Seshadri constants and geometry of surfaces

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Hiermit versichere ich, dass ich diese Doktorarbeit selbstständig und nur unter Zuhilfenahme der angegebenen Quellen erstellt habe.

Essen, im Juni 2005 Wioletta Syzdek

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Contents

1 Introduction

Seshadri constants were introduced by Demailly [De] as an attempt to tackle the Fujita Conjecture. They quickly gained remarkably interest on their own. It also quickly turned out that they are very hard to compute or even estimate. In the course of the years one has got a pretty good conjectural picture of the behavior of these invariants. It has been related to and embedded in the conjectural landscape of linear series on surfaces (e.g. Nagata Conjecture).

Systematic study of Seshadri constants on smooth surface began with the paper [EL] of Ein and Lazarsfeld and was continued by Bauer [Ba1] and Szemberg [Sz1]. Already in their first paper Ein and Lazarsfeld ask which concrete geometric properties of a surface are influenced by Seshadri constants. With the results of [HK] it became clear that there is a close relation between Seshadri constants and the fiber structure of the underlying variety.

The original aim of this thesis was to compute Seshadri constants of ruled surfaces. This was motivated by the hope that the easy fiber structure of ruled surfaces suffices for the calculations, at least of Seshadri constants at single points of such surfaces. This aim turned out to bee too ambitious, the precise geometry of curves living on ruled surfaces still remains beyond our understanding. The main results we obtained in this direction are presented in Theorem 3.27 . We note, that after this part of the thesis was finished (see also [Sy2]) a preprint [F] of Fuentes appeared. He studies also Seshadri constants on ruled surfaces, Theorems 4.14 and 4.16 in [F] verify our Theorem 3.27 and the estimates given in Lemma 4.12 [F] are in most cases worse than our estimates in Theorem 3.27.

Coming from the other end, motivated by [HK], [SzT-G] we study impact of (low) Seshadri constants on the geometry of the underlying surface. First we study multiple point Seshadri constant and give a sharp upper bound on Seshadri constants resulting in detecting a fiber structure of the surface (Theorem 4.2). Such a bound was given in the case of single point Seshadri constants by [SzT-G]. In that case we show that the only example satisfying their bound is a cubic surface in $\mathbb{P}^3$ and thus a better bound holds for all other surfaces (Theorem 4.8).

In the last chapter we study Riemann-Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of the Nagata-Biran Conjecture 2.6. This conjecture predicts that for a sufficiently large number of points multiple point Seshadri constants of an ample line bundle on algebraic surface are maximal. Biran gives an effective lover bound N_0. We construct examples verifying to the effect that the assertions of the Nagata-Biran Conjecture can not hold for small number of points (Theorem 5.33). We discuss cases where no such construction is possible (Theorem 5.20). We observe also that there is a strong connection between the Riemann-Roch expected curves on $\mathbb{P}^1 \times \mathbb{P}^1$ and the symplectic packing problem (Theorem 5.41). Biran relates the packing problem to the existence of solutions of certain Diophantine equations. We construct such solutions for any ample line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and a relatively small number of points. These solutions geometrically correspond to Riemann-Roch expected curves.

Finally we discuss in how far the Biran number N_0 is optimal in the case $\mathbb{P}^1 \times \mathbb{P}^1$. In fact we conjecture that it can be replaced by a lower number and we provide evidence justifying this conjecture.

2 Seshadri constants and the Nagata-Biran conjecture.

In this chapter we introduce the notion of the Seshadri constant, first at one point and next at $r \geq 2$ points (the multiple point Seshadri constant). We explain the concept of submaximal curves and we study them in the context of the Nagata-Biran Conjecture 2.6. More precisely, we are looking for counter-examples for the Nagata-Biran Conjecture at $r < N_0$ points. To do this, first we observe that in the cases with the submaximal Seshadri constant, the number of irreducible and reduced computing curves is bounded (Proposition 2.8). In turn, this allows to say something more about the multiplicity vector of submaximal curves on surfaces with the Picard number $\varrho \in \{1, 2, 3\}$ (Proposition 2.10).

2.1 Seshadri constants - basics definitions and properties.

The concept of Seshadri constant was introduced by Demailly in [De]. He associated a real number $\varepsilon(L; x)$ to an ample line bundle L and a point x of an algebraic variety X. This number in effect measures how much of positivity of L is concentrated at x. In general Seshadri constants are very hard to control and their exact value is known only in few examples.

Let us recall the definition and some properties of Seshadri constants.

Definition 2.1 *Let L be a nef line bundle on a smooth projective variety X over $\mathbb{C}$. Fix a point x on X. Let $\sigma : X_x \longrightarrow X$ be the blowing-up of X at the point x with the exceptional divisor $E = \sigma^{-1}(x)$. The Seshadri constant of L at x is a non-negative real number*

$$\varepsilon(L; x) = \sup\{\varepsilon \in \mathbb{R} \mid \sigma^* L - \varepsilon E \quad \textit{is nef}\}.$$

From Kleiman's nefness criterion it follows that $\varepsilon(L; x) \leq \sqrt[\dim X]{L^{\dim X}}$. If the value of $\varepsilon(L; x)$ is less than the previous upper bound, then we say that the Seshadri constant of L at x is *submaximal*.

Remark 2.2 *We can define the Seshadri constant as*

$$\varepsilon(L; x) = \inf_{D \ni x} \left\{ \frac{L.D}{\operatorname{mult}_x D} \right\}$$

where the infimum is taken over all reduced and irreducible curves D passing through the point x with the multiplicity $\operatorname{mult}_x D$ *(see [La] 5.1.5).*

If $\frac{L.D}{\operatorname{mult}_x D} = \varepsilon(L;x)$, then we say that the curve D *computes* the Seshadri constant at point x and such a curve we call *the Seshadri curve.* By *the Seshadri quotient of* G we mean $\frac{L.G}{\operatorname{mult}_x G}$, where G is a curve passing through a point x with multiplicity $\operatorname{mult}_x G$.

Assume moreover that L is an ample line bundle. For a fixed point $x \in X$, we denote by $\mathfrak{m}_x \subset \mathcal{O}_X$ its maximal ideal.

Definition 2.3 *We say that the complete linear system $|L|$ separates s-jets at x, if the natural map*

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_X/\mathfrak{m}_x^{s+1})$$

taking the sections of L to their s-jets is surjective.
By $s(L,x)$ we denote the maximal number such that $|L|$ separates s-jets at x.

Using above terminology we have the following

Proposition 2.4 *([La], 5.1.17) For an ample line bundle L on X*

$$\varepsilon(L;x) = \limsup_{k\to\infty} \frac{s(kL;x)}{k}.$$

2.2 Multiple points Seshadri constants and the Nagata-Biran conjecture.

These generalized invariants were first studied by Xu in [Xu1].

Definition 2.5 *The Seshadri constant of L at $x_1,\dots,x_r$ is the real number*

$$\varepsilon(L;x_1,\dots,x_r) = \inf_{D\cap\{x_1,\dots,x_r\}\neq\emptyset} \frac{L.D}{\sum_{i=1}^r \operatorname{mult}_{x_i} D},$$

where the infimum is taken over all reduced and irreducible curves D passing through at least one of the points $x_1,\dots,x_r$.

As a function on X^r the Seshadri constant $\varepsilon(L;\cdot,\dots,\cdot)$ is semi-continuous and has the maximal value at a very general point of X^r (i.e. on the complement of a union of at most countably many Zariski closed subsets). For more details see [Og]. We denote by $\varepsilon(L;r)$ this maximal value . It is conjectured that for r sufficiently large $\varepsilon(L;r)$ has the maximal possible value which is $\varepsilon_{\max}(L;r) = \sqrt{\frac{L^2}{r}}$. More precisely it is conjectured

Nagata-Biran conjecture 2.6 *Let (X, L) be a polarized surface. Let k_0 be the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve and let $N_0 = k_0^2 L^2$. With the above assumptions*

$$\varepsilon(L; x_1, \ldots, x_r) = \sqrt{\frac{L^2}{r}}$$

for general $x_1, \ldots, x_r \in X$ and $r \geq N_0$.

Now we introduce some notation.

Let D be a curve on a surface X passing through $x_1, \ldots, x_r$ with multiplicities $m_1 := \operatorname{mult}_{x_1} D, \ldots, m_r := \operatorname{mult}_{x_r} D$ respectively. To the curve D we assign its *multiplicity vector* $M_D := (m_1, \ldots, m_r) \in \mathbb{Z}^r$. We say that

Definition 2.7 *A curve D is almost-homogeneous if all but at most one of the coordinates of its multiplicity vector M_D are equal. In this case we can also say that the multiplicity vector is almost-homogeneous.*

Let (X, L) be a polarized surface with the Picard number ϱ. Let $L_1, \ldots, L_\varrho$ be a fixed basis of the Néron-Severi group $\mathrm{NS}(X)$ and let $x_1, \ldots, x_r$ be fixed points on X. To a curve $D \subset X$ we can assign a vector

$$v_D = (l_1, \ldots, l_\varrho, m_1, \ldots, m_r) \in \mathrm{NS}(X) \times \mathbb{Q}^r$$

such that $D \equiv l_1 L_1 + \cdots + l_\varrho L_\varrho$ and $M_D = (m_1, \ldots, m_r)$ is its multiplicity vector.
By analogy to [Sz1] Proposition 1.8 and 4.5 one can expect that the number of irreducible and reduced Seshadri submaximal curves is bounded.
A naive approach would be the following. Assume that $D_1, \ldots, D_s$ are such curves. Each of this curves has a vector

$$v_{D_i} = v_i = (l_1^{(i)}, \ldots, l_\varrho^{(i)}, m_1^{(i)}, \ldots, m_r^{(i)}) \in \mathrm{NS}(X) \times \mathbb{Q}^r \quad \text{for} \quad i = 1, \ldots, s$$

If the number $s > \varrho + r$ then the equation

$$\sum_{i=1}^{s} \lambda_i v_i = 0 \quad \text{where} \quad \lambda_i \in \mathbb{Q}$$

has a non-trivial solution. We may in fact assume that $\lambda_i \in \mathbb{Z}$ (because we can multiply both sides of this equation by the common denominator).
Now we define curves D_+ and D_- in the following way:

$$D_+ := \sum_{i=1}^{s} \beta_i D_i, \quad \text{where} \quad \beta_i = \begin{cases} \lambda_i & \text{if} \quad \lambda_i \geq 0, \\ 0 & \text{if} \quad \lambda_i < 0 \end{cases}$$
$$D_- := \sum_{i=1}^{s} \gamma_i D_i, \quad \text{where} \quad \gamma_i = \begin{cases} 0 & \text{if} \quad \lambda_i \geq 0, \\ -\lambda_i & \text{if} \quad \lambda_i < 0. \end{cases}$$

Then of course

$$D_+ \equiv D_-$$

and the multiplicity vectors

$$M_+ = (m_1^+, \ldots, m_r^+), \quad M_- = (m_1^-, \ldots, m_r^-)$$

are equal.
Let $M = (m_1, \ldots, m_r)$ be the multiplicity vector at $x_1, \ldots, x_r$ of both curves. The curves D_+ and D_- are submaximal (as combinations of submaximal curves with non-negative integer coefficients). Hence

$$\frac{L.D_+}{\sum_{i=1}^r m_i} < \sqrt{\frac{L^2}{r}}$$

and

$$\frac{L.D_-}{\sum_{i=1}^r m_i} < \sqrt{\frac{L^2}{r}}.$$

By their definition, D_+ and D_- have no common components, thus

$$\begin{aligned} D_-^2 = D_+.D_- \ & \geq \sum_{i=1}^r m_i \geq \frac{1}{r}\left(\sum_{i=1}^r m_i\right)^2 = \frac{1}{\sqrt{r}}\sum_{i=1}^r m_i \cdot \frac{1}{\sqrt{r}}\sum_{i=1}^r m_i \\ & > \frac{L.D_+}{\sqrt{L^2}} \cdot \frac{L.D_-}{\sqrt{L^2}} = \frac{(L.D_-)^2}{L^2} \geq D_-^2, \end{aligned}$$

where the last inequality follows from the Hodge Index Theorem. This is a contradiction, so s can be at most $\varrho + r$.

To obtain a better bound (in fact the optimal one) we use the Hodge Index Theorem in a slightly different way.

Proposition 2.8 *Let (X, L) be a polarized surface with Picard number ϱ and let $x_1, \dots, x_r$ be points in X such that $\varepsilon = \varepsilon(L; x_1, \dots, x_r)$ is submaximal. There are at most $\varrho + r - 1$ irreducible and reduced Seshadri curves.*

Proof. Let $\pi : Y \longrightarrow X$ be the blowing up of X at $x_1, \dots, x_r$ with exceptional divisors $E_1, \dots, E_r$ and let $H := \pi^* L$. Suppose that $C_1, \dots, C_s$ are irreducible and reduced curves computing ε and $\widetilde{C_1}, \dots, \widetilde{C_s}$ are their proper transforms. The $\mathbb{Q}$-divisor $M := H - \varepsilon \sum_{i=1}^{r} E_i$ is nef and big and for arbitrary $\lambda_i \geq 0$ we have

$$M.\left(\sum_{i=1}^{s} \lambda_i \widetilde{C}_i\right) = \sum_{i=1}^{s} \lambda_i \cdot \left(M.\widetilde{C}_i\right) = \sum_{i=1}^{s} \lambda_i \left(\pi^* L.\widetilde{C}_i - \varepsilon \sum_{j=1}^{r} E_j.\widetilde{C}_i\right) =$$

$$= \sum_{i=1}^{s} \lambda_i \left(\pi^* L.\pi^* C_i - \sum_{k=1}^{r} \operatorname{mult}_{x_k} C_i \cdot (\pi^* L.E_k) - \varepsilon \sum_{j=1}^{r} \operatorname{mult}_{x_j} C_i\right) =$$

$$= \sum_{i=1}^{s} \lambda_i \left(L.C_i - 0 - L.C_i\right) = 0.$$

The Hodge Index Theorem implies that the intersection matrix of $\widetilde{C_1}, \dots, \widetilde{C_s}$ is negative definite. Since $\varrho(Y) = \varrho + r$ then must be $s \leq \varrho + r - 1$.

□

This upper bound is in fact optimal as the following example shows

Example 2.9 *Let be $X = \mathbb{P}^2$ and the number of points $r = 7$. In this case $\varrho = 1$ and from previous proposition we have that the number of irreducible and reduced Seshadri curves is at most 7. We know that there exists an irreducible cubic D with the multiplicity vector $M_D = (1, 1, 1, 1, 1, 1, 2)$. Such a cubic computes the submaximal Seshadri constant $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); x_1, \dots, x_7) = \frac{3}{8}$. We observe that $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); x_1, \dots, x_7)$ is also computed by a cubic D_i with multiplicity vector having 2 on the i-th position (e.g. $D_7 = D$). The number of such curves is 7.*

Using the same arguments like in [Sz1] Corollary 4.6, after some elementary calculations we can prove the following

Proposition 2.10 *Let (X, L) be a polarized surface with Picard number ϱ and let $x_1, \dots, x_r$ be general points on X. If ϱ equal one, two or three and the Seshadri constant $\varepsilon(L; x_1, \dots, x_r)$ is submaximal, then any irreducible and reduced Seshadri curve is almost-homogeneous.*

Proof. Since the Seshadri constant $\varepsilon(L; x_1, \ldots, x_r)$ is submaximal then by the real valued Nakai-Moishezon criterion [CP] it follows that there exists a computing curve.
Let D be an irreducible and reduced Seshadri curve with the multiplicity vector $M_D = (m_1, \ldots, m_r)$. Since the points are general, the monodromy group acts as the full symmetric group S_r i.e. for $\sigma \in S_r$ there exists a curve D_σ with the multiplicity vector $M_{D_\sigma} = (m_{\sigma(1)}, \ldots, m_{\sigma(r)})$ which is also irreducible Seshadri curve.
Let $w = \#\{m_1, \ldots, m_r\}$ be the number of different multiplicities. Without loss of generality we can assume that $\{m_1, \ldots, m_w\} = \{m_1, \ldots, m_r\}$.
We claim that $w \leq 2$.

