

SEMISTABLE REDUCTION FOR G -BUNDLES ON CURVES

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ABSTRACT. We prove a semistable reduction theorem for principal bundles on curves in almost arbitrary characteristics. For exceptional groups we need some small explicit restrictions on the characteristic.

1. INTRODUCTION

We want to prove a semistable reduction theorem for principal G -bundles over a smooth projective curve.

Theorem 1. *Let C/k be a smooth projective curve over an algebraically closed field k and let G be a semisimple group over k .*

Assume either that Behrend's conjecture holds for G or assume that $\text{char}(k) \neq 2$ if G contains factors of type B, C, D , $\text{char}(k) > 7$ if G contains a factor of type G_2 , $\text{char}(k) > 19$ if G contains a factor of type F_4 or E_6 , $\text{char}(k) > 31$ if G contains a factor of type E_7 and $\text{char}(k) > 53$ if G contains a factor of type E_8 .

Let R be a complete discrete valuation ring over k with fraction field K . Then for any semistable principal G -bundle \mathcal{G}_K on $C \times_k \text{Spec } K$ there exists a finite extension R' of R such that the pullback of \mathcal{G}_K extends to a semistable G -bundle on $C \times_k \text{Spec } R'$.

Recall that Behrend's conjecture ([4], 7.6) says that the reduction of a principal bundle to the maximal instability parabolic has no infinitesimal deformations. This is only known in characteristic 0 and for characteristics bigger than the height $h(G)$, by Biswas and Holla [7]. For exceptional groups we need to apply this result in the last step of our proof, which causes the bounds on the characteristics. In case G is of type A , our proof also works in characteristic 2, as in Langton's theorem [16].

If the characteristic of k is 0, then the above theorem has been shown by Faltings [11] and Balaji and Seshadri [2]. This was generalized to large characteristics by Balaji and Parameswaran [1]. In their article they prove the theorem if the characteristic of k is bigger than $(h(G)^{\text{rank}(G)} \text{rank}(G)!)$ and conjecture that a better bound should be true.

Our motivation to look for another approach to this theorem came from the recent work of Gomez, Langer, Schmitt and Sols [13], who managed to construct quasi-projective coarse moduli spaces for semistable principal bundles in arbitrary characteristic. The above theorem shows that in the case of smooth projective curves their moduli spaces are projective in almost any characteristic.

Our strategy is similar to Langton's proof [16] of the corresponding theorem for vector bundles:

- (1) Take any extension \mathcal{G}_R of \mathcal{G}_K to $C \times_k \text{Spec } R$. This is possible – at least after a finite extension of R – because the affine Grassmannian is (ind-)projective.
- (2) If the special fiber \mathcal{G}_k of the principal bundle is not semistable, then we can apply Behrend's theorem to find a canonical reduction of \mathcal{G}_k to a parabolic subgroup $P \subset G$. Since \mathcal{G}_K is semistable there is a maximal n such that this canonical reduction to P can be lifted to R/\mathfrak{m}^n , where \mathfrak{m} is the maximal ideal of R .

Choose a cocycle for \mathcal{G}_R , such that it reduces to a cocycle defining the canonical reduction modulo \mathfrak{m}^n . (We even have to be a little more careful in choosing the cocycle, see Proposition 12.) Conjugating the cocycle for \mathcal{G}_R with a K -valued element of the center of the Levi subgroup L of P we find a modification \mathcal{G}'_R of \mathcal{G}_R such that the special fiber of \mathcal{G}'_R has a reduction to the opposite parabolic P^- , the associated L -bundle is isomorphic to the L -bundle defined by the canonical reduction of the special fiber of \mathcal{G} and finally this P^- -bundle has no reduction to L .

- (3) Deduce that the special fiber of \mathcal{G}'_R is less unstable than the special fiber of \mathcal{G}_R . Here “less unstable” means that the bundle lies in a component of the Harder-Narasimhan-stratification of the moduli stack of G -Bundles, which contains \mathcal{G}_R in its closure. This holds, since by our construction \mathcal{G}'_k has a reduction to a parabolic P^- , such that the associated L -bundle is the same as the one obtained from the canonical reduction of \mathcal{G}_k . We show that the deformation which deforms \mathcal{G}'_k into the bundle induced from the associated L -bundle, is transversal to the Harder-Narasimhan-stratum of \mathcal{G}_k . Only here we need our extra condition on G , because we have to avoid higher order deformations of the canonical parabolic reduction.

Let us compare this to Langton’s original argument. He performs a modification of the vector bundle, such that the new bundle has a quotient which is isomorphic to the maximal destabilizing subsheaf of the old bundle. This corresponds to the reduction to P^- in our situation. For vector bundles this new bundle is less unstable, unless the above quotient is a direct summand, which can happen only if the destabilizing subsheaf of the original bundle can be lifted to $C \times \text{Spec}(R/\mathfrak{m}^2)$. In our construction we therefore use a modification of higher order in order to obtain a bundle for which the quotient defines a nontrivial extension (condition (4) in Proposition 12). This property allows us to prove that the new bundle is always less unstable.

A second difference to Langton’s algorithm is that we use the affine Grassmannian and cocycles instead of the Bruhat-Tits building. This has the advantage that we can stay within the category of G -bundles and do not have to consider more complicated families of group schemes over C .

Acknowledgments. I would like to thank Alexander Schmitt for pointing out the problem of semistable reduction for principal bundles to me and for many discussions and comments. I would like to thank G. Harder, U. Stuhler and V. Balaji for their comments and A. Langer and Ngo Dac Tuan for pointing out a mistake in a previous version of this article. Finally the article would not be in its present form without the invaluable suggestions of the referee.

Notation. Throughout the article we will fix C/k a smooth projective curve over an algebraically closed field k , G a semisimple group scheme over k . R denotes a complete discrete valuation ring over k with residue field k and quotient field K . Finally, let π be a local parameter for the maximal ideal $\mathfrak{m} \subset R$. We will often need to replace R by a finite extension, which we will then denote by R again.

For a k -algebra A we will denote the base-change of any object over k to A by a lower index A , e.g. $C_A := C \times_k \text{Spec } A$.

Given an affine algebraic group H which acts on a scheme S by $\rho : H \rightarrow \text{Aut}(S)$ and an H -bundle \mathcal{H} on C , we will denote by $\mathcal{H} \times^{H,\rho} S := \mathcal{H} \times S/H$ the associated bundle with fiber S .

For an algebraic groups G, P, Q, U we will denote the corresponding Lie algebras by $\mathfrak{g}, \mathfrak{p}, \mathfrak{q}, \mathfrak{u}$.

Given an affine algebraic group G we will denote by Bun_G the moduli stack of principal G -bundles on C .

The group of ℓ -th roots of unity is denoted by μ_ℓ .

