_{\mathcal{O}(A_k^{K_0})} K_0$, where $\mathcal{O}(A_k^{K_0})$ acts on $\mathcal{O}(G_k^K)$ through f^*, and $q^* : \mathcal{O}(G_k^K) \to \mathcal{O}(L_{K_0}^K)$ is the projection. Define a morphism

$$\phi : \mathcal{O}(G_k^K) \otimes_{\mathcal{O}(A_k^{K_0})} \mathcal{O}(G_k^K) \to \mathcal{O}(L_{K_0}^K)_s \otimes_t \mathcal{O}(G) \tag{2.14}$$

$g \otimes h \mapsto \sum_{(g)} q^*(g_{(1)}) \otimes g_{(2)}h$.

The following lemma generalizes [4, Lem. 6.5].

Lemma 2.8. *The morphism in* (2.14) *is an isomorphism and is* $\mathcal{O}(G_k^K)$*-colinear (i.e.* G_k^K *equivariant) with respect to the right coaction of* $\mathcal{O}(G_k^K)$ *on the second tensor component as well as* $\mathcal{O}(L_{K_0}^K)$*-colinear with respect to the left coaction of* $\mathcal{O}(L_{K_0}^K)$ *on the first tensor components.*

Proof. We give the inverse map. We first define a map $\bar{\psi} : \mathcal{O}(G_k^K)_s \otimes_t \mathcal{O}(G_k^K) \to \mathcal{O}(G_k^K) \otimes_{\mathcal{O}(A_k^{K_0})} \mathcal{O}(G_k^K)$, $\psi(g \otimes h) = \sum_{(g)} g_{(1)} \otimes \iota(g_{(2)})h$, where ι is the antipode of $\mathcal{O}(G_k^K)$. Then we note that this map indeed factors through a map $\psi : \mathcal{O}(L_{K_0}^K)_s \otimes_t \mathcal{O}(G_k^K) \to \mathcal{O}(G_k^K) \otimes_{\mathcal{O}(A_k^{K_0})} \mathcal{O}(G_k^K)$ which is the inverse to ϕ. The second claim is obvious from the definition of ϕ. □

The functor res^q possesses a right adjoint which is the induced representation functor, denoted by $\mathsf{ind}_q : \mathsf{Rep}(L_{K_0}^K) \to \mathsf{Rep}(G_k^K)$. For an $\mathcal{O}(L_{K_0}^K)$ comodule U denote $U\square_{\mathcal{O}(L_{K_0}^K)}\mathcal{O}(G_k^K)$ the equalizer of the maps

$$\begin{tikzcd} U \otimes_t \mathcal{O}(G_k^K) \arrow[d, "\mathrm{id}\otimes\Delta"'] \arrow[dr, "\rho_U\otimes\mathrm{id}"] & \\ U \otimes_t \mathcal{O}(G_k^K)_s \otimes_t \mathcal{O}(G_k^K) \arrow[r, "\mathrm{id}\otimes q^*\otimes\mathrm{id}"'] & V \otimes \mathcal{O}(L_{K_0}^K)_s \otimes_t \mathcal{O}(G_k^K). \end{tikzcd} \tag{2.15}$$

Proposition 2.9. *Let* $(L = L_{K_0}^K, q)$ *be the kernel of* $f : G = G_k^K \to A = A_k^{K_0}$ *as above. Then for* $U \in \mathsf{Rep}(L)$*, we have*

(i) $\mathsf{ind}_q(U)$ *is canonically isomorphic to* $U\square_{\mathcal{O}(L)}\mathcal{O}(G)$*;*

(ii) *there is an isomorphism*

$$\big(U\square_{\mathcal{O}(L)}\mathcal{O}(G)\big) \otimes_{\mathcal{O}(A)} \mathcal{O}(G) \cong U \otimes_t \mathcal{O}(G) \tag{2.16}$$

(iii) *the functor* $\mathsf{ind}_q : \mathsf{Rep}(L) \to \mathsf{Rep}(G)$ *is exact (and hence faithfully exact) iff* $\mathcal{O}(G)$ *is flat (hence faithfully flat over* $f^*\mathcal{O}(A)$*).*

Proof. We show that the functor $U \mapsto U\square_{\mathcal{O}(L)}\mathcal{O}(G)$ is right adjoint to the restriction functor res^q, which amounts to

$$\mathrm{Hom}_G(V, U\square_{\mathcal{O}(L)}\mathcal{O}(G)) \cong \mathrm{Hom}_L(\mathsf{res}(V), U), \quad V \in \mathsf{Rep}(G), U \in \mathsf{Rep}(L)$$

The map is given by composing a morphism $U \to U\square_{\mathcal{O}(L)}\mathcal{O}(G)$ with the canonical map $\varepsilon_U : U\square_{\mathcal{O}(L)}\mathcal{O}(G) \to U$ given by $v \otimes g \mapsto v\epsilon(g)$, where ϵ denotes the counit of $\mathcal{O}(G)$. The inverse map is given by $f \mapsto (f \otimes \mathrm{id})\rho_U$. Thus (i) is proved.

To show the isomorphism in (2.16) we first tensor both sides of (2.14) with U from the left and then taking the equalizer as in (2.15). The last claim follows from (2.16) since the right hand side of (2.16) is a faithfully flat functor on U. □

Lemma 2.10. [4, Lem. 6.2] *Let* $q : L_K^K \to G_k^K$ *be the kernel for a homomorphism* $f : G_k^K \to A_k^K$ *of transitive groupoids. Then the adjoint functor* $\mathsf{ind}_q : \mathsf{Rep}(L) \to \mathsf{Rep}(G)$ *is exact.*□

Proposition 2.11. *Let $q : L^K_{K_0} \to G^K_k$ be the kernel of the morphism $f : G^K_k \to A^{K_0}_k$ of transitive groupoids. Then:*

(i) *the induced representation functor ind_q is faithfully exact;*

(ii) *each finite dimensional representation of L can be embedded into a representation of G considered as representation of L.*

Proof. We have the following commutative diagram with exact lines:

$$\begin{tikzcd}
& L^K_K \arrow[dl, hook] \arrow[dr, hook, "\bar{q}"] & & & \\
L^K_{K_0} \arrow[rr, hook, "q"'] & & G^K_k \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & A^{K_0}_k \\
& & & A^K_k \arrow[ur, two heads] &
\end{tikzcd} \tag{2.17}$$

where the right triangle was given in (2.11).

According to 2.10, the adjoint functor $\mathsf{ind}_{\bar{q}}$ is faithfully exact, hence, according to 2.9, (iii), $\mathcal{O}(G^K_k)$ is faithfully flat over $\bar{f}^*\mathcal{O}(A^K_k)$. Since $\mathcal{O}(A^K_k)$ is faithfully flat over $\mathcal{O}(A^{K_0}_k)$, we conclude that $\mathcal{O}(G^K_k)$ is faithfully flat over $f^*\mathcal{O}(A^{K_0}_k)$, hence ind_q is faithfully exact. Thus (i) is proved.

Now (ii) follows from (i) by standard argument, see e.g. [4, Lem. 5.5-5.6]. □

2.12. In the situation of Proposition 2.11, we call a representation V of G^K_k a K_0/k-representation if V is equipped with a k-linear homomorphism $K_0 \to \mathrm{End}_{G^K_k}(V)$. In other words, there exists a structure of K_0-vector space on V, which is compatible with the k-structure and commutes with the actions of K and G^K_k. In the language of comodules, denote the new action of K_0 on V by $(\lambda, v) \mapsto t(\lambda)v$, then the above assumption amounts to saying that the comodule map $\rho_V : V \to V \otimes_t \mathcal{O}(G^K_k)$ satisfies $\rho_V(t(\lambda)v) = t(\lambda)\rho_V(v)$. Denote the category of K_0/k-representations by $\mathsf{Rep}(G^K_k)_{K_0/k}$. We notice that the category $\mathsf{Rep}(G^K_k)_{K_0/k}$ is a tensor category with respect to the tensor product over K_0.