Assume to the contrary that $w \geq 3$.
Let n_i denote the number of entries m_i in the multiplicity vector M_D. We can assume $n_1 \geq \ldots \geq n_w$. We have

$$n_1 + \ldots + n_w = r$$

and

$$n_1 \leq r - (w - 1).$$

Consider $\Omega \subset \mathbb{Z}^w_{\geq 0}$ with

$$\Omega = \{(n_1, \ldots, n_w) \mid n_1 \geq \ldots \geq n_w \text{ and } \sum_{i=1}^{w} n_i = r\}.$$

Consider the function

$$\Omega \ni (n_1, \ldots, n_w) \longrightarrow \frac{r!}{n_1! \ldots n_w!} \in \mathbb{Z}.$$

This function is minimal and equal 1 if $n_1 = r$ and $n_2 = \ldots = n_w = 0$. We are interested in the subset Ω_3 consisting of all $(n_1, \ldots, n_w)$ such that $n_3 \geq 1$. On this subset the minimum is attained obviously if $n_1 = r - 2$ and $n_2 = n_3 = 1$. This minimum is equal $r(r-1)$ and this in turn is at least $r + \varrho$ since $r(r-2) \geq \varrho$ holds for $\varrho \geq 3$ as $r \geq 3$. This contradicts the bound on the number of submaximal curves obtained in Proposition 2.8.

□

3 Seshadri constants on ruled surfaces.

In this chapter we study ample line bundles on ruled surface. We observe that criterions for the ampleness of a linear system $|D|$ permit to compute the Seshadri constant only in some cases. To estimate all values we need something more. We expect that the problem can be solved by the elementary transformations, but we were not able to achieve this goal here.

3.1 Ruled surfaces - basic definitions and properties.

First we recall some basic definitions and facts from the theory of ruled surfaces.

Definition 3.1 *A geometrically ruled surface, or simply a ruled surface, is a surface X, together with a surjective morphism $\pi : X \longrightarrow C$ to a (nonsingular) curve C, such that for every point $y \in C$, the fibre X_y is isomorphic to $\mathbb{P}^1$, and such that π admits a section (i.e. a morphism $s : C \longrightarrow X$ such that $\pi \circ s = id_C$).*

Example 3.2 If C is a nonsingular curve, then $C \times \mathbb{P}^1$ with the first projection is a ruled surface.

Example 3.3 Let $\mathcal{E}$ be a vector bundle of rank 2 over a curve C. The associated projective space bundle $\mathbb{P}(\mathcal{E})$ with the projection morphism $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow C$ is a ruled surface.

The following proposition shows that all ruled surfaces are as in the above example.

Proposition 3.4 *([Ha] V, 2.2) If $\pi : X \longrightarrow C$ is a ruled surface, then there exists a vector bundle $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C. If $\mathcal{E}$ and $\mathcal{E}'$ are two vector bundles of rank 2 on C, then $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic as ruled surfaces over C if and only if there is an invertible sheaf $\mathcal{L}$ on C such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.*

Remark 3.5 *A surface X is called a birationally ruled surface if is birationally equivalent to $C \times \mathbb{P}^1$ for some curve C. Since $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, this means that every rational surface is a birationally ruled surface.*

Let $\pi : X \longrightarrow C$ be a ruled surface over a curve C of a genus g. By Proposition 3.4, we can choose $\mathcal{E}_0$ a locally free sheaf of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E}_0)$. Moreover we can assume that $\mathrm{H}^0(\mathcal{E}_0) \neq 0$ but for all

invertible sheaves $\mathcal{L}$ on C with $\deg \mathcal{L} < 0$, we have $\mathrm{H}^0(\mathcal{E}_0 \otimes \mathcal{L}) = 0$. A sheaf $\mathcal{E}_0$ with this property is called *normalized*.
In general $\mathcal{E}_0$ is not necessarily uniquely determined, but its invariant $e = -\deg(\mathcal{E}_0)$ is fixed.

Example 3.6 Let C be a curve with positive genus, and $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{L}$ where $\deg(\mathcal{L}) = 0$ but $\mathcal{L} \ncong \mathcal{O}_C$. In this case we have two choices of normalized $\mathcal{E}_0$, namely $\mathcal{E}$ and $\mathcal{E} \otimes \mathcal{L}^{-1}$.

Let $\mathfrak{e}$ be the divisor on C corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}_0$, then $e = -\deg(\mathfrak{e})$. Moreover, there exists a section $s_0 : C \longrightarrow X$ with the image C_0, such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_X(1)$, where $\mathcal{O}_X(1)$ is the Serre line bundle on X (for more details see [Ha] V, 2.8).

Proposition 3.7 *([Ha] V, 2.3) Let $\mathcal{E}_0$ be a normalized vector bundle and $X = \mathbb{P}(\mathcal{E}_0)$. Then*

$$\mathrm{Pic}(X) \cong \mathbb{Z} \cdot C_0 \oplus \pi^* Pic(C)$$

and

$$\mathrm{Num}(X) \cong \mathbb{Z} \cdot C_0 \oplus \mathbb{Z} \cdot f,$$

where f is the class of a fiber. The intersection product on X is determined by $C_0.f = 1$, $f^2 = 0$ and $C_0^2 = \deg \bigwedge^2 \mathcal{E}_0 = -e$ (see Proposition 3.9).

If $\mathfrak{b}$ is any divisor on C, then we denote the divisor $\pi^*\mathfrak{b}$ on X by $\mathfrak{b}f$. Thus from Proposition 3.7 we have that, any element of $Pic(X)$ can be written as $aC_0 + \mathfrak{b}f$ with $a \in \mathbb{Z}$ and $\mathfrak{b} \in Pic(C)$. Any element of $Num(X)$ can be written as $aC_0 + bf$ with $a, b \in \mathbb{Z}$.

Lemma 3.8 *([Ha] V, 2.20 and 2.11) Using above notations*

(1) the canonical divisor K on X is given by

$$K \sim -2C_0 + (\mathfrak{k} + \mathfrak{e})f$$

where $\mathfrak{k}$ is the canonical divisor on C.

(2) For numerical equivalence, we have

$$K \equiv -2C_0 + (2g - 2 - e)f$$

and therefore

$$K^2 = 8(1 - g).$$

Proposition 3.9 *([Ha] V, 2.6 and 2.9) Let $\mathcal{E}$ be a locally free sheaf of rank two on a curve C, and let X be the ruled surface $\mathbb{P}(\mathcal{E})$. Then there exists a one-to-one correspondence between sections $s : C \longrightarrow X$ and quotients $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{L}$ is an invertible sheaf on C, given by $\mathcal{L} = s^*\mathcal{O}_X(1)$. Furthermore, if D is any section of X corresponding to a surjection $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, and if $\mathcal{L} = \mathcal{O}_C(\mathfrak{d})$, for some divisor $\mathfrak{d}$ on C, then $\deg(\mathfrak{d}) = C_0.D$, and $D \sim C_0 + (\mathfrak{d} - \mathfrak{e})f$. In particular, we have that $C_0^2 = \deg(\mathfrak{e}) = -e$.*

From Proposition 3.7 and Proposition 3.9 it follows that C_0 is a curve on X with the minimal self-intersection. Next lemma gives us more information about a number of such curves.

Lemma 3.10 *([FP], 2.8) Let $\pi : X = \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ be a ruled surface. Then $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 2$ if and only if $\mathbb{P}(\mathcal{E}_0) \cong C \times \mathbb{P}^1$ and $\mathrm{h}^0(\mathcal{O}_X(C_0)) = 1$ in all other cases.*

This means that the curve C_0 is unique in its class of linear equivalence, except when the ruled surface is the product $C \times \mathbb{P}^1$.

Definition 3.11 *A ruled surface $X \cong \mathbb{P}(\mathcal{E}_0)$ is called decomposable if $\mathcal{E}_0$ is a direct sum of two invertible sheaves (in particular such a vector bundle $\mathcal{E}_0$ is not stable).*

Theorem 3.12 *([Ha] V, 2.12) Let X be a ruled surface over a curve C of genus g, determined by a normalized locally free sheaf $\mathcal{E}_0$.*

(1) If $\mathcal{E}_0$ is decomposable, then $\mathcal{E}_0 \cong \mathcal{O}_C \oplus \mathcal{L}$ for some $\mathcal{L}$ with $\deg(\mathcal{L}) \leq 0$. Therefore $e \geq 0$. All values of $e \geq 0$ are possible.

(2) If $\mathcal{E}_0$ is indecomposable, then $-g \leq e \leq 2g - 2$.

Let $X \cong \mathbb{P}(\mathcal{E}_0)$ be a decomposable ruled surface. Geometrically it means that, X has two disjoint unisecant curves C_0 and C_1 (i.e. $C_i.f = 1$ for each fiber f). These curves are given by surjections $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C \longrightarrow 0$ and $\mathcal{E}_0 \cong \mathcal{O}_C(\mathfrak{e}) \oplus \mathcal{O}_C \longrightarrow \mathcal{O}_C(\mathfrak{e}) \longrightarrow 0$ respectively. Moreover from Proposition 3.9, we have that $C_1 \sim C_0 - \mathfrak{e}f$.

3.2 Linear systems on ruled surfaces.

We start by recalling the following ampleness criterion.

Theorem 3.13 *([Ha] V, 2.20 and 2.21) Let X be a ruled surface over a curve C of genus g, with a fiber f, the section C_0 and $e = -\deg(\mathfrak{e}) = -C_0^2$.*

(1) If $Y \equiv aC_0 + bf$ is an irreducible curve different from C_0 and a fiber, then

(a) $a > 0$ and $b \geq ae$ for $e \geq 0$,

(b) ($a = 1$ and $b \geq 0$) or ($a \geq 2$ and $b \geq \frac{1}{2}ae$) for $e < 0$.

(2) A divisor $D \equiv aC_0 + bf$ is ample if and only if

(a) $a > 0$ and $b > ae$ for $e \geq 0$,

(b) $a > 0$ and $b > \frac{1}{2}ae$ for $e < 0$.

Remark 3.14 *There are no better numerical conditions characterizing irreducible curves on ruled surfaces as this property does not depend only on the numerical equivalence class of the considered line bundle.*

Example 3.15 Let C be a non-hyperelliptic curve of genus 3. Its canonical divisor K_C is very ample. On the other hand K_C is the only very ample divisor on C of degree 4. This follows directly from the result of Halphen [Ha] IV Proposition 6.1.

3.3 Elementary transformation of a ruled surface.

Let $\pi : X \longrightarrow C$ be a geometrically ruled surface and let x be a point on X with $\pi(x) = P$. We denote by Pf the fiber through the point x.
Let $\sigma : X_x \longrightarrow X$ be the blow-up of X at x with the exceptional divisor $E = \sigma^{-1}(x)$. We have $\sigma^*(Pf) = \widetilde{Pf} + E$, where $\widetilde{Pf} = \overline{\sigma^{-1}(Pf \setminus \{x\})}$ is the *strict transform* of the fiber Pf. Since $\widetilde{Pf} \cong \mathbb{P}^1$ and $\widetilde{Pf}^2 = -1$, we can blow-down the surface X_x along $\widetilde{Pf}$ (this follows from the Castelnuovo's criterion). We denote by $\tau : X_x \longrightarrow X'$ the blow-down of X_x along the exceptional curve $E' = \widetilde{Pf}$ (see figure 1).

Definition 3.16 *An elementary transformation of X at the point x is the birational map $\nu : X' \longrightarrow X$ where $\nu = \sigma \circ \tau^{-1}$. The surface X' is called the elementary transform of X at x.*
For a curve D on the surface X we define its strict transform as $D' = \tau_*(\widetilde{D})$, where by $\widetilde{D}$ we mean $\widetilde{D} = \overline{\sigma^{-1}(D \setminus \{x\})}$.

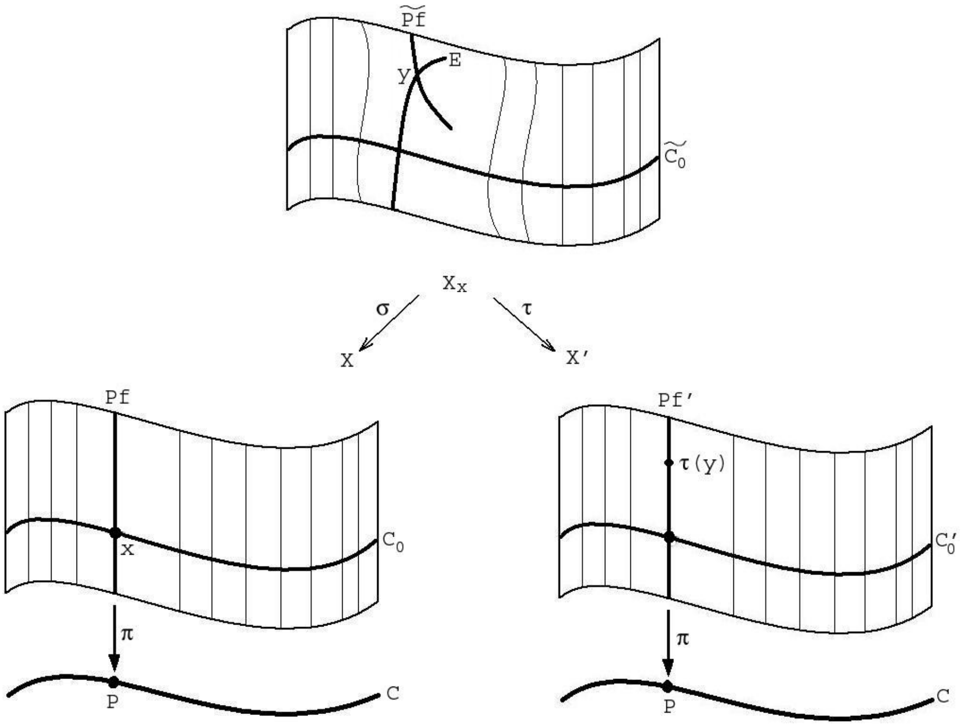

Figure 1: An elementary transformation of X at a point x

We observe that:

Remark 3.17 *If X' is an elementary transform of X at x, then X is the elementary transform of X' at $\tau(y)$, where y is the intersection of the exceptional divisors E and E' on X_x.*
Moreover, if Pf' is the fiber through the point $\tau(y)$, then $\widetilde{Pf'} = E$.

Assume that $\pi : X \longrightarrow C$ is a geometrically ruled surface over a curve C of genus g with the invariant e. Let $D \subset X$ be a curve on X. We say that D is *n-secant curve* on X if $D \equiv nC_0 + bf$ for some $b \in \mathbb{Z}$.
Let $\nu : X' \longrightarrow X$ be the elementary transformation of the surface X at a point x with $\pi(x) = P$. The question is: how the elementary transformation ν changes properties of D and X?

Proposition 3.18 *([FP], 4.4) Let $\nu : X' \longrightarrow X$ be the elementary transformation of X at a point x with $\pi(x) = P$.*

(1) If $\mathfrak{b}$ is a divisor on C, then $\nu^*(\mathfrak{b}f) = \mathfrak{b}f'$.

(2) If D is a curve on X, then $\nu^* D = D' + (\operatorname{mult}_x D) \cdot Pf'$.

(3) If D and G are respectively n and m-secant curves on X, then

$$D'.G' = D.G + nm - n \cdot \operatorname{mult}_x G - m \cdot \operatorname{mult}_x D.$$

Therefore, if D and G are unisecant curves on X then:

$$D'.G' = \begin{cases} D.G - 1 & \text{if} \quad x \in D \cap G \\ D.G + 1 & \text{if} \quad x \notin D \cup G \\ D.G & \text{if} \quad x \in D \text{ but } x \notin G \end{cases}$$

(4) If D is a unisecant curve on X, then $\nu_* \nu^* D = D + Pf$.

Since we would like to know exactly how an elementary transformation changes properties of a divisor, we go through the prove of statements claimed.