Remark. Since the algorithm will terminate after a finite number of steps, the assumption that the residue field of R is k and therefore algebraically closed is not essential, we could as well start with an arbitrary complete discrete valuation ring, and take the finite extension needed in each step to find a canonical parabolic subgroup in the special fiber.

2. STABILITY OF G -BUNDLES

Let us recall the concept of stability for G -bundles.

Definition 1 (Ramanathan). *A principal G -bundle \mathcal{G} on a curve C is called semi-stable if for all parabolic subgroups $P \subset G$, all reductions of \mathcal{G} to \mathcal{P} and all dominant characters $\alpha : P \rightarrow \mathbb{G}_m$ we have:*

$$\deg(\mathcal{P} \times^{P,\alpha} \mathbb{A}^1) \leq 0.$$

Here $\mathcal{P} \times^{P,\alpha} \mathbb{A}^1$ is the line bundle obtained from the action of P on \mathbb{A}^1 defined by α .

Behrend showed ([4] Proposition 8.2/Theorem 7.3) that every G -bundle \mathcal{G} over C has a canonical reduction \mathcal{P} to a parabolic subgroup $P \subset G$, which is characterized by:

- (1) Let $R_u(P)$ be the unipotent radical of P . Then $\mathcal{P}/R_u(P)$ is a semi-stable $P/R_u(P)$ bundle.
- (2) For all parabolic subgroups $Q \supset P$ we have $\deg(\mathcal{P} \times^P \mathfrak{q}/\mathfrak{p}) < 0$.

The type $t(\mathcal{P}) = (t(P), \deg)$ of the canonical reduction of \mathcal{G} is given by the type of the parabolic subgroup P together with the degree of \mathcal{P} , which is defined as

$$\begin{aligned} \deg : X^*(P) &\rightarrow \mathbb{Z} \\ \alpha &\mapsto \deg(\mathcal{P} \times^{P,\alpha} \mathbb{A}^1). \end{aligned}$$

The *degree of instability* of the canonical reduction of \mathcal{G} is defined as

$$\text{ideg } \mathcal{G} := \deg(\mathcal{P} \times^P \mathfrak{p}).$$

Considering the one-dimensional representation $\det(\mathfrak{p})$ of P as an element of $X^*(P)$ we may rewrite this as $\text{ideg } \mathcal{G} = \underline{\deg}(\det(\mathfrak{p}))$.

Behrend proved ([4] Proposition 7.2) that the canonical reduction can also be characterized as the reduction to a parabolic subgroup which is maximal among all subgroups such that $\deg(\mathcal{P} \times^P \mathfrak{p})$ is maximal. In particular maximality of the parabolic follows if condition (2) above is satisfied, since if we had $P \subset Q$ such that $\deg(\mathcal{P} \times^P \mathfrak{p}) = \deg(\mathcal{P} \times^P \mathfrak{q})$ then it follows that $\deg(\mathcal{P} \times^P \mathfrak{q}/\mathfrak{p}) = 0$, which would contradict (2).

To define the analog of the Harder-Narasimhan-stratification for Bun_G Behrend observed the following ([5] Proposition 7.1.3):

Proposition 2 (Behrend). *Let \mathcal{G} be a G -bundle on $C \times S$ where $S = \text{Spec}(R)$ is the spectrum of a discrete valuation ring with generic point η and special point 0 . Denote by \mathcal{G}_η and \mathcal{G}_0 the restrictions of \mathcal{G} to $C \times \eta$ and $C \times 0$. Then:*

$$\text{ideg}(\mathcal{G}_0) \geq \text{ideg } \mathcal{G}_\eta$$

and equality implies that the type of instability is the same for \mathcal{G}_0 and \mathcal{G}_η .

Finally if the canonical reduction of \mathcal{G}_η is defined over η and $\text{ideg}(\mathcal{G}_0) = \text{ideg}(\mathcal{G}_\eta)$ then there is a reduction of \mathcal{G} inducing the canonical reduction in every fiber over S .

For the convenience of the reader we briefly recall the argument in our situation.

Proof. Since the canonical reduction \mathcal{P}_η of \mathcal{G}_η is defined after a finite extension of R we may without loss of generality assume that the canonical reduction is defined over η . Let $P \subset G$ be the parabolic subgroup defining this canonical reduction, i.e. the reduction of \mathcal{G}_η is a section of \mathcal{G}_η/P .

Since G/P is projective we can extend the reduction to points of codimension 1, thus there is an open subset $U \subset C \times S$, with $C_\eta \subset U$ and $U_0 \neq \emptyset$ such that the reduction \mathcal{P}_η extends to a reduction \mathcal{P}_U of $\mathcal{G}|U$. In particular we obtain a generic reduction of \mathcal{G}_0 to P which by the same argument can be extended to a reduction \mathcal{P}_0 of the whole special fiber \mathcal{G}_0 .

Projectivity of the Quot-scheme implies furthermore that the subbundle $\mathcal{P}_\eta \times^P \mathfrak{p} \subset \mathcal{G}_\eta \times^G \mathfrak{g}$ extends to a subsheaf $\mathcal{F} \subset \mathcal{G} \times^G \mathfrak{g}$ over $C \times S$. Over U this coincides with $\mathcal{P}_U \times^P \mathfrak{p}$, thus the subbundle $\mathcal{P}_0 \times^P \mathfrak{p} \supset \mathcal{F}|_{C \times 0}$ is the saturation of $\mathcal{F}|_{C \times 0}$. Therefore we find the inequality:

$$\text{ideg}(\mathcal{G}_0) \geq \text{ideg}(\mathcal{P}_0 \times^P \mathfrak{p}) \geq \text{ideg}(\mathcal{F}|_{C \times 0}) = \text{ideg}(\mathcal{F}_\eta) = \text{ideg}(\mathcal{G}_\eta).$$

If $\text{ideg}(\mathcal{G}_0) = \text{ideg}(\mathcal{G}_\eta)$ then this implies $\mathcal{P}_0 \times^P \mathfrak{p} = \mathcal{F}|_{C \times 0}$, so that $\mathcal{F} \subset \mathcal{G} \times^G \mathfrak{g}$ is a subbundle, i.e. it defines a section of the associated bundle with fibres the Grassmannian $\text{Grass}(\mathfrak{g})$ parameterizing subspaces of \mathfrak{g} . Over U this section factors through the closed subspace $G/P \subset \text{Grass}(\mathfrak{g})$, thus this holds globally. Therefore the section defines a reduction \mathcal{P} of the family \mathcal{G} , extending \mathcal{P}_U . Since the degree of a reduction is locally constant in a family the characterization of the canonical reduction in terms of the degree implies that $\mathcal{P} \subset \mathcal{G}$ is canonical in every fiber. \square

This observation can be used to define a stratification of the moduli stack of principal G -bundles on C , the Harder-Narasimhan-stratification. First it implies that the instability degree ideg defines an upper-semicontinuous function on Bun_G . Therefore principal G -bundles such that $\text{ideg}(\mathcal{G}) \leq d$ form an open substack $\text{Bun}_G^{\leq d}$.