Consider the situation of (2.9): $G^K_{K_0} \xrightarrow{\Delta_{K_0}} G^K_k \to \mathrm{Spec}\,(K_0) \times_k \mathrm{Spec}\,(K_0)$. Then for any representation W over $G^K_{K_0}$, $\mathsf{ind}_{\Delta_{K_0}}(W) = W\square_{\mathcal{O}(G^K_{K_0})}\mathcal{O}(G^K_k)$ is a K_0/k-representation of G^K_k with the K_0-action induced from the K_0-action on W.

Lemma 2.13. *The functor $\mathsf{ind}_{\Delta_{K_0}}$ induces an equivalence of tensor categories $\mathsf{Rep}(G^K_k) \to \mathsf{Rep}(G^K_k)_{K_0/k}$. In particular, the tensor product over K_0 in $\mathsf{Rep}(G^K_k)_{K_0/k}$ is flat.*

Proof. To see that $\mathsf{ind}_{\Delta_{K_0}}$ is a tensor functor one has to check that there is an isomorphism

$$(V\square_{\mathcal{O}(G^K_{K_0})}\mathcal{O}(G^K_k))\otimes_{K_0}(W\square_{\mathcal{O}(G^K_{K_0})}\mathcal{O}(G^K_k))\cong(V\otimes_{K_0}W)\square_{\mathcal{O}(G^K_{K_0})}\mathcal{O}(G^K_k) \tag{2.18}$$

We define the map to be

$$\phi:(v\otimes h)\otimes(w\otimes g)\mapsto\sum_{(v)(w)}(v_{(0)}\otimes w_{(0)})\otimes v_{(1)}w_{(1)}gh$$

To see that this defines an isomorphism, using the fact that $\mathcal{O}(G^K_k)$ is faithfully flat over $\mathcal{O}(A^{K_0}_k)$, if suffices to show that (the cotensor product is over $\mathcal{O}(G^K_{K_0})$

$$\phi\otimes\mathrm{id}:(V\square\mathcal{O}(G^K_k))\otimes_{K_0}(W\square\mathcal{O}(G^K_k))\otimes_{\mathcal{O}(A^{K_0}_k)}\mathcal{O}(G^K_k)\to$$
$$(V\otimes_{K_0}W)\square\mathcal{O}(G^K_k)\otimes_{\mathcal{O}(A^{K_0}_k)}\mathcal{O}(G^K_k)$$

is an isomorphism. This last fact follows from the isomorphism in (2.16). □

2.14. Consider again the situation of 2.11. For a representation W of $L^K_{K_0}$, the K_0 structure on W yields a K_0-structure on $\mathsf{ind}_q(W)=W\square_{\mathcal{O}(L^K_{K_0})}\mathcal{O}(G^K_k)$ making it a K_0/k-representation of G^K_k. In particular $f^*\mathcal{O}(A^{K_0}_k)\cong\mathsf{ind}_q(K)$ is an object of $\mathsf{Rep}(G^K_k)_{K_0/k}$, moreover, it is an algebra in this tensor category. To simplify the situation, we shall assume that f^* is injective, i.e., f is a surjective homomorphism of groupoids, and identify $\mathcal{O}(A^{K_0}_k)$ with its image in $\mathcal{O}(G^K_k)$. Denote $\mathsf{Rep}(G^K_k)_{A^{K_0}_k}$ the category of $\mathcal{O}(A^{K_0}_k)$-modules in $\mathsf{Rep}(G^K_k)_{K_0/k}$. Then this is a tensor category with respect to the tensor product over $\mathcal{O}(A^{K_0}_k)$.

Proposition 2.15. *With the assumption of 2.11 and that f is surjective we have an equivalence of tensor categories $\mathsf{Rep}(L^K_{K_0})\to\mathsf{Rep}(G^K_k)_{A^{K_0}_k}$ given by the functor ind_q.*

Proof. The proof is similar to that of Lemma 2.13. □

3. QUOTIENT CATEGORIES

3.1. Tensor categories and Tannaka duality. We refer to [1, §1] for the definition of tensor categories. A tensor category $\mathcal{C}$ over a field k is a k-linear abelian category equipped with a symmetric tensor product (i.e. a symmetric monoidal structure), such that the endomorphism ring of the unit object (always denoted by I)is isomorphic to k. $\mathcal{C}$ is called rigid if each object is rigid, i.e. possesses a dual object and each object has finite length (of decomposition series).

A tensor functor between tensor categories is an additive functor that preserves the tensor product as well as the symmetry. A K-valued fiber functor of a rigid tensor category $\mathcal{C}$ over k ($K\supset k$) is k-linear exact tensor functor from $\mathcal{C}$ to Vect_K, the category of K-vector spaces, its image lies automatically in the subcategory vect_K of finite dimensional vector spaces. If such a fiber functor exists, $\mathcal{C}$ is called a Tannaka category. If, more over, $K = k$ then $\mathcal{C}$ is called neutral Tannaka.

For example, $\mathsf{Rep}_f(G_k^K)$, where G_k^K is a transitive groupoid, is a Tannaka category with the fiber functor being the forgetful functor to vect_K. The general Tannaka duality [2, Thm. 1.12] establishes a 1-1 correspondence between rigid tensor categories over k together with a fiber functor to vect_K and groupoids acting transitively over $\operatorname{Spec} K$.

Assume that $\mathcal{C}$ is rigid over k but not necessarily Tannaka. Let Ind-$\mathcal{C}$ be the Ind-category of $\mathcal{C}$, whose object are directed systems of objects of $\mathcal{C}$, and whose hom-sets are defined as follows:

$$\operatorname{Hom}(X_{i,i\in I}, Y_{j,j\in J}) := \varprojlim_i \varinjlim_j \operatorname{Hom}_{\mathcal{T}}(X_i, Y_j)$$

One can also define Ind-$\mathcal{C}$ as the category of left-exact functors from $\mathcal{T}^{\mathrm{op}}$ to vect_k, an object X of $\mathcal{C}$ can then be identified with the functor $\operatorname{Hom}_{\mathcal{C}}(-, X)$. The natural inclusion $\mathcal{T} \hookrightarrow$ Ind-$\mathcal{T}$ is exact and full. Further, $\mathcal{T}$ is closed under taking sub- and quotient objects, and each object of Ind-$\mathcal{T}$ is the limit of its subobjects which are isomorphic to objects from $\mathcal{T}$.

Definition 3.2. Let $\mathcal{T}$ be a rigid tensor category over k. Let $K \supset k$ be a field extension. A (normal) *K-quotient* of $\mathcal{T}$ is a pair $(\mathcal{Q}, \mathfrak{q} : \mathcal{T} \to \mathcal{Q})$ consisting of a K-linear rigid tensor category $\mathcal{Q}$ and k-linear exact tensor functor $\mathfrak{q}$ (the k-linear structure over $\mathcal{Q}$ is induced from the inclusion $k \subset K$), such that:

(i) for an object $X \in \mathcal{T}$ the largest trivial subobject of $\mathfrak{q}(X)$ is isomorphic to the image under $\mathfrak{q}$ of a subobject of X;
(ii) each object of $\mathcal{Q}$ is isomorphic to a subobject of the image of an object from $\mathcal{T}$ as well as a quotient of the image of an object from $\mathcal{T}$.