Proof. We follow the outline of [FP] Proposition 4.4.
Claim (1).
Any divisor $\mathfrak{b}$ on C can be written as a difference of two effective divisors. This means that it is sufficient to proof (1) in the case $\mathfrak{b}f = Qf$, for any fiber Qf.
If $x \in Qf$ then $\nu^*(Qf) = \tau_*(\sigma^*(Qf)) = \tau_*(\widetilde{Qf} + E) = E = Qf'$.
If $x \notin Qf$ then $\nu^*(Qf) = \tau_*(\sigma^*(Qf)) = \tau_*(\widetilde{Qf}) = Qf'$.

Claim (2).
For a curve D on X we have:

$$\begin{aligned} \nu^* D &= \tau_*(\sigma^* D) = \tau_*(\widetilde{D} + (\operatorname{mult}_x D) \cdot E) = \tau_* \widetilde{D} + (\operatorname{mult}_x D) \cdot \tau_* E = \\ &= D' + (\operatorname{mult}_x D) \cdot Pf'. \end{aligned}$$

Claim (3).
By Definition (3.16)

$$D'.G' = (\tau_* \widetilde{D}).(\tau_* \widetilde{G}).$$

Since $\tau : X_x \longrightarrow X'$ is the blow-up X' at the point $\tau(y)$ (see Remark 3.17), then for $\widetilde{G} \in Pic(X_x)$ and $\tau_* \widetilde{D} \in Pic(X')$ it follows that

$$\tau^*(\tau_* \widetilde{D}).\widetilde{G} = (\tau_* \widetilde{D}).(\tau_* \widetilde{G}),$$

and

$$D'.G' = \tau^* \tau_* \widetilde{D}.\widetilde{G}.$$

For

$$\tau^* \tau_* \widetilde{D} = \tau^*(\tau_* \widetilde{D}) = \widetilde{\tau_* \widetilde{D}} + (\operatorname{mult}_{\tau(y)} \tau_* \widetilde{D}) \cdot \widetilde{Pf},$$

and

$$\widetilde{\tau_* \widetilde{D}} = \overline{\tau^{-1}(\tau_* \widetilde{D} \setminus \tau(y))} = \widetilde{D}$$

with the condition

$$\begin{aligned} \operatorname{mult}_{\tau(y)} \tau_* \widetilde{D} &= \widetilde{D}.\widetilde{Pf} = D.Pf - \operatorname{mult}_x D \cdot \operatorname{mult}_x Pf = \\ &= n - \operatorname{mult}_x D, \end{aligned}$$

in result we have

$$\begin{aligned} D'.G' &= \left(\widetilde{D} + (n - \operatorname{mult}_x D) \cdot \widetilde{Pf}\right).\widetilde{G} = \widetilde{D}.\widetilde{G} + (n - \operatorname{mult}_x D)(\widetilde{Pf}.\widetilde{G}) = \\ &= D.G - \operatorname{mult}_x D \cdot \operatorname{mult}_x G + (n - \operatorname{mult}_x D)(m - \operatorname{mult}_x G) = \\ &= D.G + nm - n \cdot \operatorname{mult}_x G - m \cdot \operatorname{mult}_x D. \end{aligned}$$

Claim (4).
For a unisecant curve D on X we have:
(a) if $x \notin D$ then from (2) $\nu^* D = D'$ and $\tau(y) \in D'$, so $\nu_* \nu^* D = \nu_* D' = D'' + Pf = D + Pf$;
(b) if $x \in D$ then from (2) $\nu^* D = D' + Pf'$ and $\tau(y) \notin D'$, so $\nu_* \nu^* D = D'' + Pf = D + Pf$.

□

Let C_0 be the minimum self-intersection curve on X. We know that $C_0^2 = -e$ and for any other curve D on X, we have $D^2 \geq -e$. Moreover assume that $x \in C_0$. Let C_0' denote the strict transform of C_0 by the elementary transformation of X at x. From Proposition 3.18 it follows that $C_0'^2 = C_0^2 - 1$, but for any other unisecant curve D we have $D'^2 \geq D^2 - 1$. It follows that $D'^2 \geq C_0'^2$ and C_0' is the minimum self-intersection curve on X'. Since $C_0'^2 = -e - 1$, then $e' = e + 1$.
In this way we proved the following

Theorem 3.19 *([FP], 4.9) Let $\pi : \mathbb{P}(\mathcal{E}_0) \longrightarrow C$ be a ruled surface. Fix a point x on the minimum self-intersection curve C_0 on X, with $\pi(x) = P$. Let X' denote the elementary transform of X at x. Then X' is a ruled surface corresponding to a normalized sheaf $\mathcal{E}_0'$ with $\bigwedge^2 \mathcal{E}_0' \cong \mathcal{O}_C(\mathfrak{e}')$ satisfying $\mathfrak{e}' \sim \mathfrak{e} - P$ ($e' = e + 1$). Furthermore, the minimum self-intersection curve on X' is C_0'.*

Let X_0 be an indecomposable ruled surface over a curve C of genus g and invariant e. If we apply an elementary transformation to X at a point on C_0, then we obtain a ruled surface X_1 with invariant $e_1 = e + 1$ (from Theorem

3.19). We can take n such transformations so that $e_n = e + n > 2g - 2$. This means that after n steps the surface X_n is decomposable (see Theorem 3.12). Applying Remark 3.17 to surfaces X and X_n, we have that X can be obtained from X_n by elementary transformations. We proved the following

Remark 3.20 *([FP], 4.10) Any indecomposable ruled surface is obtained from a decomposable one by a finite number of elementary transformations.*

We can say more, namely

Remark 3.21 *([FP], 4.11) Any ruled surface over the curve C is obtained from $C \times \mathbb{P}^1$ applying a finite number of elementary transformations.*

It follows Remark 3.21 that every ruled surface is birationally ruled (compare with Remark 3.5).

Remark 3.17 and Theorem 3.19 give us useful tools to study numerical properties of transformed divisors.

Proposition 3.22 *Let $\nu : X' \longrightarrow X$ be the elementary transformation at $x \in C_0$.*

(a) Let D be a divisor on X. If $D \equiv aC_0 + bf$ with integers a and b, then $\nu^* D \equiv aC'_0 + (a+b)f'$, where C'_0 and f' generate $Num(X')$.

(b) Let G be a divisor on X'. If $G \equiv pC'_0 + qf'$ with integers p and q, then $\nu_* G \equiv pC_0 + qf$.

Proof. We are using the notation introduced in the definition of an elementary transformation and in the previous propositions.
Claim (a)
Let

$$\nu^* D \equiv pC'_0 + qf', \quad \text{with} \quad p, q \in \mathbb{Z}. \tag{3.22.1}$$

From Proposition 3.7 we have that for any fiber f'

$$(\nu^* D).f' = p,$$

but

$$(\nu^* D).f' = (\tau_* \sigma^* D).f' = (\sigma^* D).(\tau^* f').$$

Let $\tau(y) \in f'$. In our notation it means that $f' = Pf'$. Then

$$\begin{aligned}(\nu^* D).f' &= (\sigma^* D).(\widetilde{Pf'} + E') = (\sigma^* D).E + (\sigma^* D).\widetilde{Pf} = \\ &= (\sigma^* D).(\sigma^*(Pf)) - (\sigma^* D).E = D.(Pf) = a.\end{aligned}$$

If $\tau(y) \notin f'$, then

$$(\nu^* D).f' = (\sigma^* D).(\tau^* f') = (\sigma^* D).\widetilde{f'} = (\sigma^* D).\widetilde{f} = (\sigma^* D).(\sigma^* f) = D.f = a.$$

In this way we proved $p = a$.
To show that $q = a + b$ holds, it is enough to test the intersection product $(\nu^* D).C_0'$.
Since $x \in C_0$, then $\tau(y) \notin C_0'$. Moreover from Theorem 3.19 it follows that

$$C_0'^2 = C_0^2 - 1. \tag{3.22.2}$$

By Proposition 3.7, conditions (3.22.1) and (3.22.2)

$$(\nu^* D).C_0' = pC_0'^2 + q = pC_0^2 - p + q. \tag{3.22.3}$$

On the other hand

$$\begin{aligned}(\nu^* D).C_0' &= (\sigma^* D).(\tau^* C_0') = (\sigma^* D).\widetilde{C_0'} = (\sigma^* D).\widetilde{C_0} = \\ &= (\sigma^* D).(\sigma^* C_0) - (\sigma^* D).E = D.C_0 = aC_0^2 + b.\end{aligned} \tag{3.22.4}$$

Applying the equality $p = a$ for conditions (3.22.3) and (3.22.4) we have $q = a + b$.

Claim (b).
Let Pf' denote, as before, the fiber through $\tau(y)$. Moreover assume that

$$\nu_* Y \equiv aC_0 + bf. \tag{3.22.5}$$

The idea of the proof for this part is the same as in the part (a). In particular, it is not difficult to see that $a = p$. We concentrate more on the second intersection product i.e. $(\nu_* Y).C_0$.
From conditions (3.22.2) and (3.22.5) it follows that

$$(\nu_* Y).C_0 = aC_0^2 + b = aC_0'^2 + a + b. \tag{3.22.6}$$

We have also

$$\begin{aligned}(\nu_* Y).C_0 &= (\sigma_*(\tau^* Y)).C_0 = (\tau^* Y).(\sigma^* C_0) = (\tau^* Y).(\widetilde{C_0} + E) = \\ &= (\tau^* Y).\widetilde{C_0'} + (\tau^* Y).\widetilde{Pf'} = \\ &= (\tau^* Y).(\tau^* C_0') + (\tau^* Y).(\tau^* Pf' - E') = \\ &= Y.C_0' + Y.Pf' = pC_0'^2 + q + p.\end{aligned} \tag{3.22.7}$$

Applying the equality $a = p$ to (3.22.6) and (3.22.7) we see that $b = q$.

□

From this proposition it follows immediately the generalization of (4) in Proposition 3.18 for a-secant curves.

Remark 3.23 *Let X and ν be like in previous proposition. If $D \equiv aC_0 + bf$ with $a, b \in \mathbb{Z}$, then*

$$\nu_* \nu^* D \equiv \nu_*(aC_0' + (a+b)f') \equiv aC_0 + (a+b)f.$$

It means that we do not have $\nu_*\nu^*(D) \equiv D$ but the secancy of D remains invariant under the transformation $\nu_*\nu^*$.

For strict transforms it holds:

Remark 3.24 *For any a-secant curve D on X its strict transform D' on X' is still an a-secant curve.*

Proof. Let $D \equiv aC_0 + bf$ be an a-secant curve on X and let $\nu : X' \longrightarrow X$ be an elementary transformation at a point x. From Proposition 3.18 it follows that

$$D' = \nu^* D - (\operatorname{mult}_x D) \cdot Pf'.$$

Hence by Proposition 3.22 we derive immediately:
(a) if $x \in C_0$, then $D' \equiv aC_0' + (a + b - \operatorname{mult}_x D)f'$;
(b) if $x \notin C_0$, then $D' \equiv aC_0' + (b - \operatorname{mult}_x D)f'$.

□

Let $G \equiv aC_0 + bf$ be an ample divisor on X. The question we want to address now is: what happens to the ampleness of the strict transform G'? Is G' still ample?
In general G' need not to be ample. More precisely we can formulate the following

Proposition 3.25 *Let $\nu : X' \longrightarrow X$ an elementary transformation of X at a point $x \in X$ and an ample divisor $G \in \operatorname{Num}(X)$, if x is not a base point of $|G|$ then the strict transform G' is also ample.*

Proof. Let X be a ruled surface with an invariant e, and let $D \in |G|$.

Case (1).
If $x \in C_0$, then by Theorem 3.19 the surface X' is ruled with invariant $e' = e + 1$. Moreover by Proposition 3.22 the strict transform

$$D' \equiv aC_0' + (a + b - \operatorname{mult}_x D)f'.$$

The rest of the proof is purely computational and we skip it. We note only that, we should use Theorem 3.13 and we should consider all possible values of $a - b - \mathrm{mult}_x D$ with respect to e.

Case (2).
If $x \notin C_0$ then by Theorem 3.19 and Remark 3.17 the surface X' is the ruled surface with invariant $e' = e - 1$. By Proposition 3.22 the strict transform

$$D' \equiv aC'_0 + (b - \mathrm{mult}_x D)f'.$$

The rest of the proof is the same like in Case (1) but for $b - \mathrm{mult}_x D$ instead of $a - b - \mathrm{mult}_x D$.

□

We observe that doing the same calculations for unisecant divisors i.e. $G \equiv C_0 + \mu_0 f$ we can say more, namely:

Proposition 3.26 *([Sy2] Prop. 7) The strict transform G' is ample except when*

(i) in the case $e > 0$ we have $G \equiv C_0 + (e+1)f$ and we apply an elementary transformation at a point $x \in C_0$ which is also a base point of $|G|$,

(ii) in the case $e < 0$ and e odd we have $G \equiv C_0 + \frac{1}{2}(e+1)f$ and we apply an elementary transformation at a base point of $|G|$.

Proof. Let be $x \in X$ and $D \in |G|$. Since a divisor D is unisecant then it must be $mult_x D \leq 1$. The rest follows from direct calculations.

□

3.4 Seshadri Constants on ruled surfaces.

In this section we compute the Seshadri constant on ruled surfaces. We can also see that in general the conditions contain in Theorem 3.13 are to weak to make good estimations of Seshadri quotients.

Now we can formulate the following

Theorem 3.27 *Let X be a ruled surface with a polarization $L \equiv aC_0 + bf$, and let $x \in X$ be a point on X.*

(1) If $e > 0$ then the Seshadri constant at x is equal to

$$\varepsilon(L;x) = \begin{cases} L.C_0 & \text{if } x \in C_0 \text{ and } b < a(e+1) \\ L.f & \text{if } x \notin C_0 \text{ or } b \geq a(e+1). \end{cases}$$

(2) If $e = 0$ *then the Seshadri constant*

$$\varepsilon(L;x) \geq \min\{L.C_0, L.f\},$$

and

(a) if $b \geq a$*, then* $\varepsilon(L;x) = L.f$*,*

(b) if $b < a$*, then for* $x \in C_0$ *the Seshadri constant is equal to* $\varepsilon(L;x) = L.C_0$*.*

(3) If $e < 0$ *and* $b \geq \frac{1}{2}ae + a + \frac{1}{4} - \frac{1}{4}(-1)^{ae+1}$*, then the Seshadri constant is equal to* $\varepsilon(L;x) = L.f$*.*

We should also remark that the curves computing Seshadri constants are submaximal.

Proof. This theorem follows by straightforward calculations.

Case (1) ($e > 0$)
Since L is an ample line bundle on X, then from Theorem 3.13 we can write b as $b = ae+1+n$, where n is a positive integer or zero. For every irreducible curve $D \equiv \alpha C_0 + \beta f$, which is not C_0 or f, we have

$$\frac{L.D}{m} = \frac{a\beta + \alpha(ae+1+n)}{m} \geq \frac{\alpha(ae+1)}{m},$$

where $m = \operatorname{mult}_x D$. Since $m \leq D.f = \alpha$ then for $e \geq 1$

$$\frac{\alpha(ae+1)}{m} \geq ae+1 > a.$$

It means that irreducible curves different from C_0 and f never produce the Seshadri quotients smaller or equal $a = L.f$. This implies the assertion as the fiber through x and C_0 are smooth.
We observe also that for $x \in C_0$ the Seshadri quotient given by C_0 is smaller than given by the fiber f if and only if $b < a(e+1)$.

Case (2) ($e = 0$)
By Theorem 3.13 we have that $b \geq 1$. As before by $D \equiv \alpha C_0 + \beta f$, with $\beta \geq 0$, we denote an irreducible curve on X and $m = \operatorname{mult}_x D$. Using the fact that $m \leq \alpha$ we can approximate the Seshadri quotient, namely we have

$$\frac{L.D}{m} = \frac{a\beta + b\alpha}{m} \geq L.C_0 = b.$$

It means only that $\varepsilon(L;x) \geq \min\{L.C_0, L.f\}$ and if $a \leq b$ then the Seshadri constant $\varepsilon(L;x) = a$. If $a > b$ then we can compute the Seshadri constant only in the case when $x \in C_0$.