Denote by $\text{Bun}_G^{\overline{d}} \subset \text{Bun}_G^{\leq d}$ the complement $\text{Bun}_G^{\leq d} \setminus \text{Bun}_G^{\leq d-1}$ endowed with the reduced stack structure. The proposition above shows that the type of the canonical reduction can only change under specialization if the degree of instability changes. Therefore $\text{Bun}_G^{\overline{d}} = \coprod_{\underline{d}(\det(\mathfrak{p}))=d} \text{Bun}_G^{t(P), \underline{d}}$ is a disjoint union where the points of $\text{Bun}_G^{t(P), \underline{d}}$ are those G -bundles such that the canonical reduction is of type $(t(P), \underline{d})$.

It has become tradition to call this decomposition into constructible subsets a stratification even if it is not true that the closure of a stratum is a union of strata – this even fails in the case of vector bundles of rank ≥ 3 . However we just saw that a specialization of a bundle of given type of instability cannot intersect a different stratum of the same degree of instability, so that the closure of a stratum can only intersect strata for which the degree of instability is larger.

Finally we recall Behrend's conjecture ([4], Conjecture 7.6). Again let \mathcal{G} be a G -bundle on C and \mathcal{P} the canonical reduction, $P \subset G$ the corresponding parabolic subgroup. This defines an exact sequence of vector bundles:

$$(1) \quad 0 \rightarrow \mathcal{P} \times^P \mathfrak{p} \rightarrow \mathcal{P} \times^P \mathfrak{g} \rightarrow \mathcal{P} \times^P (\mathfrak{g}/\mathfrak{p}) \rightarrow 0$$

and Behrend conjectured that $H^0(C, \mathcal{P} \times^P (\mathfrak{g}/\mathfrak{p})) = 0$.

For us one implication of this conjecture is important. Denote by $\text{Bun}_P^{\underline{d}}$ the moduli stack of P -bundles of degree \underline{d} . Recall that a P -bundle \mathcal{P} is called semistable if the corresponding $P/R_u(P)$ -bundle $\mathcal{P}/R_u(P)$ is semistable. Denoted by $\text{Bun}_P^{\underline{d}, ss} \subset \text{Bun}_P^{\underline{d}}$ the open substack of semistable P -bundles.

Fix a type of instability $t(\mathcal{P}) = (t(P), \underline{d})$, i.e. assume that \underline{d} satisfies condition (2). Then the representable map $\text{Bun}_P^{\underline{d}, ss} \rightarrow \text{Bun}_G$ factors through $\text{Bun}_G^{t(P), \underline{d}}$. Uniqueness of the canonical reduction implies that this map is radiciel, i.e. every

geometric fiber of this morphism consists of a single point, which might however be non-reduced.

Lemma 3. *Assume that Behrend's conjecture holds for G . Then for every type $(t(P), \underline{d})$ of a canonical reduction the map $p : \text{Bun}_P^{\underline{d}, \text{ss}} \rightarrow \text{Bun}_G$ is an embedding.*

Proof. From the exact sequence (1) we get that Behrend's conjecture implies that the map $H^1(C, \mathcal{P} \times^P \mathfrak{p}) \rightarrow H^1(C, \mathcal{P} \times^P \mathfrak{g})$ is injective and that $H^0(C, \mathcal{P} \times^P \mathfrak{p}) \cong H^0(C, \mathcal{P} \times^P \mathfrak{g})$. This is equivalent to the statement that we get an embedding of the tangent stacks $T_{\text{Bun}_P, \mathcal{P}} \hookrightarrow T_{\text{Bun}_G, \mathcal{G}}$. So the map p is unramified. But unramified, radicial maps are embeddings (EGA IV, 17.2.6). \square

In particular given a reduced scheme S and a map $f : S \rightarrow \text{Bun}_G$, which factors through $\text{Bun}_G^{t(P), \underline{d}}$, the Lemma shows that f can be lifted to Bun_P , i.e. there is a canonical reduction for the family of G -bundles defined by f .

Lemma 4. *It is sufficient to prove our theorem for adjoint groups.*

Proof. From the definition of semistability we see that a G -bundle is semistable if and only if the induced $G/Z(G)$ -bundle (where $Z(G)$ is the center of G) is semistable. Moreover the map $\text{Bun}_G \rightarrow \text{Bun}_{G/Z(G)}$ is finite, because the obstruction to lift a $G/Z(G)$ -bundle to a G -bundle is given by a class in $H^2(C, Z(G))$, so the image of Bun_G in $\text{Bun}_{G/Z(G)}$ is closed and the fibres are a torsor for $\text{Bun}_{Z(G)}$, which is finite. \square

Remark 5. Since a semisimple adjoint group G/k is a product of simple groups ([9] Exposé XXIV, Prop. 5.9), it would be sufficient to prove our theorem for simple adjoint groups. We will use this only to simplify the last step of our proof for groups containing an exceptional factor.

3. FIRST STEP: FIND AN ARBITRARY EXTENSION OF \mathcal{G}_K

Recall the following theorems: Let S be any scheme over k .

Theorem 6 (Drinfeld–Simpson, Harder). *Let G/k be semisimple and $U \subset C$ an affine open subset. Then the restriction of any G -bundle on $C \times S$ to $U \times S$ becomes trivial after a suitable faithfully flat base change $S' \rightarrow S$, which is locally of finite presentation. If G is simply connected, then $S' \rightarrow S$ can be chosen to be étale.*

See [10], or [14] for $S = \text{Spec } k$.

Theorem 7 (Affine Graßmannian). *There is an ind-projective scheme $G((t))/G[[t]]$, which parameterizes G -bundles on C together with a trivialization on $C - p$, where p is a closed point on C .*

I am not sure who was the first to find this result. A reference in characteristic 0 is [17] Proposition 3.10 (see also [6] 2.3.4 and 4.5.1). A proof general enough to include the above formulation can be found in [12] (Corollary 3 and the remark following the proof of this corollary show the existence of the affine Graßmannian as ind-scheme over \mathbb{Z} , which comes equipped with a closed immersion into an affine Graßmannian for $G = \text{SL}_n$. For $G = \text{SL}_n$ the affine Graßmannian is constructed as a limit of projective varieties, so ind-projectivity also holds for arbitrary semisimple groups. Corollary 16 gives the functorial description in terms of G -bundles).