Our notion of normal quotient category in case $K = k$ and $\mathcal{T}$ is a Tannaka category is equivalent to Milne's notion of normal quotient [5] . For convenience we shall omit the term "normal" in the rest of the work.

Given a K-quotient $(\mathcal{Q}, \mathfrak{q})$ of $\mathcal{T}$, let $\mathcal{S}$ denote the full subcategory of $\mathcal{T}$ consisting of those objects of $\mathcal{T}$ whose images in $\mathcal{Q}$ are trivial (i.e. isomorphic to a direct sum of the unit object). It is easy to see that $\mathcal{S}$ is a tensor subcategory and is closed under taking sub- and quotient objects. We shall call $\mathcal{S}$ the *invariant subcategory* with respect to the quotient $(\mathcal{Q}, \mathfrak{q})$.

Lemma 3.3. [5, §2] *Let $(\mathcal{Q}, \mathfrak{q})$ be a K-quotient of $\mathcal{T}$ and $\mathcal{S}$ be the invariant subcategory of $\mathcal{T}$. Then $\mathcal{S}$ is a Tannaka category with a fiber functor to vect_K.*

Proof. The full subcategory of $\mathcal{Q}$ of trivial subobjects is equivalent to vect_K. The fiber functor is given by $\mathcal{S} \ni X \mapsto \mathfrak{q}(X) \mapsto \operatorname{Hom}_{\mathcal{Q}}(I, \mathfrak{q}(X))$ where I denotes the unit object in $\mathcal{Q}$. Since $\mathfrak{q}(X)$ is trivial in $\mathcal{Q}$, this functor is a fiber functor. Hence $\mathcal{S}$ is a Tannaka category. □

4. QUOTIENT CATEGORY BY A NEUTRAL TANNAKA SUBCATEGORY

Let $\mathcal{T}$ be a rigid tensor category over a field k. Let $\mathsf{q} : \mathcal{T} \to \mathcal{Q}$ be a k-quotient of $\mathcal{T}$ and $\mathcal{S}$ be the invariant category as in Definition 3.2 (with $K = k$). According to Lemma 3.3, $\mathcal{S}$ is a neutral Tannaka category with fiber functor given by $\omega(S) \cong \mathrm{Hom}_{\mathcal{Q}}(I, \mathsf{q}S)$. Let us consider the category vect_k as a full subcategory of $\mathcal{Q}$ by identifying a vector space V with $V \otimes I$ in $\mathcal{Q}$ [1]. Then we can consider the above fiber functor as the restriction of q to $\mathcal{S}$. In other words, we have the following functorial isomorphism

$$\omega(S) \otimes_k I = \mathrm{Hom}_{\mathcal{Q}}(I, \mathsf{q}(S)) \otimes_k I \cong \mathsf{q}(S), \quad S \in \mathcal{S} \tag{4.1}$$

by means of which we shall identify $\omega(S)$ with $\mathsf{q}(S)$ for $S \in \mathcal{S}$.

4.1. The existence. Assume that $\mathcal{T}$ is a Tannaka category. Thus, there exists a fiber functor $\bar{\omega} : \mathcal{T} \to \mathsf{vect}_K$ extending the fiber functor ω and $\mathcal{T} \cong \mathsf{Rep}_f(G_k^K)$, $\mathcal{S} \cong \mathsf{Rep}_f(A_k^k)$ for some groupoid scheme G_k^K and group scheme A_k^k. According to Lemma 2.5, Proposition 2.11, $\mathcal{Q} := \mathsf{Rep}_f(L_k^K)$ is a k-quotient of $\mathcal{T}$ by $\mathcal{S}$.

4.2. The algebra $\mathcal{O}$. By means of the fiber functor ω, $\mathcal{S}$ is equivalent to the category $\mathsf{Rep}_f(G(\mathcal{S}))$ of finite dimensional k-representation of $G(\mathcal{S})$, where $G(\mathcal{S})$ is an affine k-group scheme, and Ind-$\mathcal{S}$ is equivalent to $\mathsf{Rep}(G(\mathcal{S}))$. Let $\mathcal{O}$ denote the function algebra of the group $G(\mathcal{S})$. It is a k-Hopf algebra. By means of the right regular action of $G(\mathcal{S})$ on $\mathcal{O}$, that is, consider $\mathcal{O}$ as a right comodule on itself by means of the coproduct map, $\mathcal{O}$ can be considered as an object in Ind-$\mathcal{S}$. Since the unit map $u : k \to \mathcal{O}$ and the multiplication map $m : \mathcal{O} \otimes_k \mathcal{O} \to \mathcal{O}$ of $\mathcal{O}$ are compatible with the coproduct and the counit maps, they are also morphisms in Ind-$\mathcal{S}$, hence $(\mathcal{O}, m, u)$ is an algebra in Ind-$\mathcal{S}$ (it is not a Hopf algebra since the coproduct and the counit are not morphisms in Ind-$\mathcal{S}$). For convenience we shall use the same notation for denoting $\mathcal{O}$ as a k-vector space or as an object in $\mathcal{S}$ as well as an object of $\mathcal{Q}$ when we consider ω (the fiber functor of $\mathcal{O}$) as the restriction of q to $\mathcal{S}$.

4.3. The largest $\mathcal{S}$-subobject. For an object $X \in \mathcal{T}$, let $X_{\mathcal{S}}$ denote the largest subobject of X which is isomorphic to an object in $\mathcal{S}$. Since $\mathcal{S}$ is closed under taking sub- and quotient objects, we have the equality

$$\mathrm{Hom}_{\mathcal{T}}(S, X) = \mathrm{Hom}_{\mathcal{S}}(S, X_{\mathcal{S}}), \quad S \in \mathcal{S}, X \in \mathcal{T} \tag{4.2}$$

Thus we have a functor $(-)_{\mathcal{S}} : \mathcal{T} \to \mathcal{S}$ whose definition on hom-sets is just the restriction of morphisms $\mathrm{Hom}_{\mathcal{T}}(X, Y) \mapsto \mathrm{Hom}_{\mathcal{T}}(X_{\mathcal{S}}, Y) = \mathrm{Hom}_{\mathcal{T}}(X_{\mathcal{S}}, Y_{\mathcal{S}})$. Equation (4.2) also shows that this functor is right adjoint to the inclusion functor $\mathcal{S} \hookrightarrow \mathcal{T}$. The functor $(-)_{\mathcal{S}}$ is canonically extended to a functor Ind-$\mathcal{T} \to$ Ind-$\mathcal{S}$ denoted by the same symbol which is also the right adjoint to the inclusion functor Ind-$\mathcal{S} \to$ Ind-$\mathcal{T}$.

Let $X^\vee$ denote the dual object to X. The for any $S \in \mathcal{S}$, we have

$$\mathrm{Hom}_{\mathcal{S}}(X, S) \cong \mathrm{Hom}_{\mathcal{S}}(S^\vee, X^\vee) \cong \mathrm{Hom}_{\mathcal{S}}(S^\vee, (X^\vee)_{\mathcal{S}}) \cong \mathrm{Hom}_{\mathcal{S}}({(X^\vee)_{\mathcal{S}}}^\vee, S)$$

Since $X_{\mathcal{S}} \to X$ is mono, $X^\vee \to (X^\vee)_{\mathcal{S}}{}^\vee$ is epi. Thus the largest $\mathcal{S}$-quotient of X is isomorphic to $(X^\vee)_{\mathcal{S}}{}^\vee$.

Lemma 4.4. *There is a functorial isomorphism*

$$\mathrm{Hom}_{\text{Ind-}\mathcal{T}}(I, X \otimes \mathcal{O}) \xrightarrow{\cong} \omega(X_{\mathcal{S}}), \qquad X \in \mathcal{T}. \tag{4.3}$$

where ω is the fiber functor of $\mathcal{S}$ to vect_k.