Case (3) ($e < 0$)
In this case from Theorem 3.13 it follows that $b = \frac{1}{2}ae + \frac{3}{4} - \frac{1}{4}(-1)^{ae+1} + n$, with $n \in \mathbb{N}$. If $D \equiv \alpha C_0 + \beta f$ is irreducible and different from C_0 and f then
(a) for $\alpha = 1$ must be $\beta \geq 1$
(b) for $\alpha \geq 2$ we have $\beta \geq \frac{1}{2}\alpha e$.
Since $m = \operatorname{mult}_x D \leq \alpha$ then in the case (a) we have that $m = 1$ and the Seshadri quotient

$$\frac{L.D}{m} = -\frac{1}{2}ae + a\beta + 1 + n = L.C_0 > a = L.f$$

In the case (b) if we assume that $n \geq a - \frac{1}{2}$ then the Seshadri quotient

$$\frac{L.D}{m} = \frac{-a\alpha e + a\beta + b\alpha}{m} \geq \frac{\alpha(\frac{3}{4} - \frac{1}{4}(-1)^{ae+1} + n)}{m} \geq \frac{3}{4} - \frac{1}{4}(-1)^{ae+1} + n \geq a.$$

We can easily check that curves computing the Seshadri constants are submaximal.

□

This is all what we can say about the Seshadri constants using only Theorem 3.13. We expect that the remaining cases can be computed with help of elementary transformations.

This cases are somehow hard. In the recent preprint [F] Fuentes gives some estimates on $\varepsilon(L; x)$. This bounds can be covered and in some cases improved by the following considerations.
Let X be a ruled surface and $\nu : X' \longrightarrow X$ be the elementary transformation of X at a point $x \in C_0 \subset X$ contracting the fiber Pf through x to the point $y \in X'$. Let L be an a-secant bundle on X. The transformation ν induces inclusion:

$$H^0(X, L) \cong H^0(X', \nu^*L \otimes \mathfrak{m}_y^a) \subset H^0(X', \nu^*L). \qquad (3.27.1)$$

Now, one can invoke Proposition 2.4 and get the following estimate

$$\varepsilon(L, z) \leq \varepsilon(\nu^*L, z)$$

at any point $z \notin Pf$.
In general the inclusion (3.27.1) is strict so that the bundle on the right could generate more jets at z. However it is not clear to us how to control this in an effective way.

4 Seshadri fibrations.

Recently several authors studied the question of how submaximal Seshadri constant determine the geometry of the underlying variety [Nak], [HK], [SzT-G]. The results say that if the Seshadri constant of an ample line bundle at a general (and hence at every) point of a variety is relatively small with respect to the maximal possible value, then the variety is fibered by Seshadri curves. This problem was studded on surfaces in [Nak] and [SzT-G] and in arbitrary dimension in [HK]. Here we study more closely effective bounds given in [SzT-G] and [HK]. We show that their bounds are obtained only in the case of cubic surface in $\mathbb{P}^3$. This enables us to improve the bound in the case of all other surfaces. We also pass to a natural generalization to multiple point Seshadri constants. In this case we can see that not only the picture is similar to the one point situation but the Nagata-Biran conjecture holds asymptotically for a big class of surfaces. For very ample line bundles this was observed by different methods by Harbourne [Har] Theorem 1.1.

Before we pass to the general situation let us recall the following result proved in [SzT-G].

Theorem 4.1 *Let X be a smooth projective surface and L an ample line bundle on X with*

$$\varepsilon(L,1) < \sqrt{\frac{3}{4}} \cdot \varepsilon_{\max}(L;1).$$

Then there is a fibration of X given by Seshadri curves of L.

The main result of this section extends the above theorem for a higher number of points.

Theorem 4.2 *Let X be a smooth projective surface, L an ample line bundle on X and $r \geq 2$ a fixed integer. If*

$$\varepsilon(L;r) < \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L;r) \qquad (4.2.1)$$

then there exists a fibration $f : X \longrightarrow C$ over a curve C such that for given $x_1, \ldots, x_r \in X$ very general, the fiber $f^{-1}(f(x_i))$ computes $\varepsilon(L; x_1, \ldots, x_r)$ for arbitrary $i = 1, \ldots r$.
Moreover the factor $\sqrt{\frac{r-1}{r}}$ is optimal for every r.

Before we start the proof, we recall two lemmas. First lemma was proved by Xu.

Lemma 4.3 *([Xu1]) Suppose that X is a smooth projective surface and $(D_t, (x_1)_t, \ldots, (x_r)_t)_{t\in T}$ is a non-trivial one parameter family of pointed reduced and irreducible curves on X and let m_i be positive integers such that* $\operatorname{mult}_{(x_i)_t} D_t \geq m_i$ *for all $i = 1, \ldots, r$. Then*

$$D_t^2 \geq \sum_{i=1}^{r} m_i^2 - \min\{m_1, \ldots, m_r\}.$$

The second lemma, which is purely numerical, was obtained by Küchle.

Lemma 4.4 *([Ku1]) Let $r \geq 2$ and $m_1, \ldots, m_r \in \mathbb{Z}_+$ be integers with $m_1 \geq \cdots \geq m_r$ and $m_1 \geq 2$. Then we have*

$$(r+1)\sum_{i=1}^{r} m_i^2 > \left(\sum_{i=1}^{r} m_i\right)^2 + m_r(r+1).$$

Now we have all we need to prove our Theorem.

Proof of Theorem 4.2. Let $x_1, \ldots, x_r \in X$ be points in very general position. Since $\varepsilon(L; x_1, \ldots, x_r)$ is not maximal, then from the real valued Nakai-Moishezon criterion [CP] it follows that, there exists a computing curve $D_{x_1,\ldots,x_r}$. The points are in very general position, hence moving them around we obtain a non-trivial family $D_t = D_{(x_1)_t,\ldots,(x_r)_t}$ of such curves.
Let $m_1 \geq \cdots \geq m_r$ be non-negative integers such that $\operatorname{mult}_{(x_i)_t} D_t = m_i$ for general member D_t of the family.
We proceed by induction with respect to the number of points r and we begin with $r = 2$. Note that our Theorem is empty for $r = 1$ and we can not use the Hwang-Keum result as the first step of the induction.

First we assume that $m_1 \geq m_2 \geq 1$. From Lemma 4.3 we obtain

$$(m_1^2 + m_2^2 - m_2) \cdot L^2 \leq (D_t)^2 \cdot L^2.$$

On the other hand, by the assumption of our Theorem we have

$$(L.D_t)^2 < (m_1 + m_2)^2 \cdot \frac{1}{4} \cdot L^2.$$

Thanks to the Hodge Index Theorem this two inequalities can be written as

$$m_1^2 + m_2^2 - m_2 < \frac{1}{4}(m_1 + m_2)^2.$$

Rearranging the terms we arrive at

$$4m_2 > (m_1 - m_2)^2 + 2(m_1^2 + m_2^2). \qquad (4.4.1)$$

Since $m_1 \geq m_2$, then we can write $m_1 = m_2 + k$ with $k \in N$. With this assumption the inequality (4.4.1) is equivalent to

$$4(m_2 + k) > 4m_2(m_2 + k) + 3k^2,$$

which holds only for $m_2 = 0$ and $k = 1$. This is a contradiction with the condition $m_1 \geq m_2 \geq 1$.

If $m_2 = 0$, then by the assumption of the Theorem

$$\frac{L.D_t}{m_1} < \sqrt{\frac{1}{4}L^2} < \sqrt{\frac{3}{4}L^2}$$

and the Hwang-Keum theorem implies our assertion.

For the induction step we assume now that the number of r points is at least 3 and that the Theorem holds for $(r-1)$ points.
There are the following possibilities:

(a) $m_1 \geq \cdots \geq m_r \geq 1$ and $m_1 \geq 2$ or

(b) $m_1 = \cdots = m_r = 1$ or

(c) $m_r = 0$.

In case (a) the Hodge Index Theorem together with Lemma 4.3 give:

$$\frac{\sum_{i=1}^r m_i^2 - m_r}{\left(\sum_{i=1}^r m_i\right)^2} L^2 \leq \frac{L^2 \cdot (D_t)^2}{\left(\sum_{i=1}^r m_i\right)^2} \leq \frac{(L.D_t)^2}{\left(\sum_{i=1}^r m_i\right)^2} < \frac{r-1}{r^2} L^2.$$

Hence by Lemma 4.4 we obtain

$$\begin{aligned}\sum_{i=1}^r m_i^2 - m_r \quad &< \frac{r-1}{r^2}\left(\sum_{i=1}^r m_i\right)^2 < \frac{(r-1)(r+1)}{r^2}\left(\sum_{i=1}^r m_i^2 - m_r\right) < \\ &< \sum_{i=1}^r m_i^2 - m_r,\end{aligned}$$

a contradiction.

Case (b) is also immediately excluded as $(D_t)^2 \geq r - 1$ by Lemma 4.3 and thus

$$\frac{L.D_t}{\sum_{i=1}^r m_i} = \frac{L.D_t}{r} \geq \frac{\sqrt{r-1}}{r}\sqrt{L^2}$$

by the Hodge Index Theorem. In this way we obtained a contradiction with our assumption (4.2.1).

In the last case (c) we have

$$\frac{L.D_t}{\sum_{i=1}^{r-1} m_i} = \frac{L.D_t}{\sum_{i=1}^{r} m_i} < \sqrt{\frac{r-1}{r^2}L^2} < \sqrt{\frac{r-2}{(r-1)^2}L^2},$$

where the first inequality is just our assumption (4.2.1) and the second holds as $r \geq 3$. Hence the assumption (4.2.1) is satisfied for $(r-1)$ points and we conclude by induction.

□

In particular from the Theorem 4.2 it follows that:

Corollary 4.5 *If a surface X admits no fibration over a curve (e.g. a general surface of general type), then*

$$\varepsilon(L; r) \geq \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; r).$$

In particular the Nagata-Biran conjecture holds asymptotically.

The following example shows that the upper bound in our Theorem is optimal.

Example 4.6 *Let $X = \mathbb{P}^2$ with the polarization $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and let $r = 2$. Then the line through two given points x_1, x_2 computes*

$$\varepsilon(L, x_1, x_2) = \frac{1}{2} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L; 2)$$

and there is no fibration on $\mathbb{P}^2$.

More generally, let r be given and let X be a rational normal scroll in $\mathbb{P}^r$ and let $L = \mathcal{O}_X(1)$. The scroll is of curse fibered but the curves in the ruling are not the Seshadri curves. To see this let $x_1, \ldots, x_r \in X$ be points in general position. Then for a fiber F on the ruling passing through the set $x_1, \ldots, x_r$ we have

$$\frac{L.F}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} F} = 1.$$

On the other hand r points span a hyperplane in $\mathbb{P}^r$ i.e. there is a curve $D \in |L|$ passing through all of them with Seshadri quotient

$$\frac{L.D}{\sum_{i=1}^{r} \operatorname{mult}_{x_i} D} = \frac{r-1}{r} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\max}(L;r) < 1.$$

So in this case X is not fibered by the Seshadri curves.

Szemberg and Tutaj-Gasińska observe that in the case $r = 1$ the upper bound in Theorem 4.1 is optimal. They show that $\varepsilon(L;x) = \sqrt{\frac{3}{4}} \cdot \varepsilon_{max}(L;x)$ for a cubic in $\mathbb{P}^3$ which is of curse not fibered by Seshadri curves. We observe that their example is unique. This was suggested by Ein.

Theorem 4.7 *Let X be a projective surface and L an ample line bundle on X with*

$$\varepsilon(L,x) = \sqrt{\frac{3}{4}} \cdot \varepsilon_{max}(L;x), \text{ for all } x \in X. \qquad (4.7.1)$$

If there is no fibration on X then (X,L) is a cubic surface in $\mathbb{P}^3$.

Proof. The condition (4.7.1) in particular means that the Seshadri constant $\varepsilon(L;x)$ is submaximal. Hence by the real valued Nakai-Moishezon criterion [CP] it follows that there exists a computing curve. Let D_x be such a curve i.e.

$$\frac{L.D_x}{m_x} = \varepsilon(L;x),$$

where m_x denotes the multiplicity D_x at x. By Remark 2.2 we have also that D_x is irreducible and reduced.

Step 1.
We show that for every point $x \in X$ the multiplicity D_x at x is $m_x \geq 2$.

For our purposes it is enough to consider the open set $X_0 \subset X$ on which the multiplicity is constant, equal to m.

If $m = 1$ then

$$\frac{3}{4}L^2 = (L.D_x)^2 \geq L^2 D_x^2, \qquad (4.7.2)$$

where the inequality follows from the Hodge Index Theorem. The condition (4.7.2) implies that it must be $D_x^2 = 0$. A contradiction - X is not fibered by Seshadri curves.

If $m \geq 2$ then by Xu [Xu1] Lemma 1 we have that

$$D_x^2 \geq m(m-1)+1. \tag{4.7.3}$$

Applying this inequality to (4.7.1) and using the Hodge Index Theorem we obtain that

$$\frac{3}{4} \geq \frac{m(m-1)+1}{m^2}$$

or equivalently $\left(\frac{1}{2}m-1\right)^2 \leq 0$. This means that there is only one possibility namely $m=2$.

Step 2.
We show that $D_x^2 = 3$ and $\mathcal{O}_X(D_x)$ is ample.

We observe that $m=2$ implies $D_x^2 = 3$ and by (4.7.3) and (4.7.1) in the Hodge Index Theorem we have the equality. Hence there exist integers p and q such that

$$pL \equiv qD_x,$$

which in particular implies that $\mathcal{O}_X(D_x)$ is ample.

Step 3.
We show that for every x the curve D_x is rational.

We observe that on X we have two parameter family of curves $\{D_x \ni x\}_{x\in X_0}$ with $D_x^2 = 3$ and $\operatorname{mult}_x D_x = 2$. Since D_x is reduced, it can be the Seshadri curve at only finitely many points. This implies that fixing $x_0 \in X_0$ and taking $\Delta \times \Delta$ a neighborhood of x_0 (with Δ a unit disc) the deformation $\left\{D_{x_{(t,s)}} \ni x_{(t,s)}\right\}_{\Delta\times\Delta}$ determinates non-degenerate Kodaira-Spencer map

$$\rho : T_0\Delta \times T_0\Delta \longrightarrow H^0(D_{x_0}, \mathcal{O}_{D_{x_0}}(D_{x_0})).$$

This gives rise to two non-trivial sections $\rho\left(\frac{d}{dt}\right)$ and $\rho\left(\frac{d}{ds}\right)$ in $H^0(D_{x_0}, \mathcal{O}_{D_{x_0}}(D_{x_0}) \otimes \mathfrak{m}_{x_0})$ as in [EL] Corollary 1.2.
Let $f : Y \longrightarrow X$ be the blowing-up of X at $x \in X$, with the exceptional divisor E. By the projection formula we have

$$H^0(D_{x_0}, \mathcal{O}_{D_{x_0}}(D_{x_0}) \otimes \mathfrak{m}_{x_0}) \cong H^0(D'_{x_0}, f^*(\mathcal{O}_{D_{x_0}}(D_{x_0})) \otimes \mathcal{O}_Y(-E)|_{D'_{x_0}}), \tag{4.7.4}$$

where D'_{x_0} denotes the proper transform of D_{x_0}. So the line bundle

$$f^*(\mathcal{O}_{D_{x_0}}(D_{x_0})) \otimes \mathcal{O}_Y(-E)$$

has at least two independent sections. On the other hand

$$\deg\left(f^*(\mathcal{O}_{D_{x_0}}(D_{x_0}))\otimes\mathcal{O}_Y(-E)|_{D'_{x_0}}\right)=\left(f^*(\mathcal{O}_{D_{x_0}}(D_{x_0}))-E\right).D'_{x_0}=$$
$$=(f^*D_{x_0}-E).(f^*D_{x_0}-2E)=3-2=1.$$

This implies that D'_{x_0} is a rational curve, hence so is D_{x_0}.

This shows that X is covered by rational curves. By classification of surfaces X is itself a rational surface. This implies in particular that L and all $\mathcal{O}_X(D_x)$ are linearly equivalent.

Step 4.
We show that the linear system $|D_x|$ is base point free.