These theorems show that we can extend G -bundles: Let \mathcal{G}_K be a G -bundle on C_K . Choose a point $p \in C(k)$, then by the first theorem we can replace K by a finite extension (and R by its integral closure in this extension), such that $\mathcal{G}_K|_{(C-p) \times \text{Spec}(K)}$ becomes trivial. Any trivialization defines a map $\text{Spec}(K) \rightarrow G((t))/G[[t]]$. Since the affine Graßmannian is ind-projective this defines a map $\text{Spec}(R) \rightarrow G((t))/G[[t]]$ and thus a G -bundle \mathcal{G}_R on C_R , which extends \mathcal{G}_K .

Remark 8. For any linear algebraic group H over k there is an ind-scheme $H((t))$, called loop group, which is defined by $H((t))(R) = H(R((t)))$ for all k -algebras R (e.g. [12]). In the following we will often want to lift points $G((t))/G[[t]](R)$ to $G((t))(R)$. This is possible – after possibly extending the coefficients – by choosing a trivialization of the G -bundle on the formal completion at p , the glueing datum for the G -bundle is then a point of $G((t))$.

Recall that conversely, any element of $g \in G((t))(R)$ defines a glueing datum for a G -bundle on C . For noetherian R this comes from flat descent, in general this was shown by Beauville-Laszlo [3] (see [17] Proposition 3.8). From this description one also sees that if we take $g_{C-p} \in G(R[C-p])$ and $g_p \in G(R[[t]])$ then the bundles given by g and $g_{C-p}g_p$ are isomorphic.

In the following we will need a slight generalization of this construction, using a finite number of points $S \in C(k)$ instead of the single point p above. Given S choose local parameters at all points $p \in S$, then any element in $\prod_{p \in S} G((t))(R)$ defines a glueing-datum for a G -bundle on C . For example we can use the preceding construction to add the points in S successively.

4. SECOND STEP: FIND A MODIFICATION OF \mathcal{G}_R

Assume that the special fiber \mathcal{G}_k of \mathcal{G}_R is not semistable. Since k is algebraically closed, the bundle \mathcal{G}_k has a canonical reduction \mathcal{P}_k to a maximal unstable parabolic subgroup $P \subset G$ ([4] Thm 7.3, Prop 8.2).

We want to find a cocycle for \mathcal{G}_R , such that its reduction modulo π is a cocycle with values in $P \subset G$ which defines \mathcal{P}_k . To do so, first we have to note:

Lemma 9. *The bundle \mathcal{P}_k is Zariski locally trivial on C_k .*

Proof. This follows from a general theorem in SGA 3: We know that the bundle \mathcal{G}_k is Zariski locally trivial on C_k , because G is semisimple. Therefore we may assume that \mathcal{P}_k is a reduction to P of the trivial bundle over an open subset $U \subset C$, i.e. a section of $G/P \times U$. Now by [9] (Exposé XXVI Cor. 5.2) we know that over any (semi-)local base S we have $G/P(S) = G(S)/P(S)$. Therefore there is an open covering $\cup_{i=1}^n V_i = U$ and $g_i \in G(V_i)$ such that $(\mathcal{P}_k)|_{V_i} = g_i \cdot (P \times V_i) \cong P \times V_i$. \square

Since \mathcal{P}_k is Zariski locally trivial, there is a finite set of points $S \subset C(k)$ such that $\mathcal{P}|_{C-S}$ is trivial and the restrictions of \mathcal{P}_k to the complete local rings at points in S are trivial as well. To simplify notations we choose local parameters at all points in S , i.e. isomorphisms $\widehat{\mathcal{O}}_{C,p} \cong k[[t]]$. Then by Remark 8 every element $\prod_{p \in S} G((t))(R)$ defines a G -bundle on C_R . With this notation we can conclude:

Corollary 10. *There exists a finite set $S \subset C(k)$ and an element $\bar{p} \in \prod_{p \in S} P((t))(k)$ such that \mathcal{P}_k is isomorphic to the bundle defined by \bar{p} .*

Furthermore there exists $g \in \prod_{p \in S} G((t))(R)$ such that g defines the bundle \mathcal{G}_R and $\bar{p} \equiv g \pmod{(\pi)}$.

Proof. The first part has been shown above. For the second part we first choose trivializations of \mathcal{P}_k at the completed local rings of the points in S and a trivialization on $C-S$, which is also affine. Now a local trivialization of \mathcal{P}_k is the same as a section of $\mathcal{P}_k \subset \mathcal{G}_k$. Thus it is sufficient to lift these local sections to \mathcal{G}_R . Since G is smooth \mathcal{G}_R is also smooth over C_R . Therefore the lifting criterion for smoothness tells us that we can inductively lift any local section of \mathcal{G}_k to $\mathcal{G}_R/(\pi^n)$ for all n . Using our simplifying assumption that R is complete we end up with a local trivialization of \mathcal{G}_R . \square

To find a modification of \mathcal{G}_R with less unstable special fiber we choose a maximal parabolic $Q \supset P$. The canonical reduction \mathcal{P}_k defines a reduction \mathcal{Q}_k of \mathcal{G}_k to Q .

As indicated in the introduction we want to choose our cocycle such that it defines a lifting of this reduction to $\mathcal{G}|_{C \times \text{Spec}(R/(\pi^n))}$ for n maximal. Such an n exists, because the generic fiber of \mathcal{G}_R is semistable, therefore we cannot obtain a lift to C_R .

Let us briefly explain the following construction in the case of $G = \text{SL}_m$, before we set up the notation for the general case. We view SL_m -bundles as vector bundles with a fixed trivialization of the determinant. Thus we replace \mathcal{G}_R by a vector bundle \mathcal{E}_R over C_R . Now the canonical reduction of \mathcal{E}_k defines a subbundle $\mathcal{E}'_k \subset \mathcal{E}_k$ (we replaced the canonical flag by a subbundle in the general case by the choice of the maximal parabolic $Q \supset P$). In Langton's argument the bundle \mathcal{E} would now be replaced by $\ker(\mathcal{E}_R \rightarrow \mathcal{E}_k/\mathcal{E}'_k)$, which has the same generic fiber as \mathcal{E}_R . In terms of our cocycle g this corresponds to conjugation of g by the diagonal matrix $(\pi, \dots, \pi, 1, \dots, 1)$ in which the first $r := \text{rk}(\mathcal{E}'_k)$ entries are equal to π . This still produces an integral cocycle because we chose g such that the lower diagonal block is 0 mod π . Moreover, the new cocycle has an upper block which is 0 mod π , the diagonal blocks are unchanged whence these still define the bundles \mathcal{E}'_k and $\mathcal{E}_k/\mathcal{E}'_k$. Thus the new special fiber has a quotient isomorphic to \mathcal{E}'_k , and therefore the new bundle is less unstable unless the quotient is a direct summand. The extension class in $\text{Ext}^1(\mathcal{E}'_k, \mathcal{E}_k/\mathcal{E}'_k)$ of our new special fiber is described by the Čech-cocycle given by the lower diagonal block of our glueing cocycle. The next proposition shows that if this class is 0 then we can modify our original choice of g in such a way that conjugation with $(\pi^2, \dots, \pi^2, 1, \dots, 1)$ still produces an integral cocycle. Inductively we will see that there is a maximal n such that there exists a choice of g in Corollary 10 for which the lower diagonal block of g is 0 mod π^n . Conjugating this element by the diagonal matrix $(\pi^n, \dots, \pi^n, 1, \dots, 1)$ we obtain a non-zero extension class in $\text{Ext}^1(\mathcal{E}'_k, \mathcal{E}_k/\mathcal{E}'_k)$.