Proof. If $X = S \in \mathcal{S}$ we have the following well-known isomorphism

$$S \otimes \mathcal{O} \cong \omega(S) \otimes_k \mathcal{O} \tag{4.4}$$

Indeed, by applying ω on both side it suffices to exhibit a $G(\mathcal{S})$-linear isomorphism $\omega(S) \otimes_k \mathcal{O} \to \mathcal{O}^{\dim_k \omega(S)}$, where $G(\mathcal{S})$ as on the source by the diagonal action. Let $\delta : \omega(S) \to \omega(S) \otimes_k \mathcal{O}$ denote the coaction of $\mathcal{O}$ on $\omega(S)$ induced from the action of $G(\mathcal{S})$, then the following map

$$\omega(S) \otimes_k \mathcal{O} \overset{\delta\otimes\mathrm{id}}{\to} \omega(S) \otimes_k \mathcal{O} \otimes_k \mathcal{O} \overset{\mathrm{id}\otimes m}{\to} \omega(S) \otimes_k \mathcal{O} \cong \mathcal{O}^{\dim_k \omega(S)}$$

is $G(\mathcal{S})$-linear and bijective with the inverse given by

$$\begin{aligned} \mathcal{O}^{\dim_k \omega(S)} &\cong \omega(S) \otimes_k \mathcal{O} \overset{\delta\otimes\mathrm{id}}{\to} \omega(S) \otimes_k \mathcal{O} \otimes_k \mathcal{O} \\ &\overset{\mathrm{id}\otimes\iota\otimes\mathrm{id}}{\to} \omega(S) \otimes_k \mathcal{O} \otimes_k \mathcal{O} \overset{\mathrm{id}\otimes m}{\to} \omega(S) \otimes_k \mathcal{O} \end{aligned}$$

Since $\mathrm{Hom}_{G(\mathcal{S})}(k, \mathcal{O}) \cong k$, we obtain

$$\mathrm{Hom}_{\text{Ind-}\mathcal{T}}(I, S \otimes \mathcal{O}) \cong \mathrm{Hom}_{G(\mathcal{S})}(k, \mathcal{O}^{\dim_k \omega(S)}) \cong \omega(S)$$

In the general case, let $X^\vee$ be the dual object to X, then morphisms $I \to X \otimes \mathcal{O}$ are in 1-1 correspondence with morphisms $X^\vee \to \mathcal{O}$, which, are in 1-1 correspondence with morphisms from the largest $\mathcal{S}$-quotient of $X^\vee$ to $\mathcal{O}$, since $\mathcal{O}$ is an object of Ind-$\mathcal{S}$. Notice that the largest $\mathcal{S}$-quotient of $X^\vee$ is isomorphic to $(X_{\mathcal{S}})^\vee$, as for $X \in \mathcal{T}$ one has $(X^\vee)^\vee \cong X$. Thus we have

$$\mathrm{Hom}_{\text{Ind-}\mathcal{T}}(I, X \otimes \mathcal{O}) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X^\vee, \mathcal{O}) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}((X_{\mathcal{S}})^\vee, \mathcal{O}) \cong \omega(X_{\mathcal{S}})$$

Therefore (4.3) is proved. The functoriality of (4.3) is obvious since any morphism $f : X \to Y$ restricts to a morphism $f : X_{\mathcal{S}} \to Y_{\mathcal{S}}$. □

Corollary 4.5. *There exists a functorial isomorphism*

$$\mathrm{Hom}_{\mathcal{Q}}(I, \mathfrak{q}(X^\vee \otimes Y)) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X \otimes Y^\vee, \mathcal{O}) \tag{4.5}$$

Proof. By definition of quotient category, the largest trivial subobject of an object $\mathfrak{q}(X) \in \mathcal{Q}$ has the form $\mathfrak{q}(X')$, where X' is a subobject of X (in $\mathcal{T}$). By definition of $\mathcal{S}$, X' is in $\mathcal{S}$ and should be the largest $\mathcal{S}$-subobject of X, that is $X' = X_{\mathcal{S}}$. We have

$$\mathrm{Hom}_{\mathcal{Q}}(\mathfrak{q}(X), \mathfrak{q}(Y)) \cong \mathrm{Hom}_{\mathcal{Q}}(I, \mathfrak{q}(X \otimes Y)) \cong \mathrm{Hom}_{\mathcal{Q}}(I, \mathfrak{q}((X^\vee \otimes Y)_{\mathcal{S}})$$

These isomorphisms together with (4.1) and Lemma 4.4 imply (4.5). □

The isomorphism in (4.5) implies the following

$$\mathrm{Hom}_{\mathcal{Q}}(\mathfrak{q}(X), \mathfrak{q}(Y)) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, Y \otimes \mathcal{O}) \tag{4.6}$$

Let us denote this map by

$$f \mapsto \bar{f}$$

By the functoriality of (4.3) we see that, for $X = Y$,

$$\overline{\mathrm{id}}_X = \mathrm{id}_X \otimes u : X \to X \otimes \mathcal{O} \tag{4.7}$$

where $u : I \to \mathcal{O}$ is the unit map of $\mathcal{O}$.

4.6. The adjoint functor to $\mathfrak{q}$. The quotient functor $\mathfrak{q}$ extends to a functor from Ind-$\mathcal{T} \to$ Ind-$\mathcal{Q}$, denoted by the same symbol

$$\mathfrak{q}(\varinjlim_i X_i) := \varinjlim \mathfrak{q}(X_i)$$

$\mathfrak{q}$ possesses a right adjoint, denoted by $\mathfrak{p}$. Indeed, for an object U of $\mathcal{Q}$, $\mathfrak{p}(U)$ is determined by the condition

$$\mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, \mathfrak{p}(U)) \cong \mathrm{Hom}_{(}\mathfrak{q}(X), U), \quad \forall X \in \mathcal{T}$$

and for an object $\varinjlim_i U_i \in$ Ind-$\mathcal{Q}$,

$$\mathfrak{p}(\varinjlim_i U_i) := \varinjlim_i \mathfrak{p}(U_i)$$

Thus $\mathfrak{p}$ satisfies the required functorial isomorphism:

$$\mathrm{Hom}_{\text{Ind-}\mathcal{Q}}(\mathfrak{q}(X), U) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, \mathfrak{p}(U)), \quad \forall X \in \text{Ind-}\mathcal{T}, U \in \text{Ind-}\mathcal{Q} \tag{4.8}$$

For $X = \mathfrak{p}(U)$ in (4.8), the identity of $\mathfrak{p}(U)$ corresponds to a map $\varepsilon_U : \mathfrak{q}\mathfrak{p}(U) \to U$, and (4.8) given by composing a morphism on the right hand side with ε_U.

$$\begin{tikzcd}
& \mathfrak{q}\ \mathfrak{p}(U) \arrow[d, "\varepsilon_U"] \\
\mathfrak{q}(X) \arrow[ur, "\mathfrak{q}(\bar{f})"] \arrow[r, "f"'] & U
\end{tikzcd}$$

Lemma 4.7. *For an object $X \in$ Ind-$\mathcal{T}$ the object $\mathfrak{p}\mathfrak{q}(X)$ in Ind-$\mathcal{T}$ is canonically isomorphic to $X \otimes \mathcal{O}$ and the map*

$$\mathfrak{q}(X) \otimes \mathcal{O} \cong \mathfrak{q}\mathfrak{p}\mathfrak{q}(X) \overset{\varepsilon_X}{\to} \mathfrak{q}(X)$$

is given by $\mathrm{id}_X \otimes \epsilon$, where ϵ is the counit of $\mathcal{O}$ (considering ω as the restriction of $\mathfrak{p}$ to $\mathcal{S}$). With respect to the isomorphism $\mathfrak{p}\mathfrak{q}(X) \cong X \otimes \mathcal{O}$, (4.8) for $U = \mathfrak{q}(Y)$ reduces to (4.6).