Let $y\in X$. With the point y we can associate the curve $D_y\in|D_x|$ with $\operatorname{mult}_y D_y\geq 2$. Let y_1 be a general smooth point on D_y. There exists the irreducible curve $D_{y_1}\in|D_x|$ such that $\operatorname{mult}_{y_1} D_{y_1}\geq 2$. Since D_y and D_{y_1} have no common irreducible component and they are numerically equivalent, then

$$3=D_y.D_{y_1}=\sum_{p\in D_y\cap D_{y_1}}(D_y.D_{y_1})_p, \tag{4.7.5}$$

where by $(D_y.D_{y_1})_p$ we denote the intersection multiplicity of D_y and D_{y_1} at p. Observe that $(D_y.D_{y_1})_{y_1}\geq 2$ implies $(D_y.D_{y_1})_y=0$. Indeed, if not then $(D_y.D_{y_1})_y\geq 2$ and we would obtain a contradiction with (4.7.5). This shows that $y\notin D_{y_1}$, hence y is not a base point of $|D_x|$.

Step 5.
We observe that for all x we have $h^0(X,\mathcal{O}_X(D_x))\geq 4$.

Note that D_x is ample and base point free, so the image of the induced map is 2-dimensional. Hence we can apply Bertini Theorem ([Ha] III 7.9.1) which tells us that a general member of the linear system D_x is smooth and irreducible. We have a 2-dimensional family of singular divisors D_x which implies that $h^0(X,\mathcal{O}_X(D_x))\geq 4$.

Step 6.
Finally we show that $|D_x|$ is very ample.

(a) $|D_x|$ separates points.
Let y_1 and y_2 be different points on X. With points y_1 and y_2 we associate the singular curves $D_{y_1}\in|D_x|$ and $D_{y_2}\in|D_x|$ respectively. If $y_1\notin D_{y_2}$ then we are done, if not then using the same arguments like in Step 4 we obtain that $y_2\notin D_{y_1}$.

(b) $|D_x|$ separates tangent vectors.
Let $y \in X$ and $\vec{v} \in T_y(X)$. With the point y we associate the singular curve $D_y \in |D_x|$. If $\vec{v} \notin T_y(D_y)$ then we are done. Suppose that all curves in $|D_x|$ passing through y have tangent vector $\vec{v}$ at y. This implies that these curves have intersection multiplicity at y with D_y at least 3 so they can not have any other point in common with D_y (unless they have a whole component in common). Since

$$h^0(\mathcal{O}_X(D_x) \otimes \mathfrak{m}_y) = h^0(\mathcal{O}_X(D_x)) - 1 \geq 3,$$

then there is an irreducible curve C in $|D_x|$ passing through y and general points $y_1 \in D_y$ and $y_2 \notin D_y$. The last condition implies that C and D_y have no common components. On the other hand $y_1 \in D_y \cap C$ forces $C.D_y \geq 4$, a contradiction.

This means that $|D_x|$ gives the embedding X as the surface of degree three in the projective space, hence X it must be the cubic surface in $\mathbb{P}^3$.

□

We conclude by proving the following improvement of Theorem 4.1.

Theorem 4.8 *Suppose that X is a projective surface and L is an ample line bundle on X. If*

$$\varepsilon(L;1) < \sqrt{\frac{7}{9}} \cdot \varepsilon_{max}(L;1),$$

then

(a) either X is a cubic in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$,

(b) or X is fibered by Seshadri curves of L.

Proof. We assume to the contrary that X is neither fibered nor the cubic. From the proof of Theorem 4.7 it follows immediately that the multiplicity of Seshadri curve D in general point of X is in this case $m \geq 3$. Applying (4.7.3) and the Hodge Index Theorem as usual, we get

$$L^2(m^2 - m + 1) \leq (L.D)^2 < \frac{7}{9} m^2 L^2$$

or equivalently

$$\frac{m^2 - m + 1}{m^2} < \frac{7}{9}.$$

It is elementary to see that this is not possible for $m \geq 3$.

□

5 R-R expected curves, Nagata-Biran conjecture and symplectic packing.

In this chapter we study R-R expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of Nagata-Biran Conjecture 2.6. Our objective here is to check in how far the number N_0 appearing in the conjecture is optimal. We construct examples showing that the Nagata-Biran Conjecture is not valid for small number of points and discuss cases where no such construction is possible.

5.1 Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$.

By a *polarization of type* (a, b) or by a *curve of type* (a, b) in the product $\mathbb{P}^1 \times \mathbb{P}^1$ we mean a curve of bidegree a, b.

Definition 5.1 *Let $D \subset X$ be a curve passing through the points $x_1, \ldots, x_r$ with multiplicities at least $m_1, \ldots, m_r$ respectively. We say that D is Riemann-Roch expected (for short R-R expected) if*

$$h^0(\mathcal{O}_X(D)) - \sum_{i=1}^{r} \binom{m_i + 1}{2} > 0.$$

This simply means that a curve D is R-R expected if its existence follows from the naive dimension count (note that it takes at most $\binom{m+1}{2}$ independent linear conditions on a linear system passing through a given point with multiplicity at least m).

Using this terminology in the context of multiple point Seshadri constants and Nagata-Biran Conjecture 2.6 we remark that

Remark 5.2

(1) On $(\mathbb{P}^2, \mathcal{O}(1))$ we have $N_0 = 9$ and the curves computing the Seshadri constant for $r \leq N_0$ points are R-R expected.

(2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with the polarization of type $(1, 1)$ we have $N_0 = 8$ and again all curves computing the Seshadri constant for at most 8 points are R-R expected.

This implies that in these two examples the number N_0 suggested by Biran cannot be lowered. However there are cases (e.g. $(1, 2)$ polarization, see [Sy1]) suggesting the Biran number N_0 might not be optimal even in the simple case of $\mathbb{P}^1 \times \mathbb{P}^1$. We address this question in this part. Before proceeding, we need some more notation. For a vector $M = (m_1, \ldots, m_r) \in \mathbb{Z}^r$ we define

$$
\begin{aligned}
|M| &:= \sum_{i=1}^{r} m_i, \\
\alpha(M) &:= \max\{|m_i - m_j| : \; i,j = 1,\dots,r\}, \\
\mathbf{l}(M) &:= \sum_{i=1}^{r} \binom{m_i+1}{2}.
\end{aligned}
$$

Lemma 5.3 *Let $M_1 = (m,\dots,m,m+\delta) \in \mathbb{Z}^r$ with $r \geq 2$ and an integer δ. If $|\delta| = c \cdot r + q$, with $c \in \mathbb{N}$, $0 \leq q < r$ and*

$$
M_2 = \\
(\underbrace{m + sgn(\delta)\cdot c,\dots,m+sgn(\delta)\cdot c}_{r-q}, \underbrace{m+sgn(\delta)\cdot(c+1),\dots,m+sgn(\delta)\cdot(c+1)}_{q}),
$$

then $\mathbf{l}(M_2) \leq \mathbf{l}(M_1)$ and the equality holds if and only if $|\delta| = 0$ or $|\delta| = 1$.

Proof. This is a simple computation:

$$
\begin{aligned}
\mathbf{l}(M_2) - \mathbf{l}(M_1) &= sgn(\delta)\cdot mcr - m\delta + sgn(\delta)\cdot qm + qc + \\
&\quad + \frac{1}{2}\left(q - \delta + sgn(\delta)\cdot cr + c^2 r + sgn(\delta)\cdot q - \delta^2\right).
\end{aligned}
$$

Substituting $\delta = sgn(\delta)\cdot c \cdot r + sgn(\delta) \cdot q$ and rearranging terms we obtain that

$$
\mathbf{l}(M_2) - \mathbf{l}(M_1) = -\frac{1}{2}\left[q(q-1) + c^2 r(r-1)\right] - qc(r-1).
$$

We observe that the equality holds only for $q = 0$ and $c = 0$ or $q = 1$ and $c = 0$.

□

An obvious consequence of this lemma is

Corollary 5.4 *Let $\mathcal{M}_p = \{M \in \mathbb{Z}^r : \; |M| = p\}$. Let M_0 be an element in $\mathcal{M}_p$ imposing the least theoretical number of conditions i.e. $\mathbf{l}(M_0) = \min\{\mathbf{l}(M) \mid M \in \mathcal{M}_p\}$. Then either $\alpha(M_0) = 0$, or if this is not the case, then $\alpha(M_0) = 1$.*

We have also

Corollary 5.5 *Let (X, L) be a polarized surface with Picard number $\varrho = 2$ and let $x_1,\dots,x_r \in X$ be fixed general points. If $M_D = (m_1,\dots,m_r) \in \mathbb{Z}^r$ is the multiplicity vector of a R-R expected submaximal irreducible and reduced curve D at $x_1,\dots,x_r$, then up to permutation M_D is of the form*

$$
M_D = (m,\dots,m,m+\delta) \text{ with } \delta \in \{-1,0,1\}.
$$

Proof. Since the Picard number $\varrho = 2$ and D is reduced and irreducible submaximal curve, then by Proposition 2.10 its multiplicity vector M_D, up to permutation, is of the form

$$M_D = (m, \ldots, m, m + \delta).$$

Suppose that $|\delta| \geq 2$. Then as the points are general, we have r different submaximal curves. By Lemma 5.3 there exists also a R-R expected submaximal curve D' with $\alpha(D') \leq 1$. This implies, again by generality of the points $x_1, \ldots, x_r$ the existence of at least $(r-1)r$ additional submaximal curves which contradicts Proposition 2.8. Hence $|\delta| \leq 1$.

□

5.2 Symplectic packing and the Nagata-Biran conjecture.

First we recall some basic definitions and properties.

Definition 5.6 *A symplectic manifold is a smooth n-dimensional manifold M over $\mathbb{C}$ with a closed and non-degenerate 2-form ω on it i.e. $d\omega = 0$ and on any tangent space T_pM, with $p \in M$, if for all $y \in T_pM$ we have $\omega\big|_{T_pM}(x, y) = 0$ then $x = 0$.*
The volume of (M, ω) we define as $Vol(M, \omega) = \dfrac{1}{n!}\displaystyle\int_M \omega \wedge \omega$.

Example 5.7 *The linear complex space $\mathbb{C}^n$ with $\omega_{std} = \sum_{k=1}^{n} dx_k \wedge dy_k$, where $z_k = x_k + iy_k$ $(k = 1, \ldots, n)$ are coordinates on $\mathbb{C}^n$, is a symplectic manifold of real dimension equal to $2n$. The form ω_{std} is called the standard symplectic form on $\mathbb{C}^n$.*

Let $\coprod_{q=1}^{N} B^{2n}(\lambda_q, \omega_{std})$ be the disjoint union of N balls of radii λ_q and (M, ω) be a symplectic manifold of real dimension equal to $2n$. By φ we denote a map $\varphi = \coprod_{q=1}^{N} \varphi_q : \coprod_{q=1}^{N} B^{2n}(\lambda_q, \omega_{std}) \longrightarrow (M, \omega)$ whose restriction to the q-th ball coincides with $\varphi_q : B^{2n}(\lambda_q, \omega_{std}) \longrightarrow (M, \omega)$.

Definition 5.8 *We say that φ is a symplectic embedding (or symplectic packing) if φ is an embedding and for all q we have $\varphi_q^* \omega = \omega_{std}$.*

Now consider a symplectic packing φ_λ of (M, ω) with N equal balls of radius λ. For a symplectic manifold of finite volume McDuff and Polterovich in [MP] introduced

$$v_N(M, \omega) = \sup_\lambda \frac{Vol(\text{Image } \varphi_\lambda)}{Vol(M, \omega)},$$

where the supremum we take over all $\lambda \in \mathbb{R}_+$ such that φ_λ exists. If $v_N(M,\omega) = 1$ then there exists a *full filling*, in the other case, i.e. $v_N(M,\omega) < 1$, there is a *packing obstruction. The packing number* of (M,ω) we define as

$$P_{(M,\omega)} := \\ 1 + \max\{N \in \mathbb{N} : \text{ there does not exists a full packing by } N \text{ equal balls}\}$$

(we use the convention that $\max \emptyset = 0$ and the maximum of an unbounded set is ∞).

In [Bi1] Biran proved the following

Theorem 5.9 *([Bi1]) Theorem 6.1.A 2) On $\mathbb{P}^1 \times \mathbb{P}^1$ with polarization of type (a,b) we have*

$$v_N = \min\left\{1, \frac{N}{2ab} \cdot \inf_{(\alpha,\beta)\in D_N}\left(\frac{a\alpha + b\beta}{2\alpha + 2\beta - 1}\right)^2\right\},$$

where D_N is the set of all non-negative solutions $\alpha, \beta, m_1, \ldots, m_N \geq 0$ for the system of Diophantine equations:

$$\begin{cases} 2\alpha\beta &= \sum_{q=1}^{N} m_q^2 - 1 \\ 2\alpha + 2\beta &= \sum_{q=1}^{N} m_q + 1 \end{cases}$$

In particular on $\mathbb{P}^1 \times \mathbb{P}^1$ with polarization of type $(1,1)$ we have:
$v_1 = \frac{1}{2},\ v_2 = 1,\ v_3 = \frac{2}{3},\ v_4 = \frac{8}{9},\ v_5 = \frac{9}{10},\ v_6 = \frac{48}{49},\ v_7 = \frac{224}{225}$ and $v_N = 1$ for any $N \geq 8$ ([Bi1]).
McDuff and Polterovich in [MP] for $(\mathbb{P}^2, \omega_0)$ obtained the following values:
$v_1 = 1,\ v_2 = \frac{1}{2},\ v_3 = \frac{3}{4},\ v_4 = 1,\ v_5 = \frac{20}{25},\ v_6 = \frac{63}{64},\ v_8 = \frac{288}{289}$ and $v_N = 1$ for any $N \geq 9$.

Later Biran proved that

Theorem 5.10 *([Bi2] Theorem 1.A.) For (M,ω) a closed symplectic 4-manifold with $[\omega] \in H^2(M,\mathbb{Q})$ there exists N_0 such that for every $N \geq N_0$, (M,ω) admits full symplectic packing by N equal balls. In fact, if for some $k_0 \in \mathbb{Q}$ the Poincaré dual to $k_0[\omega]$ can be represented by a symplectic submanifold of genus at least 1, then one can assume that $N_0 = 2k_0^2 \cdot Vol(M,\omega)$.*

In the language of the linear systems it means that for a polarized surface (X, L) there exists N_0 such that for all $N \geq N_0$ we have $v_N = 1$. If by k_0 we denote the smallest integer such that in the linear system $|k_0 L|$ there exists a smooth non-rational curve, then $N_0 = k_0^2 L^2$.

Now we want to study the surface $\mathbb{P}^1 \times \mathbb{P}^1$ in the context of Theorem 5.9. More precisely we want to find a relation between the number v_N and the existence R-R expected curves at N points.

Let L on $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a polarization of type (a, b). Since a generic member of $|L|$ is smooth on X, then $g_L = ab - a - b + 1$ and

$$\min\{k \in \mathbb{N} : \ g_{kL} > 0\} = \begin{cases} 2 & \text{if} \quad a = 1 \quad \text{or} \quad b = 1 \\ 1 & \text{if} \quad a \geq 2 \quad \text{or} \quad b \geq 2. \end{cases}$$

Definition 5.11 *For a polarization L of type (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$ we define the following constants:*

(1) $N_0 := \begin{cases} 8ab & \textit{for} \quad a = 1 \quad \textit{or} \quad b = 1 \\ 2ab & \textit{for} \quad a \geq 2 \quad \textit{and} \quad b \geq 2 \end{cases}$*,*

(2) $R_0 := \dfrac{3a^2 + 2ab + 3b^2}{2ab} + \dfrac{(a+b)\sqrt{2(a^2+b^2)}}{ab}$*,*

(3) $r_0 := \left\lfloor \dfrac{2(a+b)^2}{ab} \right\rfloor$*, where by $\lfloor . \rfloor$ we mean the round down.*

We observe that

Lemma 5.12 *For every positive integers a and b we have*

(1)

$$r_0 \leq R_0 \tag{5.12.1}$$

and the equality holds only for $a = b$,

(2)

$$R_0 \leq N_0, \tag{5.12.2}$$

and the equality holds if and only if $a = 1$ and $b = 1$ or $a = 2$ and $b = 2$.