For general G we encounter two problems, first the diagonal matrix above is not in $\text{SL}_m(K)$ but only in $\text{GL}_m(K)$. We can avoid this by adjoining a root $\pi^{\frac{1}{N}}$ to K . Then we can use the diagonal matrix $(\pi^{(m-r)/N}, \dots, \pi^{(m-r)/N}, \pi^{-r/N}, \dots, \pi^{-r/N})$ in the argument above. The second problem is that for general groups the structure of the "lower diagonal block" is more complicated.

We will need some standard notation for algebraic groups (see [9] Exp. XXII, 1.1 and Exp. XXVI):

We choose an épingle of G adapted to P , i.e.: We choose $B \subset P \subset G$, a Borel subgroup of G contained in P and $T \subset B$ a maximal torus, which is split because k was assumed to be algebraically closed. For all roots α , we denote by $U_\alpha \subset G$ the root-subgroup of G (it is isomorphic to \mathbb{G}_a and conjugation by T acts on U_α via the character α).

Let $\Delta \subset X^*(T)$ be the set of roots of T , Δ^+ the set of positive roots, $I \subset \Delta^+$ the set of positive, simple roots determined by B . We denote by $\Delta_Q \subset \Delta$ the roots of Q and $I_Q := I \cap -\Delta_Q$ be the simple roots α such that $-\alpha$ is also a root of Q and let $\beta \in I - I_Q$ be the unique positive simple root, not contained in I_Q .

Let $L \subset Q$ be a Levi subgroup. Then $Z(L)^\circ := (\bigcap_{\alpha \in I_Q} \ker(\alpha))^\circ$ is the connected component of the center $Z(L)$ of L . Since the positive simple roots form a basis of $X^*(T) \otimes_{\mathbb{Z}} \mathbf{R}$ we have surjections with finite kernels:

$$\begin{array}{ccc} T & \longleftarrow & Z(L)^\circ \cong \mathbb{G}_m \\ \downarrow \prod \alpha & & \downarrow \lambda := \beta|_{Z(L)^\circ} \\ \prod_{\alpha \in I} \mathbb{G}_m & \xleftarrow{id_\beta} & \mathbb{G}_m \end{array}$$

In the following we fix the above isomorphism $\mathbb{G}_m \cong Z(L)^\circ$ to identify $K^* = Z(L)^\circ(K)$ and define $m \in \mathbb{N}$ by $\mu_m = \ker(\lambda)$.

Recall that the unipotent radical U of Q has a filtration $U = U_1 \supset U_2 \supset \cdots \supset U_h$ by normal subgroups, where $U_i := \prod_{\alpha \in \Delta_{\beta,i}} U_\alpha$, where $\Delta_{\beta,i} = \{\alpha = \sum_{\alpha_k \in I_Q} n_k \alpha_k + m\beta \in \Delta^+ | m \geq i\}$ and the product is a product of schemes. Thus this filtration can also be obtained by decomposing U into eigenspaces for the action of $Z(L)^\circ$ which acts by conjugation on U . We write $U = \prod_{i=1}^h U_{[i]}$ for the corresponding decomposition. So $U_{[i]} \cong U_i/U_{i+1}$ and $\mathbb{G}_m = Z(L)^\circ$ acts on $U_{[i]}$ via the character $z \mapsto z^{m_i}$.

Similarly for the opposite parabolic Q^- of Q (defined by $-\Delta_P$), we have a filtration $U^- = U_1^- \supset U_2^- \supset \cdots \supset U_h^-$ and $U^- = \prod_{i=1}^h U_{[i]}^-$. Finally denote by $LU_i^- := L \times U_i^- \subset Q^-$ the subgroup generated by L and U_i^- .

Remark 11. Note that the length of the above filtration is bounded by the largest coefficient occurring when writing the roots as linear combination of simple roots. In particular for all groups of type B, C, D only coefficients ≤ 2 occur ([8]), so $U_3 = (1)$ in these cases. For groups of type A we have $U_2 = (1)$.

To define our modification of \mathcal{G}_R we will have to extend R to $R' := R[\pi^{1/N}]$, where N is chosen minimal such that for all $i \in \{1, 2, \dots, h\}$ there exist k_i such that $(\lambda(\pi^{k_i/N}))^i = \pi$ in $\mathbb{G}_m(K')$, where K' is the field of fractions of R' .

Let $\mathcal{L}_k := \mathcal{Q}_k \times^Q L = \mathcal{Q}_k/U$ be the L -bundle obtained from \mathcal{Q}_k . Then any LU_i^- -bundle \mathcal{Q}_i^- on C with $\mathcal{Q}_i^-/U_i^- \cong \mathcal{L}$ defines an element in $H^1(C, {}^{\mathcal{L}}U_i^-)$, where ${}^{\mathcal{L}}U_i^- := \mathcal{L} \times^L U_i^-$ is the group scheme over C defined by the conjugation action of L on U_i .

Proposition 12. *There is a cocycle $g \in \prod G((t))(R)$ for \mathcal{G}_R and $z = \pi^{\ell/N} \in \mathbb{G}_m(K') = Z(L)^\circ(K')$ such that the following holds:*

- (1) $g \bmod \pi \in \prod_{p \in S} P((t))(k)$ and this cocycle defines the canonical reduction of \mathcal{G}_k to P .
- (2) $g' := zgz^{-1} \in \prod_{p \in S} G((t))(R[\pi^{1/N}])$.
- (3) Denote $\bar{g}' := g' \bmod \pi^{1/N}$. Then $\bar{g}' \in \prod_{p \in S} Q^-((t))(k)$.
- (4) Let i be maximal, such that $\bar{g}' \in \prod_{p \in S} (LU_i^-((t))(k))$. Then the class $[\bar{g}'] \in H^1(C, {}^{\mathcal{L}}U_i^- / {}^{\mathcal{L}}U_{i+1}^-)$ is non-zero.