Proof. First assume $X \in \mathcal{T}$. According to Lemma 4.4 and (4.8) we have, for $X, Y \in \mathcal{T}$,

$$\mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, \mathfrak{p}(Y)) \cong \mathrm{Hom}_{\mathcal{Q}}(\mathfrak{q}X, \mathfrak{q}Y) \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, Y \otimes \mathcal{O}) \tag{4.9}$$

Since this isomorphism holds for any X, Y we conclude that $\mathfrak{p}(Y)$ is canonically isomorphism to $Y \otimes \mathcal{O}$. The map ε_Y induces a morphism $\mathfrak{q}(Y) \otimes \mathcal{O} \to \mathfrak{q}(Y)$ which will be denoted by the same symbol. Thus we have the following diagram

(4.10)
$$\begin{tikzcd}
& \mathfrak{q}(Y) \otimes \mathcal{O} \arrow[d, "\varepsilon_Y"] \\
\mathfrak{q}(Y) \arrow[ur, "\mathfrak{q}(\bar{f})"] \arrow[r, "f"'] & \mathfrak{q}(Y)
\end{tikzcd}$$

Set $X = Y$ in (4.9), then, according to (4.7), the identity on Y is mapped under the second isomorphism of (4.9) to the morphism $\mathrm{id}_Y \otimes u : Y \to Y \otimes \mathcal{O}$. Since the inverse map to the second isomorphism in (4.9) is given by $f \mapsto (\mathrm{id} \otimes \epsilon) f$, we conclude that $\varepsilon_Y(\mathrm{id}_{\mathfrak{q}(Y)} \otimes u) = \mathrm{id}_{\mathfrak{q}(Y)}$. Therefore, $\varepsilon_Y = \mathrm{id}_{\mathfrak{q}(Y)} \otimes \epsilon$ (recall that we identify ϵ with $\mathfrak{q}(\epsilon)$).

For the general case we note that the tensor product in Ind-$\mathcal{T}$ commutes with direct limits, hence

$$\mathfrak{p}\mathfrak{q}(\varinjlim_i X_i) \cong \varinjlim_i (X_i \otimes \mathcal{O}) \cong (\varinjlim_i X_i) \otimes \mathcal{O}.$$

□

Corollary 4.8. *In the isomorphism (4.6) the composition*

$$\mathfrak{q}(X) \xrightarrow{f} \mathfrak{q}(Y) \xrightarrow{g} \mathfrak{q}(Z)$$

corresponds to the morphism

$$X \xrightarrow{f} Y \otimes \mathcal{O} \xrightarrow{g \otimes \mathrm{id}} Z \otimes \mathcal{O} \otimes \mathcal{O} \xrightarrow{\mathrm{id} \otimes m} Z \otimes \mathcal{O}$$

Proof. Since $\mathfrak{q}$ is a faithful functor, it suffices to check that the outer diagram below commutes.

$$\begin{tikzcd}
\mathfrak{q}(X) \arrow[r, "\mathfrak{q}(\bar{f})"] \arrow[dr, "f"'] & \mathfrak{q}(Y) \otimes \mathcal{O} \arrow[r, "\mathfrak{q}(\bar{g}) \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \epsilon"] & \mathfrak{q}(Z) \otimes \mathcal{O} \otimes \mathcal{O} \arrow[r, "m"] \arrow[d, "\mathrm{id} \otimes \epsilon \otimes \epsilon"'] & \mathfrak{q}(Z) \otimes \mathcal{O} \arrow[dl, "\mathrm{id} \otimes \epsilon"] \\
& \mathfrak{q}(Y) \arrow[r, "g"'] & \mathfrak{q}(Z) &
\end{tikzcd}$$

The commutativity of the first triangle and the middle square follow from (4.10) and of the right triangle is by the multiplicativity of the counit map ϵ. □

Using the same method we can prove the following fact.

Corollary 4.9. *Let $f_i : \mathfrak{q}(X_i) \to \mathfrak{q}(Y_i)$, $i = 0, 1$ be morphisms in $\mathcal{Q}$. Then the morphism $\overline{f_0 \otimes f_1}$ is given by*

$$\begin{tikzcd}
X_0 \otimes X_1 \arrow[r, "\overline{f_0 \otimes f_1}"] \arrow[d, "\bar{f}_0 \otimes \bar{f}_1"'] & Y_0 \otimes Y_1 \otimes \mathcal{O} \\
Y_0 \otimes \mathcal{O} \otimes Y_1 \otimes \mathcal{O} \arrow[r, "\tau_{(23)}"'] & Y_0 \otimes Y_1 \otimes \mathcal{O} \otimes \mathcal{O} \arrow[u, "m"']
\end{tikzcd}$$

where the map $\tau_{(23)}$ interchanges the second and the third tensor terms.

Proposition 4.10. *Let $f:\mathfrak{q}(X)\to\mathfrak{q}(Y)$ be a morphism in $\mathcal{Q}$. Then*

$$\mathfrak{p}(f)=X\otimes\mathcal{O}\overset{\bar{f}\otimes\mathrm{id}}{\longrightarrow}Y\otimes\mathcal{O}\otimes\mathcal{O}\overset{\mathrm{id}\otimes m}{\longrightarrow}Y\otimes\mathcal{O}\tag{4.11}$$

Proof. The morphism $\mathfrak{p}(f)$ fits in to the following commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{Q}}(\mathfrak{q}(Z),\mathfrak{q}(X)) \arrow[r, "\cong"] \arrow[d, "f\circ-"'] & \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{T}}(Z,\mathfrak{p}\mathfrak{q}(X)) \arrow[d, "\mathfrak{p}(f)\circ-"] \\
\mathrm{Hom}_{\mathcal{Q}}(\mathfrak{q}(Z),\mathfrak{q}(Y)) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathrm{Ind}\text{-}\mathcal{T}}(Z,\mathfrak{p}\mathfrak{q}(X))
\end{tikzcd}$$

and is indeed determined by this square (for all $Z\in\mathcal{T}$). Thus, in terms of (4.8) $\mathfrak{p}(f)$ is uniquely determined by the following commuting triangle

$$\begin{tikzcd}
Z \arrow[r, "\bar{g}"] \arrow[dr, "\overline{fg}"'] & \mathfrak{p}\mathfrak{q}(X) \arrow[d, "\mathfrak{p}(f)"] \\
 & \mathfrak{p}\mathfrak{q}(Y)
\end{tikzcd} \qquad \text{forall } Z\in\mathcal{T}, g:\mathfrak{q}(Z)\to\mathfrak{q}(X)$$

According to Corollary 4.8, the morphism on the right hand side of (4.11) satisfies this property, hence is equal to $\mathfrak{p}(f)$. □

Recall that $\mathcal{O}$ is a commutative algebra in Ind-$\mathcal{T}$. We can thus consider the category $\mathsf{Mod}_{\mathcal{O}}$ of $\mathcal{O}$-modules in Ind-$\mathcal{T}$, which is an abelian category equipped with a tensor product over $\mathcal{O}$.