Proof. (1) From direct calculations we obtain $R_0 \geq \frac{2(a+b)^2}{ab}$.
(2) Without loss of generality in the first case i.e. $a = 1$ or $b = 1$ we can assume that $b = 1$. The condition (5.12.2) is equivalent to

$$\frac{3a^2 + 2a + 3}{2a} + \frac{\sqrt{2}(a+1)\sqrt{(a^2+1)}}{a} \leq 8a.$$

Rearranging terms we obtain that

$$161\,(a-1)\left(a+\frac{1}{7}\right)\left(a+\frac{4\sqrt{3}+5}{23}\right)\left(a-\frac{4\sqrt{3}-5}{23}\right)\geq 0.$$

Now we can easily see that for all $a \geq 1$ and $b = 1$ the condition (5.12.2) is true and the equality holds only in one case, namely for $a = 1$.
In the second case we have to show that for all $a \geq 2$ and $b \geq 2$ we have

$$\frac{3a^2+2ab+3b^2}{2ab}+\frac{\sqrt{2(a^2+b^2)}(a+b)}{ab}\leq 2ab,$$

or equivalently

$$(4a^2-4a-1)b^2-(4a^2-2a)b-a^2\geq 0. \tag{5.12.3}$$

We observe that for

$$b\geq\frac{a(2a-1)+2a\sqrt{2(a^2-a)}}{4a^2-4a-1}$$

the condition (5.12.3) is true. On the other hand

$$\frac{a(2a-1)+2a\sqrt{2(a^2-a)}}{4a^2-4a-1}\leq 2$$

and the equality holds only for $a = 2$ and $b = 2$.

□

We would like to say more about values of R_0 and r_0 and relations between them. To do this, we introduce the following

Notation 5.13 *Since conditions in the last definition are symmetric, without loss of generality we can assume that* $a \geq b$. *We can write* a *in the unique way as* $a = k \cdot b + j$, *with* $k \geq 1$ *and* $j \in \{0, \ldots, b-1\}$.

We keep this notation for this part.

Now we prove the following

Lemma 5.14 *For integers* k, b, j *as above we have:*

$$\left(\frac{3}{2}+\sqrt{2}\right)k+\sqrt{2}+1<R_0<\left(\frac{3}{2}+\sqrt{2}\right)k+\sqrt{2}+6. \tag{5.14.1}$$

Proof. Since $a = k \cdot b + j$, then

$$R_0 = \frac{3k^2b^2 + 6kbj + 3j^2 + 2kb^2 + 2bj + 3b^2 + 2((k+1)b+j)\sqrt{(2(kb+j)^2 + 2b^2)}}{2(kb+j)b}.$$

We see that

$$R_0|_{j=0} \leq R_0 \leq R_0|_{j=b-1},$$

where by $.|_{j=\widetilde{j}}$ we mean the substitutions $j = \widetilde{j}$. Rearranging terms in $R_0|_{j=0}$ we get

$$R_0|_{j=0} = \frac{3}{2}k + \sqrt{2(k^2+1)} + \frac{3 + 2\sqrt{2(k^2+1)}}{2k} + 1.$$

To obtain our thesis it is enough to observe that

$$\sqrt{2} < \frac{3 + 2\sqrt{2(k^2+1)}}{2k} < 2$$

and

$$R_0|_{j=b-1} - R_0|_{j=0} < 3.$$

□

In this way we obtain a bound on R_0, which in fact depends only on k. Now we compute the value of r_0.

Lemma 5.15 *For any polarization of type* (a, b) *we have*

$$r_0 = \begin{cases} 2k+4 & \text{for} \quad j \in \left\langle 0, \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\ b \right) \cap \mathbb{N} \text{ and } k \geq 3 \\ 2k+5 & \text{for} \quad j \in \left\langle \frac{\sqrt{4k^2+4k-15}-2k+1}{4}\ b, \frac{1+\sqrt{k^2+2k-3}-k}{2}\ b \right) \cap \mathbb{N} \text{ and } k \geq 2 \\ 2k+6 & \text{for} \quad j \in \left\langle \frac{1+\sqrt{k^2+2k-3}-k}{2}\ b, b-1 \right\rangle \cap \mathbb{N} \text{ and } k \leq b-1+\frac{1}{b}. \end{cases}$$

Proof. Since $a = k \cdot b + j$, then from the Definition 5.11 it follows that

$$r_0 = 2k + 4 + \left\lfloor \frac{2kbj + 2b^2 + 2j^2}{b^2k + bj} \right\rfloor.$$

To prove our claim it is enough to show that for all $w \in \langle 0, b-1 \rangle$

$$\frac{2kbw + 2b^2 + 2w^2}{b^2k + bw} < 3, \tag{5.15.1}$$

or equivalently

$$2w^2 + b(2k-3)w - (3k-2)b^2 < 0. \tag{5.15.2}$$

If $k = 1$ then (5.15.2) becomes true for all $w \in \langle 0, b-1 \rangle$.
Let us assume that $k \geq 2$. We look on (5.15.2) as for the inequality with w as the variable. We calculate that for $w \in \left\langle 0, \frac{\sqrt{4k^2+12k-7}-2k+3}{4}\, b \right\rangle \cap \mathbb{N}$ the condition (5.15.2) holds.
Now we observe that for all $k \geq 2$ we have

$$\frac{\sqrt{4k^2+12k-7}-2k+3}{4} > 1,$$

but it means that (5.15.2) is true for every $w \in \langle 0, b-1 \rangle$.

Since the condition (5.15.1) holds, then to calculate r_0 it is enough to solve two inequalities

$$\frac{2kbw + 2b^2 + 2w^2}{b^2k + bw} \geq 2 \quad \text{and} \quad \frac{2kbw + 2b^2 + 2w^2}{b^2k + bw} < 1$$

as inequalities with the indeterminate w. This leads immediately to the conditions in the lemma.

□

Remark 5.16 *Note that the above lemma covers all possible situations. In particular $r_0 = 2k + 6$ happens only in the case when k is bounded i.e. a and b are relatively close.*

We should also note that

Remark 5.17 *Only for $k = 1$ we have that*

$$\frac{1 + \sqrt{k^2 + 2k - 3} - k}{2} b$$

is an integer.

Proof. It follows from the proof of Lemma 5.15 by easy computation.

□

In Lemma 5.12 we noted that $r_0 \leq R_0$. Thanks to the Lemma 5.14 we can better approximate the difference $R_0 - r_0$, namely

Remark 5.18 *For every polarization on* $\mathbb{P}^1 \times \mathbb{P}^1$ *of type* (a, b) *it holds*

$$R_0 - r_0 > \left(\sqrt{2} - \frac{1}{2}\right) k + \sqrt{2} - 5. \tag{5.18.1}$$

In particular it means that r_0 *and* R_0 *are "close" if and only if* (a, b) *is "close" to the diagonal* (b, b) *.*

On the boundary of the ample cone of $\mathbb{P}^1 \times \mathbb{P}^1$ we obtain another bound, which depends only on r_0. More precisely

Remark 5.19 *For every polarization on* $\mathbb{P}^1 \times \mathbb{P}^1$ *of type* (a, b) *we have*

$$R_0 < \frac{3}{2} r_0 + 6.$$

Proof. It follows from Lemmas 5.14 and 5.15.

□

Now we are in a good position to formulate the following

Theorem 5.20 *Let* $X = \mathbb{P}^1 \times \mathbb{P}^1$*. If* L *is a polarization of type* (a, b) *then there are no R-R expected submaximal curves on* X *through* $r \geq R_0$ *points.*

Proof. Fix r and suppose to the contrary that $D \subset X$ of type (α, β) is R-R expected and submaximal. We can assume that the multiplicity vector of D is $M_D = (m, \ldots, m, m + \delta)$, where $\delta \in \{-1, 0, 1\}$ and $m \in \mathbb{Z}$ (by Corollary 5.5). Hence the number of independent conditions imposed by M_D is

$$\mathbf{l}(M) = (r - 1)\binom{m+1}{2} + \binom{m+\delta+1}{2} = \frac{1}{2}[rm^2 + rm + 2m\delta + \delta^2 + \delta].$$

Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 2.8 there is no continuous family of submaximal curves, we must have

$$\alpha\beta + \alpha + \beta = \frac{1}{2}[rm^2 + rm + 2m\delta + \delta^2 + \delta],$$

or equivalently

$$\beta = \frac{rm^2 + rm + 2m\delta + \delta^2 + \delta - 2\alpha}{2(\alpha + 1)}. \tag{5.20.1}$$

The submaximality of D means that

$$\frac{a\beta + \alpha b}{rm + \delta} < \sqrt{\frac{2ab}{r}}. \tag{5.20.2}$$

Substituting $\sqrt{r} = t$, conditions (5.20.1) and (5.20.2) give us the inequality

$$\begin{aligned} 2tb\alpha^2 \quad & -(2\sqrt{2ab}t^2m + 2ta - 2tb + 2\sqrt{2ab}\delta)\alpha + \\ & +(at^2m + a\delta^2 + at^2m^2 + 2am\delta + a\delta - 2\sqrt{2ab}tm)t - 2\sqrt{2ab}\delta < 0. \end{aligned}$$

We view it as an inequality in the variable α. We know that the set of solutions is non-empty, hence

$$-2abt^3(t - \frac{2(a+b)}{\sqrt{2ab}})m + ((a-b)^2 - 2ab(1+\delta)\delta)t^2 + 2\sqrt{2ab}(a+b)\delta t + 2ab\delta^2 > 0. \tag{5.20.3}$$

If we assume that $t > \frac{2(a+b)}{\sqrt{2ab}} = \sqrt{r_0}$, then (5.20.3) is equivalent to

$$m < \frac{((a-b)^2 - 2ab(1+\delta)\delta)t^2 + 2\sqrt{2ab}(a+b)\delta t + 2ab\delta^2}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{5.20.4}$$

In the case $\delta = 0$ the inequality (5.20.4) is equivalent to

$$m < \frac{(a-b)^2}{2abt^2 - 2\sqrt{2ab}(a+b)t}. \tag{5.20.5}$$

If $t \geq \sqrt{R_0}$ then the right side of (5.20.5) is at most equal 1 and it must be $m = 0$, but this contradicts the definition of the multiple point Seshadri constant.

In the case $\delta = -1$ the inequality (5.20.4) is equivalent to

$$m < \frac{(a-b)^2t^2 - 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{5.20.6}$$

Since $\sqrt{r_0} \geq 1$, then $t \geq \sqrt{r_0}$ implies also $t \geq \dfrac{1}{\sqrt{r_0}}$ and

$$(a-b)^2t^2 - 2\sqrt{2ab}(a+b)t + 2ab \leq (a-b)^2t^2.$$

Applying the last inequality to (5.20.6) we obtain the condition (5.20.5) and we reduce our problem to previous one.

In the case $\delta = 1$, the inequality (5.20.4) is equivalent to

$$m < \frac{((a-b)^2 - 4ab)t^2 + 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{5.20.7}$$

Since our condition is still symmetric, then without loss of generality we may use Notation 5.13. We observe that for $t \geq \sqrt{k+4}$ there is the inequality:

$$\frac{((a-b)^2-4ab)t^2+2\sqrt{2ab}(a+b)t+2ab}{2t^3(abt-\sqrt{2ab}(a+b))} \leq \frac{(a-b)^2}{2abt^2-2\sqrt{2ab}(a+b)t}. \tag{5.20.8}$$

If $t \geq \sqrt{R_0}$ then (5.20.8) holds and

$$\frac{(a-b)^2}{2abt^2-2\sqrt{2ab}(a+b)t} < 1,$$

and in this case it can happen that (5.20.7) has a solution, namely $m = 0$. Since D is R-R expected then (5.20.1) holds and we obtain that only a fiber through one of the points $x_1, \ldots, x_r$ comes into consideration. It is easy to see that the Seshadri quotient given by the fiber is submaximal for at most $2k+2-\frac{2}{b}$ points, which gives a contradiction with our assumption $t \geq \sqrt{R_0}$.
□

To complete Theorem 5.20, we should find R-R submaximal curves for $r < R_0$ points. Before we begin, we make some trivial

Observation 5.21 *Let (X, L) be a polarized surface. Let $D \subset X$ be a curve which at r points gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$. If $\sqrt{\frac{L^2}{r}}$ is non-rational then D is submaximal.*

Proof. By assumption $\frac{L.D}{\sum_{i=1}^r m_i} \leq \sqrt{\frac{L^2}{r}}$. Since the number on the left side is always rational, then the equality can hold only in the case when $\sqrt{\frac{L^2}{r}}$ is rational.
□

It means only that in practice we will be looking for R-R curves which at r points give the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.

Analyzing the proof of Theorem 5.20 we observe that

Remark 5.22 *If $t < \sqrt{r_0}$ then (5.20.3) is equivalent to*

$$m > \frac{((a-b)^2 - 2ab(1+\delta)\delta)t^2 + 2\sqrt{2ab}(a+b)\delta t + 2ab\delta^2}{2t^3(abt - \sqrt{2ab}(a+b))}, \tag{5.22.1}$$

which for $\delta = -1$ gives a lower bound for m:

$$m > \frac{(a-b)^2 t^2 - 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}, \tag{5.22.2}$$

for $\delta = 0$

$$m > \frac{(a-b)^2}{2t(abt - \sqrt{2ab}(a+b))} \tag{5.22.3}$$

and for $\delta = 1$

$$m > \frac{(a^2 + b^2 - 6ab)t^2 + 2\sqrt{2ab}(a+b)t + 2ab}{2t^3(abt - \sqrt{2ab}(a+b))}. \tag{5.22.4}$$

Next, analyzing the value of the quotients in (5.22.2), (5.22.3) and (5.22.4) for $r \leq 2k+5$ we find R-R expected curves which give the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$. We observe that these curves depend on k and sometimes on j. More precisely we have the following

Proposition 5.23 *Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the polarization of type $(a,b) = (k \cdot b + j, b)$, where $k \geq 1$ and $j \in \{0, \ldots, b-1\}$. If $r \leq 2k+5$ then R-R curves which give the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$ are:*

(a) in the case $k = 1$

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	*The Seshadri quotient*	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(b+j)b}$
2	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{(b+j)b}$
3	$(1,1)$	0	$b-1$	1	0	$\frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1,1)$	0	$b-1$	1	-1	$\frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2,1)$	0	$b-1$	1	0	$\frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2,2)$	0	$\frac{1}{3}b$	1	1	$\frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2,1)$	$\frac{1}{3}b$	$b-1$	1	-1	$\frac{3b+j}{5}$	
7	$(4,4)$	0	$\frac{1}{7}b$	2	1	$\frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4,3)$	$\frac{1}{7}b$	$\frac{5}{9}b$	2	-1	$\frac{7b+3j}{13}$	
	$(3,1)$	$(3-\sqrt{7})b$	$b-1$	1	0	$\frac{4b+j}{7}$	

(b) in the case $k \geq 2$

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	*The Seshadri quotient*	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{2(kb+j)b}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$2k$	$(1,0)$	0	$b-1$	0	1	b	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\frac{(k+1)(2kb+j)}{2k^2+4k+1}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$\frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$

Proof. Since all curves from the tables fulfil the condition (5.20.1), then they are R-R expected. One can also check that for respective j we have

$$\frac{L.D}{\sum_{i=1}^{r} m_i} \leq \sqrt{\frac{L^2}{r}}.$$

□

As we observed, R-R submaximal curves depends sometimes on j. We see also that only in one case it can happen that for some $r \leq 2k+5$ and for some polarization we obtain two different types of submaximal curves namely

Remark 5.24 *In the case $k = 1$, if we take b such that*

$$\left(\left(3-\sqrt{7}\right)b,\ \frac{5}{9}b\right) \cap \mathbb{N} \neq \emptyset$$

then for $r = 7$ points and $(3-\sqrt{7})b < j < \frac{5}{9}b$ we have two types of R-R submaximal curves coming from type $(3,1)$ and $(4,3)$. The number of submaximal curves is altogether 14. Since we can have at most 8 reduced, irreducible and submaximal, it means that at least one of them is reducible. We see that the curve of type $(3,1)$ is a component of a curve of type $(4,3)$. Moreover we observe that if $j \leq \frac{3}{8}b$ then $\frac{7b+3j}{13} \leq \frac{4b+j}{7}$.