Proof. We will inductively modify the cocycle g we found in Corollary 10. First note that each component of g is an $R((t))$ -valued point of G contained in the open subset $U^- \times L \times U \subset G$, since $g \bmod \pi \in \prod_{p \in S} P((t))(k)$. Therefore we can decompose $g = v \cdot l \cdot u$ with $v \in \prod_{p \in S} U^-(R((t)))$, $l \in \prod_{p \in S} L(R((t)))$, $u \in \prod_{p \in S} U(R((t)))$ with $v \equiv 1 \pmod{\pi}$ by property (1).

We want to choose ℓ be maximal such that condition (2) holds for $z := \pi^{\ell/N} \in Z(L)^\circ(K)$, i.e. such that $g' := zgz^{-1}$ is an $R[\pi^{1/N}]$ -valued point of $\prod_{p \in S} G((t))$. Since $\mathbb{G}_m = Z(L)^\circ$ acts via positive characters on U we always know that $zuz^{-1} \in \prod_S U(R[\pi^{1/N}]((t)))$ and if $\ell > 0$ then this is even congruent $1 \pmod{\pi^{1/N}}$. On the other hand $v \bmod \pi = 1$ thus our choice of N implies that there exists $\ell > 0$ such that zuz^{-1} is $R[\pi^{1/N}]$ -valued. Finally, if n is the maximal integer such that we can lift of the canonical reduction of \mathcal{G}_k to Q to R/π^n , then $\ell \leq nN$, because otherwise (2) implies that $\pi^n v \pi^{-n} \in \prod_{p \in S} U^-(R((t)))$ is still congruent $1 \pmod{\pi}$, which implies that $v \equiv 1 \pmod{\pi^{n+1}}$ and therefore $g \bmod \pi^{n+1} \in \prod_{p \in S} Q((t))(R/(\pi^{n+1}))$, contradicting the maximality of n . Thus there is a maximal $\ell > 0$ such that $g' = zgz^{-1}$ is $R[\pi^{1/N}]$ -valued. Since $\ell > 0$ this cocycle automatically satisfies condition (3). Furthermore, for this ℓ we have $zuz^{-1} \not\equiv 1 \pmod{\pi^{1/N}}$ since otherwise our choice of N implies that we could increase ℓ .

Thus it is sufficient to show, that if the pair (g, z) does not satisfy condition (4) of the proposition, then we can find another cocycle for \mathcal{G}_R for which we can

increase ℓ . Write $\bar{v}' := zvz^{-1} \pmod{\pi^{1/N}}$ and $\bar{l} := l \pmod{\pi^{1/N}}$. Then $\bar{g}' := \bar{v}'\bar{l} = g' \pmod{\pi^{1/N}}$.

Let i be maximal such that $\bar{v}' \in \prod_{p \in S} U_i^-(k((t)))$. Since $\bar{v}' \neq 1$ we see that $U_i^- \neq \{1\}$ and that $im\ell/N$ is an integer.

Recall that U_i^- has a filtration such that the subquotients are isomorphic to \mathbb{G}_a , so the group $H^1(C, \mathcal{L}U_i^-/\mathcal{L}U_{i+1}^-)$ can be calculated from the covering of C given by $C - S$ and the completed local rings at S . Thus, if the class $[\bar{v}'\bar{l}] \in H^1(C, \mathcal{L}U_i^-/\mathcal{L}U_{i+1}^-)$ is zero, then the cocycle $\bar{v}'\bar{l}$ is a boundary, i.e. there exist $\bar{v}_{C-S} \in U_{[i]}^-(k[C-S])$, $\bar{v}_S \in \prod_{p \in S} U_{[i]}^-(k[[t]])$ such that $\bar{v}_{C-S}\bar{v}'\bar{l}\bar{v}_S = \bar{v}_{i+1}\bar{l}$ with $\bar{v}_{i+1} \in \prod_{p \in S} U_{i+1}^-(k((t)))$.

Take any lift of $v_{C-S} \in U_{[i]}^-(R[C-S])$ and $v_S \in \prod_{p \in S} U_{[i]}^-(R[[t]])$. Then $v_{C-S}g'v_S \pmod{\pi^{1/N}} = \bar{v}_{C-S}\bar{g}'\bar{v}_S \in \prod LU_{i+1}^-(t)(k)$. Now

$$v_{C-S}g'v_S = v_{C-S}zgz^{-1}v_S = z \left((z^{-1}v_{C-S}z)g(z^{-1}v_Sz) \right) z^{-1}.$$

Since z^{-1} acts on $U_{[i]}^-$ as multiplication with elements of valuation $im\ell/N$, we have $z^{-1}v_{C-S}z \in U_{[i]}^-(R((t)))$ and it is even congruent $1 \pmod{\pi^{im\ell/N}}$. The same holds for $z^{-1}v_Sz$.

Therefore $g_1 := (z^{-1}v_{C-S}z)g(z^{-1}v_Sz)$ is a new R -valued cocycle for \mathcal{G}_R . This also satisfies (1)-(3): $g_1 \pmod{\pi} = g \pmod{\pi}$, because $z^{-1}v_{C-S}z$ and $z^{-1}v_Sz$ are congruent $1 \pmod{\pi}$. Further, zg_1z^{-1} is still R' -valued, since this holds for zgz^{-1} , v_{C-S} and v_S . Finally

$$zg_1z^{-1} \pmod{\pi^{1/N}} = v_{C-S}zgz^{-1}v_S \pmod{\pi^{1/N}} \in \prod_{p \in S} LU_{i+1}^-(t)(k),$$

because this holds for $\bar{v}_{C-S}\bar{g}'\bar{v}_S$. Now either (4) holds for g_1 and $i+1$ or we may continue inductively. If at some point $U_{i+1}^- = \{1\}$, this implies that we can find a larger ℓ such that 2 and 3 hold. Since ℓ is bounded by nN this implies that this way we finally find a cocycle g satisfying (1)-(4). \square

Now we replace R by $R[\pi^{1/N}]$ and denote by \mathcal{G}'_R be the G -bundle defined by the cocycle g' constructed in the above lemma.

5. THIRD STEP: SHOW THAT THE MODIFIED BUNDLE \mathcal{G}'_R IS LESS UNSTABLE

To show that the modified bundle is less unstable we will need to compare the stability of a G -bundle induced from a Q (or Q^-) bundle to the one induced from the Levi-quotient. We will first deform a Q bundle to a bundle induced from the Levi quotient. Then we will compare the cohomology class defined by this deformation to the class defined in Proposition 12 (4).

Since $Z(L)^\circ \cong \mathbb{G}_m$ and $\beta \in \Delta_Q^+$, the conjugation action $Z(L)^\circ \times Q \rightarrow Q$ given by $(z, q) \mapsto zqz^{-1}$ extends to a map $\tilde{\lambda} : \mathbb{A}^1 \times Q \rightarrow Q$. From the decomposition $Q = L \times U$ we see furthermore that $\lambda|(0 \times Q)$ is the composition of the natural maps $Q \rightarrow Q/U = L \hookrightarrow Q$.