Definition 4.11. Let $(\mathcal{S},\omega:\mathcal{S}\to\mathsf{vect}_k)$ be a neutral Tannaka subcategory of a rigid tensor category $\mathcal{T}$ which is closed under taking sub- and quotient objects. Denote by $\mathcal{O}$ the function algebra over its Tannaka group, viewed as an object in Ind-$\mathcal{S}\subset$ Ind-$\mathcal{T}$. The category $\mathcal{T}$ is said to be *flat* over $\mathcal{S}$ if for any $\mathcal{O}$-linear morphism $f:X\otimes\mathcal{O}\to Y\otimes\mathcal{O}$ in Ind-$\mathcal{T}$, $X,Y\in\mathcal{T}$, the kernel of f is flat with respect to tensor product in $\mathsf{Mod}_{\mathcal{O}}$.

Theorem 4.12. *Let $(\mathcal{Q},\mathfrak{q})$ be a k-quotient category of a tensor category $\mathcal{T}$ over k and denote by $\mathcal{S}$ the corresponding invariant category. Let $\mathcal{O}$ denote the function algebra over the Tannaka group of $\mathcal{S}$. Let $\mathfrak{p}$ be the right adjoint to the quotient functor $\mathfrak{q}$. Then:*

1. *For any object $U\in\mathcal{Q}$, there exists a morphism $\mu_U:\mathfrak{p}(U)\otimes\mathcal{O}\to\mathfrak{p}(U)$ making $\mathfrak{p}(U)$ an $\mathcal{O}$-module.*
2. *Assume that $\mathcal{T}$ is flat over $\mathcal{S}$. Then $\mathfrak{p}$ is a tensor functor from $\mathcal{Q}$ to the category of $\mathcal{O}$-modules with the tensor product being the tensor product over $\mathcal{O}$. Consequently, $\mathfrak{p}$ is exact.*

Proof. Using (4.8), we define μ_U as the unique morphims in Ind-$\mathcal{T}$ making the following diagram commutative

$$\begin{tikzcd}
& \mathfrak{q}\mathfrak{p}(U) \arrow[d, "\varepsilon_U"'] \\
\mathfrak{q}\mathfrak{p}(U) \otimes \mathcal{O} \arrow[ur, "\mathfrak{q}(\mu_U)"] \arrow[r, "\mathfrak{q}(\varepsilon_U \otimes \epsilon)"'] & U
\end{tikzcd}$$

This definition is functorial hence the action of μ commutes with morphisms in $\mathcal{Q}$. The associativity of this action can also be checked by this method. Thus $\mathfrak{p}$ factors through a functor to Mod-$\mathcal{O}$, denoted by the same notation.

For objects in $\mathcal{Q}$ of the form $\mathfrak{q}(X)$, $\mathfrak{p}\mathfrak{q}(X) \cong X \otimes \mathcal{O}$, we see that the multiplication map on $\mathcal{O}$ makes the following diagram commutative

$$\begin{tikzcd}
& \mathfrak{q}(X) \otimes \mathcal{O} \arrow[d, "\varepsilon_{\mathfrak{q}(X)}"'] \\
\mathfrak{q}(X) \otimes \mathcal{O} \otimes \mathcal{O} \arrow[ur, "\mathrm{id} \otimes \mu_X"] \arrow[r, "\varepsilon_X \otimes \epsilon"'] & \mathfrak{q}(X)
\end{tikzcd}$$

Thus, the action of $\mathcal{O}$ on $\mathfrak{p}(X) \cong X \otimes \mathcal{O}$ is induced from the action of $\mathcal{O}$ on itself.

Assume now that $\mathcal{T}$ is flat over $\mathcal{S}$. We want to show that $\mathfrak{p}$ is a monoidal functor from $\mathcal{Q}$ to $\mathcal{O}$-modules. This is so for objects of the form $\mathfrak{q}(X)$, according to Corollary 4.9. For an arbitrary object U of $\mathcal{Q}$, there exists objects X, Y in $\mathcal{T}$ and a morphism $f : \mathfrak{q}(X) \to \mathfrak{q}(Y)$ such that $U = \ker f$. Since the functor $\mathfrak{p}$ is left exact (being a right adjoint functor), $\mathfrak{p}(U)$ is isomorphic to the kernel of $\mathfrak{p}(f)$. Now, for $f_i : \mathfrak{q}(X_i) \to \mathfrak{q}(Y_i)$ we have $\mathfrak{p}(f_1 \otimes f_2) = \mathfrak{p}(f_1) \otimes_{\mathcal{O}} \mathfrak{p}(f_2)$. Since for $U_i := \ker f_i$ are flat over $\mathcal{O}$ by assumption, we have an $\mathcal{O}$-linear isomorphism

$$\mathfrak{p}(U_1) \otimes_{\mathcal{O}} \mathfrak{p}(U_2) \cong \mathfrak{p}(U_1 \otimes U_2)$$

Thus $\mathfrak{p}$ is a tensor functor to $\mathcal{O}$-modules. Since $\mathcal{Q}$ is rigid with the endomorphism ring of the unit object isomorphic to k, we conclude that $\mathfrak{p}$ is an exact functor, cf. [2, 2.10]. □

4.13. A description of the quotient. We define a category $\mathcal{P}$, whose objects are triples

$$(X, Y, f : X \to Y \otimes \mathcal{O})$$

where X, Y are objects of $\mathcal{T}$ and f is a morphism in Ind-$\mathcal{T}$. The morphisms $f : X \to Y \otimes \mathcal{O}$ defines an $\mathcal{O}$-linear morphism $\hat{f}$

$$\hat{f} : X \otimes \mathcal{O} \overset{f \otimes \mathrm{id}}{\to} Y \otimes \mathcal{O} \otimes \mathcal{O} \overset{\mathrm{id} \otimes m}{\to} Y \otimes \mathcal{O} \tag{4.12}$$

The image of $\hat{f}$ is thus an $\mathcal{O}$-module.

We define morphisms in $\mathcal{P}$ between (X_i, Y_i, f_i), $i = 0, 1, 2$ as $\mathcal{O}$-linear morphisms $\phi : \mathrm{im}\hat{f}_0 \to \mathrm{im}\hat{f}_1$. Thus we obtain a category $\mathcal{P}$ which is k-linear and additive. Further, the direct sum of objects in $\mathcal{P}$ exists:

$$(X_0, Y_0, f_0) \oplus (X_1, Y_1, f_1) := (X_0 \oplus X_1, Y_0 \oplus Y_1, f_0 \oplus f_1). \tag{4.13}$$

The tensor structure on $\mathcal{P}$ is defined as follows.

$$(X_0, Y_0, f_0) \otimes (X_1, Y_1, f_1) := (X_0 \otimes X_1, Y_0 \otimes Y_1, (\mathrm{id} \otimes m)(f_0 \otimes f_1)) \tag{4.14}$$

The unit object is $I = (I, I, u : I \to \mathcal{O})$. Finally, we define a functor $\mathfrak{q}' : \mathcal{T} \to \mathcal{P}$ sending X to $\mathfrak{q}'(X) = (X, X, \mathrm{id} \otimes u : X \to X \otimes \mathcal{O})$. It is easy to see that $\mathfrak{q}'$ is a k-linear, additive tensor functor.

Proposition 4.14. *Let $\mathcal{T} \xrightarrow{\mathfrak{q}} \mathcal{Q}$ be k-quotient and $\mathcal{S}$ be the corresponding invariant subcategory. Assume that $\mathcal{T}$ is flat over $\mathcal{S}$. Then the category $\mathcal{P}$ constructed above is equivalent to $\mathcal{Q}$.*

Proof. We construct a functor $F : \mathcal{P} \to \mathcal{Q}$ such that $F\mathfrak{q}' = \mathfrak{q}$. For any $X, Y \in \mathcal{T}$, according to 4.9, morphisms $f : X \to Y \otimes \mathcal{O}$ in $\mathcal{T}$ are in 1-1 correspondence with morphisms $f_{\mathfrak{q}} : \mathfrak{q}(X) \to \mathfrak{q}(Y)$ in $\mathcal{Q}$. This allows us to define the image of an object $U = (X, Y, f : X \to Y \otimes \mathcal{O}) \in \mathcal{Q}'$ as the image of $f_{\mathfrak{q}}$ in $\mathcal{Q}'$.