In another cases it can happen that different types of R-R expected curves give the same Seshadri quotient but this quotient is no longer submaximal.

Let L be polarization of type $(a,b) = (k \cdot b + j, b)$, with $k, b \geq 1$ and $j \in \{0, \ldots, b-1\}$. Now we want to show that for $r = 2k+6$ points there exist R-R submaximal curves at least for (a,b) such that $r_0 = 2k+6$ (see Lemma 5.15). We observe that R-R submaximal curves still depend on k and j and in general case we can not write an explicit form, as we could do this for $r \leq 2k+5$ points.
First we construct a sequence of R-R expected curves and later we compute their *submaximality area* i.e. we estimate polarizations for which our curves are submaximal.

Lemma 5.25 *Let $r = 2l+6$ with $l \in \mathbb{Z}_+$ and let D_n be a curve of type (α_n, β_n) with the multiplicity vector*

$$M_{D_n} = (m_n, \ldots, m_n, m_n + \delta_n)$$

and

$$h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha_n, \beta_n)) = \mathbf{l}(M_{D_n}) + 1. \qquad (5.25.1)$$

Let

(a) $\alpha_{n+1} := \beta_n + \alpha_n(l+3) - m_n(2l+6) - \delta_n$,

(b) $\beta_{n+1} := \alpha_n$,

(c) $m_{n+1} := \alpha_n - m_n$,

(d) $\delta_{n+1} := -\delta_n$,

With above assumptions a curve D_{n+1} *of type* $(\alpha_{n+1}, \beta_{n+1})$ *with the multiplicity vector*

$$M_{D_{n+1}} = (m_{n+1}, \ldots, m_{n+1}, m_{n+1} + \delta_{n+1}) \qquad (5.25.2)$$

fulfills the condition

$$h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\alpha_{n+1}, \beta_{n+1})) = \mathbf{l}(M_{D_{n+1}}) + 1. \qquad (5.25.3)$$

In particular it means that D_{n+1} *is R-R expected.*

Proof. Let D_{n+1} be a curve of type $(\alpha_{n+1}, \beta_{n+1})$ with the multiplicity vector as in (5.25.2). For such a vector we have

$$\mathbf{l}(M_{D_{n+1}}) = \sum_{i=1}^{2l+5} \binom{m_{n+1}+1}{2} + \binom{m_{n+1}+\delta_{n+1}+1}{2}. \qquad (5.25.4)$$

Rearranging terms in (5.25.4) and using (5.25.1), we obtain (5.25.3).

□

Remark 5.26 *Since for a curve* D *of type* $((l+1)(l+2), l+2)$ *with the multiplicity vector* $M_D = (l+1, \ldots, l+1, l+2)$ *at* $r = 2l+6$ *points we have* $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(D)) = \mathbf{l}(M_D)+1$*, then the construction in the previous lemma has non-empty set of solutions.*

Thanks to Lemma 5.25 and Remark 5.26 we define the following sequence of R-R expected curves

Definition 5.27 *Let* $n \in \mathbb{Z}_+$ *be a positive integer. The sequence* $\{D_n\}_{n\in\mathbb{Z}_+}$ *of R-R expected curves is given by the construction in Lemma 5.25 with*

- ◇ $\alpha_1 = (l+1)(l+2)$,
- ◇ $\beta_1 = l+2$,
- ◇ $m_1 = l+1$,
- ◇ $\delta_1 = 1$.

We observe that curves from Definition 5.27 have the following property:

Lemma 5.28 *For every positive integer n we have*

$$2\alpha_n + 2\beta_n - 2(l+3)m_n - \delta_n - 1 = 0. \tag{5.28.1}$$

Proof. The proof is very easy, we have to use the induction on n.

□

Previous lemma has a nice consequence, namely

Remark 5.29 *Rearranging terms in Lemma 5.25 for $n \geq 2$ we obtain:*

◇ $\alpha_{n+1} = (l+1)\alpha_n - \alpha_{n-1} + 1 = (l+1)\alpha_n - \beta_n + 1,$

◇ $m_{n+1} = \dfrac{(2l+4)\alpha_n - 2\alpha_{n-1} + 1 + \delta_n}{2l+6} = \dfrac{(2l+4)\alpha_n - 2\beta_n + 1 + \delta_n}{2l+6}.$

In particular it means that

Lemma 5.30 *For every positive integer $n \in \mathbb{Z}_+$ we have*

$$\frac{\alpha_n}{\beta_n} > l.$$

Proof. We use the induction on n.
Step 1.
For $n = 1$ we have

$$\frac{\alpha_1}{\beta_1} = \frac{(l+1)(l+2)}{l+2} = l + 1 > l.$$

Step 2.
If we assume the assertion for some $n \in \mathbb{Z}_+$ then for $n+1$ from Remark 5.29 we obtain

$$\begin{aligned} \frac{\alpha_{n+1}}{\beta_{n+1}} &= \frac{(l+1)\alpha_n - \beta_n + 1}{\alpha_n} = \frac{l\alpha_n + \alpha_n - \beta_n + 1}{\alpha_n} > \frac{l\alpha_n + l\beta_n - \beta_n + 1}{\alpha_n} \\ &= \frac{l\alpha_n + (l-1)\beta_n + 1}{\alpha_n} > l, \end{aligned}$$

where the first inequality follows from the inductive assumption.

□

Under the assumptions of Lemma 5.25 we obtain the following

Lemma 5.31 *Let $l, c \in \mathbb{Z}_+$ and $z \in \langle 0, c-1\rangle$. Let z_n be the smaller solution of*

$$\frac{(lc+z)\beta_n + c\alpha_n}{(2l+6)m_n + \delta_n} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{5.31.1}$$

with z as the indeterminate. The sequence $\{z_n\}_{n\in\mathbb{N}} \subset \langle 0, c-1\rangle$ is strictly decreasing and

$$\lim_{n\to\infty} z_n = \frac{1+\sqrt{l^2+2l-3}-l}{2}c$$

Proof. Let $\widetilde{z_1}$ and $\widetilde{z_2}$ be solutions of the equation

$$\frac{(lc+z)\beta_{n+1} + c\alpha_{n+1}}{(2l+6)m_{n+1} + \delta_{n+1}} = \sqrt{\frac{(lc+z)c}{l+3}} \tag{5.31.2}$$

with z as the indeterminate. Without loss of generality we may assume that $\widetilde{z_1} < \widetilde{z_2}$. By definition we have $z_{n+1} = \widetilde{z_1}$. We show that $\widetilde{z_2} = z_n$. If

$$\begin{aligned}
\gamma_n &= \frac{-6\alpha_n^3\beta_n - 2l\alpha_n^3\beta_n - 12l\alpha_n^2\beta_n^2 - 2l^2\alpha_n^2\beta_n^2 - 18\alpha_n^2\beta_n^2 + 4l^2\alpha_n^2m_n^2}{6\alpha_n^2m_n + 2l\alpha_n^2m_n + \alpha_n^2\delta_n + 2l\alpha_n\beta_n^2 + 6\alpha_n\beta_n^2 - 2l\beta_n^2m_n - 6\beta_n^2m_n - \beta_n^2\delta_n} \\
&+ \frac{24l\alpha_n^2m_n^2 + 36\alpha_n^2m_n^2 + 4l\alpha_n^2m_n\delta_n + 12\alpha_n^2m_n\delta_n + \alpha_n^2\delta_n^2 + 2l\alpha_n\beta_n^3 + 6\alpha_n\beta_n^3}{6\alpha_n^2m_n + 2l\alpha_n^2m_n + \alpha_n^2\delta_n + 2l\alpha_n\beta_n^2 + 6\alpha_n\beta_n^2 - 2l\beta_n^2m_n - 6\beta_n^2m_n - \beta_n^2\delta_n} \\
&+ \frac{4l^2\alpha_n\beta_n^2m_n + 36\alpha_n\beta_n^2m_n + 24l\alpha_n\beta_n^2m_n + 2l\alpha_n\beta_n^2\delta_n + 6\alpha_n\beta_n^2\delta_n - 24l\beta_n^2m_n^2}{6\alpha_n^2m_n + 2l\alpha_n^2m_n + \alpha_n^2\delta_n + 2l\alpha_n\beta_n^2 + 6\alpha_n\beta_n^2 - 2l\beta_n^2m_n - 6\beta_n^2m_n - \beta_n^2\delta_n} \\
&+ \frac{-36\beta_n^2m_n^2 - 4l^2\beta_n^2m_n^2 - 12\beta_n^2m_n\delta_n - 4l\beta_n^2m_n\delta_n - \beta_n^2\delta_n^2}{6\alpha_n^2m_n + 2l\alpha_n^2m_n + \alpha_n^2\delta_n + 2l\alpha_n\beta_n^2 + 6\alpha_n\beta_n^2 - 2l\beta_n^2m_n - 6\beta_n^2m_n - \beta_n^2\delta_n}
\end{aligned}$$

then

$$\begin{aligned}
\widetilde{z_1} &= \frac{18\alpha_n^2 + 6l\alpha_n^2 - 6\alpha_n\beta_n - 2l\alpha_n\beta_n - 24l\alpha_nm_n - 4l^2\alpha_nm_n - 36\alpha_nm_n}{2(l+3)\alpha_n^2}c \\
&+ \frac{-2l\alpha_n\delta_n - 6\alpha_n\delta_n + 24lm_n^2 + 4l^2m_n^2 + 36m_n^2 + 4lm_n\delta_n + 12m_n\delta_n + \delta_n^2}{2(l+3)\alpha_n^2}c \\
&- \frac{(6\alpha_n + 2l\alpha_n - 2lm_n - 6m_n - \delta_n)\gamma_n}{2(l+3)\alpha_n^2}c
\end{aligned}$$

and

$$\widetilde{z}_2 = \frac{18\alpha_n^2 + 6l\alpha_n^2 - 6\alpha_n\beta_n - 2l\alpha_n\beta_n - 24l\alpha_n m_n - 4l^2\alpha_n m_n - 36\alpha_n m_n}{2(l+3)\alpha_n^2}c$$

$$+\frac{-2l\alpha_n\delta_n - 6\alpha_n\delta_n + 24lm_n^2 + 4l^2m_n^2 + 36m_n^2 + 4lm_n\delta_n + 12m_n\delta_n + \delta_n^2}{2(l+3)\alpha_n^2}c$$

$$+\frac{(6\alpha_n + 2l\alpha_n - 2lm_n - 6m_n - \delta_n)\gamma_n}{2(l+3)\alpha_n^2}c$$

By definition we have that z_n is the smaller solution of (5.31.1) and it means that

$$z_n = \frac{-2l\alpha_n\beta_n - 6\alpha_n\beta_n - 2l^2\beta_n^2 - 6l\beta_n^2 + 36m_n^2 + 4l^2m_n^2 + 24lm_n^2}{2(l+3)\beta_n^2}c$$

$$+\frac{12m_n\delta_n + 4lm_n\delta_n + \delta_n^2 - (2lm_n + 6m_n + \delta_n)\gamma_n}{2(l+3)\beta_n^2}c.$$

After the substitution of γ_n into $\widetilde{z}_1, \widetilde{z}_2$ and z_n, we obtain that

$$\widetilde{z}_2 = \frac{-2l\alpha_n^2\beta_n - 6\alpha_n^2\beta_n + 6l\alpha_n^2 m_n + 18\alpha_n^2 m_n + 3\alpha_n^2\delta_n}{6\alpha_n^2 m_n + 2l\alpha_n^2 m_n + \alpha_n^2\delta_n + 6\alpha_n\beta_n^2 + 2l\alpha_n\beta_n^2 - 6\beta_n^2 m_n - 2l\beta_n^2 m_n - \beta_n^2\delta_n}c$$

$$+\frac{-6l\alpha_n\beta_n^2 - 2l^2\alpha_n\beta_n^2 + 2l^2\beta_n^2 m_n + 6l\beta_n^2 m_n + l\beta_n^2\delta_n}{6\alpha_n^2 m_n + 2l\alpha_n^2 m_n + \alpha_n^2\delta_n + 6\alpha_n\beta_n^2 + 2l\alpha_n\beta_n^2 - 6\beta_n^2 m_n - 2l\beta_n^2 m_n - \beta_n^2\delta_n}c$$

$$= z_n.$$

Since $\widetilde{z}_1 < \widetilde{z}_2$ and n was arbitrary, then the sequence $\{z_n\}_{n\in\mathbb{N}}$ is strongly decreasing.

On the other hand for every $n \in \mathbb{Z}_+$ we have

$$z_n + z_{n-1} = \frac{-(2l+6)\alpha_n\beta_n - (2l+6)l\beta_n^2 + (2l+6)^2m_n^2 + 2(2l+6)m_n\delta_n + 1}{(l+3)\beta_n^2}c$$

$$= \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + (2l+6)m_n^2 + 2m_n\delta_n] + 1}{(l+3)\beta_n^2}c.$$

Then from (5.25.1) we obtain that

$$z_n+z_{n-1} = \frac{(2l+6)[-\alpha_n\beta_n - l\beta_n^2 + 2\alpha_n\beta_n + 2\alpha_n + 2\beta_n - (2l+6)m_n - \delta_n - 1] + 1}{(l+3)\beta_n^2}c.$$

By (5.28.1) we have

$$z_n + z_{n-1} = \frac{(2l+6)[\alpha_n\beta_n - l\beta_n^2] + 1}{(l+3)\beta_n^2}c = \frac{(2l+6)\beta_n(\alpha_n - l\beta_n) + 1}{(l+3)\beta_n^2}c > 0,$$

and the inequality holds thanks to Lemma 5.30. Since $\{z_n\}_{n\in\mathbb{Z}_+}$ is strongly decreasing, then for every $n \in \mathbb{Z}_+$ we have $z_n > 0$. In particular it means that the sequence $\{z_n\}_{n\in\mathbb{Z}_+}$ is convergent. If this is the case, then

$$\lim_{n\to\infty} z_n = \frac{1}{2}\lim_{n\to\infty}(z_n + z_{n-1}) = \frac{1}{2}c\lim_{n\to\infty}\left(2\frac{\alpha_n}{\beta_n} - 2l + \frac{1}{(l+3)\beta_n^2}\right).$$

From Lemma 5.25 and Lemma 5.30 it follows that

$$\lim_{n\to\infty}\beta_n = +\infty \quad \text{and} \quad \lim_{n\to\infty}\frac{\alpha_n}{\beta_n} \geq l \geq 1. \tag{5.31.3}$$

Since $\lim_{n\to\infty}\frac{1}{(l+3)\beta_n^2} = 0$ and $\lim_{n\to\infty} z_n$ exists, then exists also $\lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$.
Let $g := \lim_{n\to\infty}\frac{\alpha_n}{\beta_n}$. From Remark 5.29 we obtain that

$$g = \lim_{n\to\infty}\frac{\alpha_{n+1}}{\beta_{n+1}} = \lim_{n\to\infty}\frac{(l+1)\alpha_n - \beta_n + 1}{\alpha_n} = \lim_{n\to\infty}\left((l+1) - \frac{\beta_n}{\alpha_n} + \frac{1}{\alpha}\right). \tag{5.31.4}$$

On the other hand by (5.31.3) we have $g \geq 1$ and hence there exists

$$\lim_{n\to\infty}\frac{\beta_n}{\alpha_n} = \frac{1}{g}.$$

Combining this fact with (5.31.4) we obtain the following equality

$$g = l + 1 - \frac{1}{g}.$$

Solutions are

$$g = \frac{l+1+\sqrt{l^2+2l-3}}{2} \quad \text{or} \quad g = \frac{l+1-\sqrt{l^2+2l-3}}{2}.$$

Since $\lim_{n\to\infty} z_n = (g-l)c$ we have either

$$\lim_{n\to\infty} z_n = \frac{1+\sqrt{l^2+2l-3}-l}{2}c \tag{5.31.5}$$

or

$$\lim_{n\to\infty} z_n = \frac{1-\sqrt{l^2+2l-3}-l}{2}c.$$

We proved before $z_n > 0$ for all $n \in \mathbb{Z}_+$, hence $\lim_{n\to\infty} z_n \geq 0$. This implies that (5.31.5) holds.