For any Q -bundle \mathcal{Q}_k this induces an action of Q on $\mathcal{Q}_k \times \mathbb{A}^1 \times Q$, defined as $(x, s, q).q' := (x.(q')^{-1}, s, \tilde{\lambda}(s, q').q)$. The quotient by this action is a Q -bundle on $C \times \mathbb{A}^1$:

$$\mathcal{Q}_\lambda := \mathcal{Q}_k \times^Q (\mathbb{A}^1 \times Q).$$

Similarly, for a Q^- -bundle \mathcal{Q}_k^- we use the action of \mathbb{G}_m on Q^- given by λ^{-1} to define:

$$\mathcal{Q}_\lambda^- := \mathcal{Q}_k^- \times^{Q^-} (\mathbb{A}^1 \times Q^-).$$

Lemma 13. *There are canonical isomorphisms of Q -bundles:*

- (1) $\mathcal{Q}_\lambda|_{C \times \mathbb{G}_m} \cong \mathcal{Q}_k \times \mathbb{G}_m$.
- (2) *Let s be a coordinate of \mathbb{A}^1 and $\ker(\lambda) \cong \mu_m$, then $\mathcal{Q}_\lambda|_{C \times \text{Spec } k[s]/s^m} \cong (\mathcal{Q}_k/U \times^L Q) \times \text{Spec}(k[s]/s^m)$.*

Proof. For the first part an isomorphism $\mathcal{Q}_k \times \mathbb{G}_m \rightarrow \mathcal{Q}_k \times^Q (\mathbb{G}_m \times Q)$ is given by $(x, s) \mapsto (x.s, s, 1)$ (we used the identification $Z(L)^\circ \cong \mathbb{G}_m$).

For the second part we only need to notice that, the conjugation action of $Z(L)^\circ$ on Q factors through λ . Thus $\tilde{\lambda}$ also factors through the map $s \mapsto s^m$. This implies that $\tilde{\lambda}|_{\text{Spec } k[s]/s^m} \times Q = i \circ p$ is again the composition of the projection $p: Q \rightarrow Q/U = L$ and the inclusion $i: L \rightarrow Q$. \square

Remark 14. If \mathcal{P}_k is given as the canonical reduction of a G -bundle \mathcal{G}_k to the instability parabolic P , and $\mathcal{Q}_k := \mathcal{P} \times^P Q$, then the canonical reduction of $\mathcal{G}_L := \mathcal{Q}_k/U \times^L G$ is given by $\mathcal{P}_k/(U \cap P) \times^{L \cap P} P$. In particular, \mathcal{G}_L and \mathcal{G}_k lie in the same HN-stratum of Bun_G .

This holds because, by definition of the canonical reduction $\mathcal{P}_k/R_u(P)$ is a semistable bundle, and the degree of the line bundles obtained from characters of P are constant in the family defined above.

Corollary 15. *The closure of the HN-stratum of \mathcal{G}'_k intersects the HN-stratum of \mathcal{G}_k .*

Proof. By the last remark, it is sufficient to show that $\mathcal{G}_L = \mathcal{Q}_k/U \times^L G$ is contained in the closure of \mathcal{G}'_k in Bun_G . By the definition of our cocycle, \mathcal{G}'_k has a reduction \mathcal{Q}_k^- to Q^- such that $\mathcal{Q}_k^-/U^- \cong \mathcal{L}$. Applying the previous lemma to the Q^- -bundle \mathcal{Q}_k^- defined above, we get a family of G -bundles on \mathbb{A}^1 , such that the restriction to \mathbb{G}_m is isomorphic to $\mathcal{G}'_k \times \mathbb{G}_m$ and the fiber over 0 is isomorphic to $\mathcal{Q}'_k/U^- \times^L G = \mathcal{G}_L$. \square

By Proposition 12 (4) we know that \mathcal{Q}_k^- has a natural reduction to a LU_i^- -bundle $\mathcal{Q}_{i,k}^-$. Let $\mu_j := \ker(Z(L)^\circ \rightarrow \text{Aut}(LU_i^-))$ be the kernel of the conjugation action and let $\bar{\lambda}: \mathbb{G}_m \cong Z(L)^\circ/\mu_j \rightarrow \text{Aut}(LU_i^-)$ be the induced map, which again extends to a map $\tilde{\lambda}: \mathbb{A}^1 \rightarrow \text{End}(LU_i^-)$.

As before we define $\mathcal{Q}_{i,\bar{\lambda}}^- := (\mathcal{Q}_{i,k}^- \times \mathbb{A}^1) \times^{LU_i^-} Q$ the deformation of $\mathcal{Q}_{i,k}^-$ into the bundle induced from its Levi quotient. Finally, denote by μ_l be the kernel of the action on U_i^-/U_{i+1}^- induced by $\bar{\lambda}$, i.e. $l = mi/j$.

Remark 16. If $i = 1$ then $l = 1$ since $Z(L)^\circ$ acts on $U_{[1]}$ via the character λ . Also by definition of $\bar{\lambda}$ we have $l = 1$ if $U_{i+1} = \{1\}$. Thus $l \neq 1$ can only happen if and $U_3 \neq \{1\}$, which by Remark 11 implies that G has a factor which is an exceptional group.

Next we will link the deformation class of $\mathcal{Q}_{i,\bar{\lambda}}^-$ to the class defined in condition (4) of Proposition 12. As in Lemma 13 we claim that the family $\mathcal{Q}_{i,\bar{\lambda}}^-|_{C \times \text{Spec}(k[s]/s^l)}$ is constant. This holds since the action of \mathbb{G}_m on LU_i^- given by $\bar{\lambda}$ respects the decomposition $L \times \prod_{k \geq i} U_{[k]}^-$ and on each of the factors $U_{[k]}^-$ the action of \mathbb{G}_m factors through $s \mapsto s^{mk/j}$.

Recall how deformation theory associates to the family $\mathcal{Q}_{i,\bar{\lambda}}^-|_{\text{Spec}(k[s]/s^{l+1})}$ an element in $H^1(C, \mathcal{L} \mathbf{u}_i^-)$. Since the family is constant mod s^l the map $\text{Spec}(k[s]/s^{l+1}) \rightarrow \text{Bun}_{LU_i^-}$ is constant mod s^l , thus it factors through the map $\text{Spec}(k[s]/s^{l+1}) \rightarrow \text{Spec}(k[\epsilon]/\epsilon^2)$, given by $\epsilon \mapsto s^l$. The map $\text{Spec}(k[\epsilon]/\epsilon^2) \rightarrow \text{Bun}_{LU_i^-}$ defines a tangent vector to $\mathcal{Q}_{i,\bar{\lambda}}^-|_{C \times 0}$ which defines the element we are looking for.