According to Theorem 4.12, $\mathfrak{p}$ is exact. Hence for any morphism $f : \mathfrak{q}(X) \to \mathfrak{q}(Y)$ in $\mathcal{Q}$ with $U = \mathrm{im} f$ one has

$$\mathfrak{p}(U) \cong \mathrm{im}\mathfrak{p}(f)$$

Consequently, there is a 1-1 correspondence between morphisms $U \to V$ in $\mathcal{Q}$ and morphisms between $\mathfrak{p}(U) \to \mathfrak{p}(V)$ in $\mathcal{T}$ which are $\mathcal{O}$-linear. This allows us to define F on morphisms and to check that F is a fully faithful functor. Further it is easy to see that the image of F is essential in $\mathcal{Q}$. Thus F is an equivalence. □

As a consequence of the above theorem and Proposition 2.15 we have the following.

Corollary 4.15. *Let $f : G^K_k \to A^k_k$ be a homomorphism of transitive groupoids and L^K_k be the kernel of f. Then $\mathsf{Rep}_f(L^K_k)$ is equivalent to the category whose objects are triples $(U, V, f : U \to V \otimes \mathcal{O}(A))$, where $U, V \in \mathsf{Rep}_f(G)$, f is G-equivariant (G acts diagonally on $V \otimes \mathcal{O}(A)$).*

Remark 4.16. *There is an alternative description of objects of $\mathcal{Q}$, proposed by Deligne. Notice that morphisms $f : X \to Y \otimes \mathcal{O}$ are in 1-1 correspondence with morphisms $f^\sharp : I \to X^\vee \otimes Y \otimes \mathcal{O}$. As noticed in 4.4, such a morphism $f^\sharp$ corresponds to an element of $\omega((X^\vee \otimes Y)_{\mathcal{S}})$. Thus objects of $\mathcal{Q}$ can be characterized as triples $(X, Y, f \in \omega((X^\vee \otimes Y)_{\mathcal{S}}))$.*

5. Quotient category by a not necessary neutral Tannaka subcategory

5.1. Base change for tensor category. Let $\mathcal{T}$ be a tensor category over k. Assume that for a field extension $k \subset K$, a K-quotient $\mathcal{T}_{(K)}$ of $\mathcal{T}$ exists such that the invariant subcategory is the trivial subcategory of $\mathcal{T}$, i.e. equivalent to vect_k. $\mathcal{T}_{(K)}$ is called the tensor category over K obtained from $\mathcal{T}$ by base change. Denote the quotient functor by $-_{(K)} : X \mapsto X_{(K)}$. Thus, the largest trivial subobject ${X_{(K)}}^{\text{triv}}$ of $X_{(K)}$ in $\mathcal{T}_{(K)}$ is isomorphic to the image of the largest trivial object X^{triv} of X in $\mathcal{T}$. Hence

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}_{(K)}}(I, X_{(K)}) &= \mathrm{Hom}_{\mathcal{T}_{(K)}}(I, {X_{(K)}}^{\text{triv}})\\ &\cong \mathrm{Hom}_{\mathcal{T}}(I, X^{\text{triv}}) \otimes_k K\\ &\cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(I, X^{\text{triv}}) \otimes_k K\\ &= \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(I, X) \otimes_k K\end{aligned}$$

On the other hand, we have an isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(X, Y) \otimes_k K \cong \mathrm{Hom}_{\text{Ind-}\mathcal{T}}(X, Y \otimes_k K) \tag{5.1}$$

given explicitly as follows. Fix a basis $\{e_i, i \in I\}$ of K over k. An element $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y) \otimes_k K$, represented as $f = \sum_{i \in I_0} f_i \otimes e_i$, for a certain finite subset $I_0 \subset I$, is mapped to the morphism

$$X \xrightarrow{\Delta} \bigoplus_{i \in I_0} X_i \xrightarrow{\oplus_i f_i} \bigoplus_{i \in I_0} Y_i \hookrightarrow Y \otimes_k K$$

where X_i (resp. Y_i) are copies of X (resp. Y) and the last inclusion in given by the chosen basis of K.

Extend $-_{(K)}$ to a functor Ind-$\mathcal{T} \to$ Ind-$\mathcal{T}_{(K)}$ and let $-^K$: Ind-$\mathcal{T}_{(K)} \to$ Ind-$\mathcal{T}$ denote the adjoint functor to $-_{(K)}$.

Lemma 5.2. *For $X \in \mathcal{T}$ holds: $(X_{(K)})^K \cong X \otimes_k K$ and the map ε_X is given by $\varepsilon_X : X_{(K)} \otimes_k K \to X_{(K)} \otimes_K K \cong X_{(K)}$.*

Other claims similar to those of Corollaries 4.8, 4.9 and of Prop. 4.10 also hold.

By identifying K with $K \otimes_k I$, we can consider K as an algebra in Ind-$\mathcal{T}$ and consider the category Ind-$\mathcal{T}_K$ of K-modules. Then the functor $-^K$ factors through a functor to Ind-$\mathcal{T}_K$, denoted by the same symbol.

Definition 5.3. We say that $\mathcal{T}$ is flat over K if for any $X, Y \in \mathcal{T}$ and K-linear morphism $f : X \otimes_k K \to Y \otimes_k K$ in Ind-$\mathcal{T}_K$, the kernel of f is flat with respect to the tensor product over K.

Consequently we have the following theorem, which is analogous to Theorem 4.12.

Theorem 5.4. *Assume that $\mathcal{T}$ is flat over a field $K \supset k$ and that the base change $\mathcal{T}_{(K)}$ exists. Then, for an object $U \in \mathcal{T}_{(K)}$, there exists a map $\mathfrak{p}(U) \otimes_k K \to \mathfrak{p}(U)$ making $\mathfrak{p}(U)$ a K-module. Further $\mathfrak{p}$ is a K-linear tensor functor from $\mathcal{T}_{(K)}$ to the category* Ind-$\mathcal{T}_K$ *of K-modules in $\mathcal{T}$. Consequently $-^K$ is exact.*

Corollary 5.5. *Assume that $\mathcal{T}$ is flat over K and that $\mathcal{T}_{(K)}$ exists. Then $\mathcal{T}_{(K)}$ is equivalent to the category of triples $(X, Y, f : X \to Y \otimes_k K)$, $X, Y \in \mathcal{T}$,*

$f \in$ Mor Ind-$\mathcal{T}$, and morphism defined as in section 4. Consequently, any k-linear tensor functor ω from $\mathcal{T}$ to a K-linear tensor category $\mathcal{C}$ factors through a K-linear tensor ω_K from $\mathcal{T}_{(K)}$ to $\mathcal{C}$.

Proof. The first claim is proved analogously as in the proof of Proposition 4.14. The second claim is a consequence of the first one. In deed, the functor ω induces a K-linear map

$$\omega \otimes K : \mathrm{Hom}_{\mathcal{T}}(X,Y) \otimes_k K \to \mathrm{Hom}_{\mathcal{C}}(\omega(X), \omega(Y))$$

Thus, considering f as an element of $\mathrm{Hom}_{\mathcal{T}}(X,Y) \otimes_k K$ by means of (5.1), we can define ω_K on an object $(X, Y, f : X \to Y \otimes K)$ as the image of $(\omega \otimes K)f : \omega(X) \to \omega(Y)$. □

Corollary 5.6. *Let $\mathcal{T}$ be a Tannaka category over k with fiber functor to vect_K, $K \supset k$ and G_k^K the corresponding Tannaka groupoid. Let $k \subset K_0 \subset K$ be an intermediate field. Then the K_0-base change of $\mathcal{T}$ exists: $\mathcal{T}_{K_0} \cong \mathsf{Rep}_f(G_{K_0}^K)$.*

Proof. This follows from Proposition 2.15 for $A = \operatorname{Spec} K_0 \times_k \operatorname{Spec} K_0$ and Theorem 5.4. □

The category $\mathcal{T}_K$ for K a finite extension of k was studied by Deligne-Milne [1, Sect. 3].

5.7. The description of the quotient category. Assume now that $(\mathcal{Q}, \mathfrak{q})$ is a K-quotient of $\mathcal{T}$ with $\mathcal{S}$ being the invariant category and assume that the base change $\mathcal{T}_{(K)}$ of $\mathcal{T}$ exists and that $\mathcal{T}$ is flat over K. By Corollary 5.5, $\mathfrak{q}$ factors through a K-linear tensor functor $\mathfrak{q}_K : \mathcal{T}_{(K)} \to \mathcal{Q}$, which can easily shown to be a quotient functor of tensor categories over K and the invariant category of $\mathfrak{q}_K$ is equivalent to $\mathcal{S}_K$.

Let $\mathcal{O}$ denote the function algebra of the Tannaka groupoid A_k^K of $\mathcal{S}$. Consider $\mathcal{O}$ as an object of Ind-$\mathcal{S}$ by means of the right regular coaction $\Delta : \mathcal{O} \to \mathcal{O}_s \otimes_t \mathcal{O}$. Then $\mathcal{O}$ is a K-object in the sense of 5.1 where K acts through the map t. Further $\mathcal{O}$ is a K-algebra, i.e. an algebra in Ind-$\mathcal{T}_K$, which in this case means that $\mathcal{O}$ is an algebra over $K \otimes K$ through the map $s \otimes t$ in the usual sense. We define by $\mathsf{Mod}_{\mathcal{O},K}$ the category of $\mathcal{O}$-modules in Ind-$\mathcal{T}_K$, whose objects are thus K-object equipped with an action of $\mathcal{O}$. This is a tensor category with respect to the tensor product over $\mathcal{O}$.

Definition 5.8. With the assumption of 5.7, $\mathcal{T}$ is said to be flat over $\mathcal{S}$ if for any $\mathcal{O}$-linear morphism $f : X \otimes \mathcal{O} \to Y \otimes \mathcal{O}$, $X, Y \in \mathcal{T}$, the kernel of f is flat with respect to the product over $\mathcal{O}$.

Theorem 5.9. *Let $\mathcal{T}$ be a rigid tensor category over k. Let $(\mathcal{S}, \omega : \mathcal{S} \to \mathsf{vect}_K)$ be a Tannaka subcategory, which is closed under taking sub- and quotient objects. Assume that*

(i) *$\mathcal{T}$ is flat over K and the base change $\mathcal{T}_{(K)}$ exists;*
(ii) *$\mathcal{T}$ is flat over $\mathcal{S}$ and a K-quotient $(\mathcal{Q}, \mathfrak{q})$ of $\mathcal{T}$ by $\mathcal{S}$ exists.*

Then the right adjoint functor $\mathfrak{p}$ *to* $\mathfrak{q}$ *induces an exact tensor functor to the category* $\mathsf{Mod}_{\mathcal{O},K}$.

Proof. Denote by $\mathcal{O}_K$ the function algebra of A_K^K - the diagonal subgroup of A_k^K, where A_k^K is the Tannaka groupoid of $\mathcal{S}$. Let $\mathfrak{p}_K$ denote the right adjoint functor to $\mathfrak{q}_K$: Ind-$\mathcal{T}_{(K)} \longrightarrow$ Ind-$\mathcal{Q}$. Then according to 4.12, $\mathfrak{q}_K$ is an exact tensor functor to the category of $\mathcal{O}_K$-modules in Ind-$\mathcal{T}_{(K)}$.

On the other hand, since $\mathcal{S} \cong \mathsf{Rep}_f(A_k^K)$, $\mathcal{S}_{(K)} \cong \mathsf{Rep}(A_K^K)$, the functor $-^K$ in in fact the functor ind_{Δ_K} in the situation of 2.12. Thus

$$(\mathcal{O}_K)^K \cong \mathcal{O}_K \square_{\mathcal{O}_K} \mathcal{O} \cong \mathcal{O}$$

This isomorphism gives us a morphism

$$(U \otimes_{\mathcal{O}_K} V)^K \to U^K \otimes_{\mathcal{O}} V^K$$

We don't know if this morphism is an isomorphism for an arbitrary pair $U, V \in \mathcal{T}_{(K)}$ but we know it is for U, V in the image of $\mathfrak{q}_K$, thanks to the assumption that $\mathcal{T}$ is flat over $\mathcal{S}$. In deed, for $X \in \mathcal{T}$, we have $\mathfrak{p}\mathfrak{q}(X) \cong X \otimes \mathcal{O}$, using the method as in the proof of 4.12 we deduce the required isomorphism. □.

Corollary 5.10. *Let* $f : G_k^K \longrightarrow A_k^{K_0}$ *be a homomorphism of transitive groupoids and* $L_{K_0}^K$ *be the kernel of* f. *Then* $\mathsf{Rep}_f(L_{K_0}^K)$ *is equivalent to the category whose objects are triples* $(U, V, f : U \longrightarrow V \otimes \mathcal{O}(A))$, *where* $U, V \in \mathsf{Rep}_f(G)$, f *is* G-*equivariant (*G *acts diagonally on* $V \otimes \mathcal{O}(A)$*).*

Since morphisms $f : X \longrightarrow Y \otimes \mathcal{O}$ are in 1-1 correspondence with morphism $f^\sharp : I \longrightarrow X^\vee \otimes Y \otimes \mathcal{O}$. As noticed in 4.4, such a morphism $f^\sharp$ corresponds to an element of $\omega((X^\vee \otimes Y)_{\mathcal{S}})$. Thus objects of $\mathcal{Q}$ can be characterized as triples $(X, Y, f \in \omega((X^\vee \otimes Y)_{\mathcal{S}}))$.

Another consequence of the above theorem which may be useful in checking whether a sequence of groupoids $L_{K_0}^K \longrightarrow G_k^K \longrightarrow A_k^{K_0}$ is exact is the following

Corollary 5.11. *Assume we are given a sequence of homomorphisms of groupoids* $L_{K_0}^K \xrightarrow{q} G_k^K \xrightarrow{f} A_k^{K_0}$ *with* fq *trivial. Then the sequence is exact in the sense that* q *is the kernel of* f *iff the representation categories of these groups satisfy the condition (i) and (ii) of Definition 3.2*

Remark 5.12. *The following questions are open to us:*

1. *Is* $\mathcal{T}$ *always flat over* $\mathcal{S}$, K*?*
2. *Assume that* $\mathcal{T}$ *is flat over* $\mathcal{S}$ *(and* K*), does a quotient exist, in particular, does the base change to* K *exist?*
3. *Assume that* $\mathcal{T}$ *is flat over* $\mathcal{S}$ *(and* K*), does the functor* $\mathfrak{p}$ *induce an equivalence of tensor categories between* Ind-$\mathcal{Q}$ *and* $\mathsf{Mod}_{\mathcal{O},K}$*?*

Finally we mention a related question raised by Deligne, namely whether the the fundamental group of $\mathcal{T}$ is flat over $\mathcal{S}$, see [2] for definition of the fundamental group of a rigid tensor category.

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