We can describe this explicitly. In terms of cocycles the bundle $\mathcal{Q}_{i,k}$ is defined by the element \bar{g}' from Proposition 12, which we write as before as $\bar{g}' = \bar{l}\bar{v}$ with $\bar{l} \in L$ and \bar{v} in U_i^- .

The deformation $\mathcal{Q}_{i,\lambda}^-$ is defined via the action of \mathbb{G}_m on LU_i^- . We restrict to LU_i^-/U_{i+1}^- , which is isomorphic to $L \times U_{[i]}^-$ and the épinglage defines an isomorphism $U_{[i]}^- \cong \mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-$. Using these identifications the cocycle for $\mathcal{Q}_{i,\lambda}^-/U_{i+1}^-$ is given by $l.(s^l)\bar{v} \in \prod_{p \in S} L \times \mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-(k((t))[s]/s^{l+1})$. Thus setting $s^l = \epsilon$ we obtain an element in the tangent space $(\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-)(k((t)))$ of $LU_i^-(k((t)))$ at \bar{l} and considered as Čech-cocycle this is the same as the cocycle defining the element in $H^1(C, \mathcal{L} U_{[i]}^-) = H^1(C, \mathcal{L} (\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-))$ in Proposition 12. Thus we obtain:

Lemma 17. *The family $\mathcal{Q}_{i,\lambda}^-|_{\text{Spec}(k[t]/t^l)}$ is constant. Thus $\mathcal{Q}_{i,\lambda}^-|_{\text{Spec}(k[t]/t^{l+1})}$ defines an element in $H^1(C, \mathcal{L} (\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-))$. This element is the same as the one given by $\mathcal{Q}_i^-/U_{i+1}^-$ under the canonical isomorphism $\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^- \rightarrow U_i^-/U_{i+1}^-$ defined by the épinglage of G .*

Now we apply Lemma 4 and assume that G is adjoint and either contains no factor that is an exceptional group, or that G is an exceptional simple group.

Corollary 18. \mathcal{G}'_k is not contained in the same HN-stratum as \mathcal{G}_k .

Proof. $\mathcal{G}'_\lambda := (\mathcal{Q}_{i,\lambda}^- \times^{LU_i^-} G)$ is a family of G -bundles on $C \times \mathbb{A}^1$, which defines the trivial deformation of \mathcal{G}_L on $\text{Spec } k[s]/s^l$. Thus, as in Lemma 17 the restriction of this bundle to $C \times \text{Spec}(k[s]/s^{l+1})$ defines an element in $H^1(C, \mathcal{L} \times^L \mathfrak{g})$ which lies in the image of $H^1(C, \mathcal{L} (\mathfrak{u}_i^-))$. Since L respects the decomposition of $\mathfrak{g} = \mathfrak{q} \oplus \bigoplus_{i=1}^n \mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-$ into $Z(L)^\circ$ eigenspaces $H^1(C, \mathcal{L} (\mathfrak{u}_i^-/\mathfrak{u}_{i+1}^-))$ is a direct summand of $H^1(C, \mathcal{L} \times^L \mathfrak{g})$ so that again by Lemma 17 and 12(4), we know that this element is non-zero. We claim that this implies that \mathcal{G}'_k is not contained in the same HN-stratum as \mathcal{G}_k :

The group $H^1(C, \mathcal{L} \times^L \mathfrak{q})$ describes the deformations of \mathcal{G}_L which lift to $\text{Bun}_Q \rightarrow \text{Bun}_G$. Since $H^1(C, \mathcal{L} \times^L \mathfrak{g}) = H^1(C, \mathcal{L} \times^L \mathfrak{q}) \oplus H^1(C, \mathcal{L} \times^L \mathfrak{u}^-)$ we know that the above deformation of \mathcal{G}_L does not lift to Bun_Q in such a way that it defines the constant deformation of $\mathcal{L} \times^L Q \bmod s^l$.

If \mathcal{G}'_k was contained in the same HN-stratum as \mathcal{G}_k , then every point of the family \mathcal{G}'_λ would be contained in this stratum, since the fibres are isomorphic to either \mathcal{G}'_k or \mathcal{G}_L , the latter lies in the HN-stratum of \mathcal{G}_k by Remark 14.

Now, if we can show that a canonical reduction exists for this family, then the whole family can be lifted to Bun_Q and we obtain a contradiction if $l = 1$. Furthermore, $l \neq 1$ can only happen if G is an exceptional group (Remark 16), in which case we may assume Behrend's conjecture for G . But then Lemma 3 implies that the canonical reduction exists for the family \mathcal{G}'_λ and this reduction is again constant mod s^l . This contradicts our assumption.

Thus we may assume that $l = 1$ and have to construct a canonical parabolic for the family \mathcal{G}'_λ . Since it is not known whether the map $\text{Bun}_P^{ss} \rightarrow \text{Bun}_G$ is an embedding in positive characteristic in general, we do this by hand: We know that \mathcal{G}'_λ is a principal G -bundle over $\mathbb{A}^1 \times C$ such that the fiber over 0 is \mathcal{G}_L and all other fibers are isomorphic to \mathcal{G}'_k . We claim that under our assumptions we always have a section over \mathbb{G}_m :

First note that the family \mathcal{G}'_λ is constant when we pull it back to $Z(L)^\circ \rightarrow Z(L)^\circ/\mu_j \cong \mathbb{G}_m$ (by Lemma 13(1)). Therefore there exists a canonical reduction of the family over $Z(L)^\circ$. Now, since G is adjoint and not exceptional λ is an isomorphism and $j \in \{1, 2\}$. If $l = j = 1$ then we have found a reduction over \mathbb{G}_m .

If $\text{char}(k) \neq 2$ and $j \leq 2$ then the canonical reduction on $Z(L)^\circ$ descends to \mathbb{G}_m , because the map to \mathbb{G}_m is étale.

Since we assumed that the special fiber of our family lies in the same HN-stratum as \mathcal{G}'_k , we can apply Proposition 2 to see that the canonical reduction extends to the whole family, which is what we needed to show. \square

6. CONCLUSION

The above proves our Theorem 1. Namely we take any extension \mathcal{G}_R of \mathcal{G}_K to $C \times R$. If this is not semistable, then we can modify \mathcal{G}_R as above to obtain another extension \mathcal{G}'_R such that the special fiber \mathcal{G}'_k lies in a Harder-Narasimhan-stratum which contains \mathcal{G}_k in its closure, and therefore satisfies $\text{ideg}(\mathcal{G}'_k) < \text{ideg}(\mathcal{G}_k)$. Thus after finitely many modifications we obtain a semistable bundle.

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