□

As a simple consequence of the previous two lemmas we obtain the following

Proposition 5.32 *Let $r = 2k+6$ be the number of points on $\mathbb{P}^1 \times \mathbb{P}^1$. Let L be a polarization of type $(a,b) = (k \cdot b + j, b)$ with $k, b \in \mathbb{Z}_+$ and $j \in \{0, \dots, b-1\}$. Let $\{z_n\}_{n\in\mathbb{N}}$ and (α_n, β_n) with m_n and δ_n be like in Lemma 5.31 and Definition 5.27 respectively. If for some n_0 there is $z_{n_0} < j < z_{n_0-1}$, then the curve D_{n_0} of type $(\alpha_{n_0}, \beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0}, \dots, m_{n_0}, m_{n_0} + \delta_{n_0})$ is R-R submaximal at r points. If $j = z_{n_0}$ or $j = z_{n_0-1}$ then $\sqrt{\frac{L^2}{r}}$ is rational and D_{n_0} computes this quotient.*

Proof. Since $z_{n_0} < j < z_{n_0-1}$, then by Lemma 5.31 we have

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} < \sqrt{\frac{(kb+j)b}{k+3}}.$$

This inequality means that the curve D_{n_0} of type $(\alpha_{n_0}, \beta_{n_0})$ with the multiplicity vector $M_{D_{n_0}} = (m_{n_0}, \dots, m_{n_0}, m_{n_0}+\delta_{n_0})$ is submaximal. By Lemma 5.25 the curve D_{n_0} is also R-R expected.
If $j = z_{n_0}$ or $j = z_{n_0-1}$ then

$$\frac{(kb+j)\beta_{n_0} + b\alpha_{n_0}}{(2k+6)m_{n_0} + \delta_{n_0}} = \sqrt{\frac{(kb+j)b}{k+3}}.$$

and $\sqrt{\frac{(kb+j)b}{k+3}} = \sqrt{\frac{L^2}{r}}$ must be rational. Previous equality also means that the curve of type $(\alpha_{n_0}, \beta_{n_0})$ computes the quotient $\sqrt{\frac{L^2}{r}}$.

□

In this way we obtain the following

Theorem 5.33 *Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ be a surface with the polarization of type (a, b). If $r \le r_0$ and $\sqrt{\frac{L^2}{r}}$ is non-rational, then there exist R-R expected submaximal curves at r points.*

Proof. If $r_0 \le 2k + 5$ then expected curves are given in Proposition 5.23. If $r_0 = 2k + 6$ then by Lemma 5.15

$$j \in \left\langle \frac{1 + \sqrt{k^2 + 2k - 3} - k}{2}\, b, b - 1 \right\rangle \cap \mathbb{N} \text{ and } k \le b - 1 + \frac{1}{b}.$$

From Remark 5.17 we have that $\frac{1+\sqrt{k^2+2k-3}-k}{2}\, b$ is integer only for $k = 1$. In this special case the number $\sqrt{\frac{L^2}{r}}$ is rational.
We should observe that the sequence from Lemma 5.31 is in fact a partition of the interval

$$\left\langle \frac{1 + \sqrt{k^2 + 2k - 3} - k}{2}\, b, b - 1 \right\rangle$$

and the rest of the proof follows from Proposition 5.32.

□

Using the algorithm introduced in Lemma 5.25 and Lemma 5.31 we calculate for $k = 1, 2, 3$ R-R expected curves at $r = 2k + 6$ points and their submaximality area.

Example 5.34 *For $k = 1$ we have that a polarization is of type $(a, b) = (b + j, b)$ with $j \in \{0, \dots, b - 1\}$ and $r = 8$.*
From Definition 5.27 and Lemma 5.31 and we have that

$$\alpha_1 = 6 \quad \beta_1 = 3 \quad m_1 = 2 \quad \delta_1 = 1 \quad j_1 = \tfrac{7}{9} b \quad \textit{and} \quad j_0 := b - 1.$$

From lemmas 5.25 and 5.31 it follows that

$$\alpha_2 = 10 \quad \beta_2 = 6 \quad m_2 = 4 \quad \delta_2 = -1 \quad j_2 = \tfrac{9}{16} b$$

$$\alpha_1 = 15 \quad \beta_3 = 10 \quad m_3 = 6 \quad \delta_3 = 1 \quad j_3 = \tfrac{11}{25} b \quad \textit{etc.}$$

For simplicity of our notation we write curves in the following table

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	*The Seshadri quotient*	$\sqrt{\frac{L^2}{r}}$
8	$(6,3)$	$\frac{7}{9}b$	$b-1$	2	1	$\frac{9b+3j}{17}$	$\sqrt{\frac{(b+j)b}{4}}$
	$(10,6)$	$\frac{9}{16}b$	$\frac{7}{9}b$	4	-1	$\frac{16b+6j}{31}$	
	$(15,10)$	$\frac{11}{25}b$	$\frac{9}{16}b$	6	1	$\frac{25b+10j}{49}$	
	$(21,15)$	$\frac{13}{36}b$	$\frac{11}{25}b$	9	-1	$\frac{36b+15j}{71}$	
	$(28,21)$	$\frac{15}{49}b$	$\frac{13}{36}b$	12	1	$\frac{49b+21j}{97}$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

We observe that in this case we can express the formula describing (α_n, β_n) *using an inductive notation, namely if* $j \in \left\langle \frac{2n+7}{(n+3)^2}b, \frac{2n+5}{(n+2)^2}b \right\rangle$, *with* $n \geq 0$, *then*

$\diamond$ $m_n = \begin{cases} (u+2)^2 & \textit{for} \quad n = 2u+1 \\ (u+1)(u+2) & \textit{for} \quad n = 2u \qquad u \in \mathbb{N}, \end{cases}$

$\diamond$ $(\alpha_n, \beta_n) = (m_n + m_{n+1}, m_{n-1} + m_n)$

$\diamond$ $\delta_n = (-1)^n$.

Moreover we have that

(1) $\lim\limits_{n \to \infty} j_n = 0$.

(2) For $j = 0$ *we get that* $\sqrt{\frac{L^2}{r}} = \frac{b}{2}$ *is rational and a curve of type* $(2,2)$ *with* $m = 1$ *and* $\delta = 0$ *computes this quotient.*

(3) For $j = 1$ *a curve of type* $((b+1)(2b+1), b(2b+1))$ *with* $m = b(b+1)$ *and* $\delta = 1$ *is submaximal.*

Doing the same type of calculations for $k = 2, 3$ we obtain the following curves

Example 5.35 *(a) for $k = 2$*

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	*The Seshadri quotient*	$\sqrt{\frac{L^2}{r}}$
10	$(12, 4)$	$\frac{13}{16}b$	$b-1$	3	1	$\frac{20b+4j}{31}$	$\sqrt{\frac{(2b+j)b}{5}}$
	$(33, 12)$	$\frac{31}{45}b$	$\frac{13}{16}b$	9	-1	$\frac{57b+12j}{89}$	
	$(88, 33)$	$\frac{78}{121}b$	$\frac{31}{45}b$	24	1	$\frac{154b+33j}{421}$	
	$(232, 88)$	$\frac{201}{320}b$	$\frac{78}{121}b$	64	-1	$\frac{408b+88j}{639}$	
	$(609, 232)$	$\frac{523}{841}b$	$\frac{201}{320}b$	168	1	$\frac{408b+88j}{1681}$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

(b) for $k = 3$

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	*The Seshadri quotient*	$\sqrt{\frac{L^2}{r}}$
12	$(20, 5)$	$\frac{21}{25}b$	$b-1$	4	1	$\frac{35b+5j}{7}$	$\sqrt{\frac{(3b+j)b}{6}}$
	$(76, 20)$	$\frac{73}{96}b$	$\frac{21}{25}b$	16	-1	$\frac{136b+20j}{191}$	
	$(285, 76)$	$\frac{267}{361}b$	$\frac{73}{96}b$	60	1	$\frac{513b+76j}{721}$	
	$(1065, 285)$	$\frac{991}{1350}b$	$\frac{267}{361}b$	225	-1	$\frac{1920b+285j}{2699}$	
	$(3976, 1065)$	$\frac{3693}{5041}b$	$\frac{991}{1350}b$	840	1	$\frac{7171b+1065j}{10081}$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

Assume that the number of points r is at least $r_0 + 1$ but smaller then R_0. We observe that in this case the situation seems to be out of control. We have conditions (5.20.5), (5.20.6) and (5.20.7) which should eliminate the most of multiplicities m. On the other hand, for r from the neighborhood of r_0 functions on the right side can obtain very high values. We observe that sometimes for $r_0 < r < R_0$ there are no R-R submaximal curves and this shows the following

Example 5.36 *Let L be polarization of type $(9,5)$. In this case $k=1, b=5$, $j=4$ and $R_0 = \frac{68}{15} + \frac{28}{45}\sqrt{53} \approx 9.063$. Analyzing conditions (5.20.5), (5.20.6) and (5.20.7) we obtain*
(1) $m < 0$, which is absurd, or
(2) $m = 1$, for $\delta = 0$.
Since now $r = 9$, then in the last case we have only one possibility: $\alpha = 4$ and $\beta = 1$. We see that this curve gives the quotient

$$\frac{L.D}{\sum m_i} = \frac{29}{9} > \sqrt{10} = \sqrt{\frac{L^2}{r}},$$

which is not submaximal.

On the other hand

Example 5.37 *Let L be polarization of type $(3,1)$. We have $k=3, b=1$, $j=0$, $R_0 = 6 + \frac{8}{3}\sqrt{2} \approx 11.962$ and hence $r = 11$. Analyzing conditions (5.20.5), (5.20.6) and (5.20.7) we obtain*
(1) $m < 0$, which is absurd, or
(2) $m \leq 3$, for $\delta = 0$.
We see that curve D of type $(5,1)$ with $m = 1$ gives the quotient

$$\frac{L.D}{\sum_{i=1}^{11} m_i} = \frac{8}{11} < \sqrt{\frac{6}{11}} = \sqrt{\frac{L^2}{r}},$$

which is submaximal.

This examples show that in general for in the range between r_0 and R_0 is difficult to prove for which number of points there are R-R submaximal curves. We can only generalized Examples 5.37 and 5.36, namely

Proposition 5.38 *Let L be a polarization of type (a,b). Let $r = 2k+2n+1$ with $n \in \mathbb{N}$. If $(1+n-\sqrt{2k+2n+1})b \leq j \leq b-1$, then a curve of type $(k+n,1)$ with $m=1$ and $\delta = 0$ gives the Seshadri quotient at most $\sqrt{\frac{L^2}{r}}$.*

Proof. By D we denote the curve of type $(k+n,1)$. D has the multiplicity vector $M_D = (1,\dots,1)$. Since $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k+n,1)) = 2k+2n+2 = \mathbf{l}(M_D)-1$, then D is R-R expected. We compute that

$$\frac{L.D}{\sum_{i=1}^{r} m_i} = \frac{(2k+n)b+j}{2k+2n+1},$$

hence $\frac{L.D}{\sum_{i=1}^{r} m_i}$ is at most $\sqrt{\frac{L^2}{r}}$ if and only if

$$(1+n-\sqrt{2k+2n+1})b \leq j \leq b-1.$$

The Seshadri quotient given by D is submaximal for $j \neq (1+n-\sqrt{2k+2n+1})b$ □

In this place we should note that $\langle(1+n-\sqrt{2k+2n+1})b, b-1\rangle \cap \mathbb{N} \neq \emptyset$ only for

$$0 \leq n < \frac{b+\sqrt{2(k+1)b^2-2b}-1}{b}.$$

We observe that for $k=1$ we have

$$\frac{b+\sqrt{4b^2-2b}-1}{b} < 3$$

and n can be at most 2. This means that curve of type $(n+1,1)$ with the multiplicity vector $(1,\dots,1)$ can be submaximal only for $r \leq 5$ points.

Now we are in a good position to formulate the following

Lemma 5.39 *Let D_h be a R-R expected curve of type $(h,1)$ through r points in general position. If $r \geq 2h+1$ and the multiplicity vector $M_{D_h} = (1,\dots,1)$ then D_h is irreducible.*

Proof. We proof this lemma by induction on h.

Step 1.
For $h=1$ we have that D_1 is of type $(1,1)$ with the multiplicity vector $M_{D_1}(1,1,1)$. If D_1 is reducible then D_1 decomposes in the sum of two fibers. Since points are in the general position, then the sum of two fibers gives the multiplicity vector $(1,1,0) \neq M_{D_1}$, a contradiction.

Step 2.
We assume our thesis for $h < h_0$.

Step 3.
We want to show that a curve D_{h_0} of type $(h_0,1)$ through $r \geq 2h_0+1$ points with the multiplicity vector $M_{D_{h_0}} = (1,\dots,1)$ is irreducible.
We assume to the contrary that D_{h_0} is reducible. Then we take the decomposition on irreducible components. There are two possibilities:

(1) D_{h_0} is the sum of curves of type $(1,0)$ and $(h,1)$ with $h < h_0$, or

(2) D_{h_0} is the sum of curves of type $(1,0)$ and $(0,1)$.

In first case we have

$$D_{h_0} = (h_0 - h) \cdot (1,0) + (h,1).$$

Since points are in the general position, then the multiplicity vector for a curve $(1,0)$ at r points is $(0,\ldots,0,1)$. Since $(h,1)$ is irreducible, then by inductive assumption we have that it goes through at least $2h+1$ points with multiplicities 1. Finely we obtain that curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ go through at least $(h_0-h)+(2h+1) = h_0+h+1$ points. Since $h_0+h+1 < 2h_0+1$ then the multiplicity vector of the sum of curves $(h_0-h)\cdot(1,0)$ and $(h,1)$ is different from $M_{D_{h_0}}$. A contradiction.

In second case we have

$$D_{h_0} = h_0 \cdot (1,0) + (0,1).$$

Since points are in the general position, then the multiplicity vector of the sum $h_0\cdot(1,0)+(0,1)$ at r points is $(\underbrace{1,\ldots,1}_{h_0+1},\underbrace{0,\ldots,0}_{r-h_0-1}) \neq M_{h_0}$. A contradiction.

□

Thanks to this lemma we can say more about the Seshadri constant on $\mathbb{P}^1 \times \mathbb{P}^1$. More precisely we have the following

Theorem 5.40 *For $(\mathbb{P}^1 \times \mathbb{P}^1, L)$ with L of type $(a,b) = (k\cdot b + j, b)$, with $b,k \geq 1$ and $j \in \{0,\ldots,b-1\}$ the Seshadri constants are like in the following tables*

(1) for $k=1$

r	*Type of curve*	*The submaximality area* $\ldots \le j \le \ldots$		m	δ	$\varepsilon(L;\ x_1,\ldots,x_r)$		$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	$=$	b	$\sqrt{2(b+j)b}$
2	$(1,0)$	0	$b-1$	0	1	$=$	b	$\sqrt{(b+j)b}$
3	$(1,1)$	0	$b-1$	1	0	$=$	$\frac{2b+j}{3}$	$\sqrt{\frac{2(b+j)b}{3}}$
4	$(1,1)$	0	$b-1$	1	-1	$=$	$\frac{2b+j}{3}$	$\sqrt{\frac{(b+j)b}{2}}$
5	$(2,1)$	0	$b-1$	1	0	$=$	$\frac{3b+j}{5}$	$\sqrt{\frac{2(b+j)b}{5}}$
6	$(2,2)$	0	$\frac{1}{3}b$	1	1	$\le$	$\frac{4b+2j}{7}$	$\sqrt{\frac{(b+j)b}{3}}$
	$(2,1)$	$\frac{1}{3}b$	$b-1$	1	-1	$=$	$\frac{3b+j}{5}$	
7	$(4,4)$	0	$\frac{1}{7}b$	2	1	$\le$	$\frac{8b+4j}{15}$	$\sqrt{\frac{2(b+j)b}{7}}$
	$(4,3)$	$\frac{1}{7}b$	$\frac{3}{8}b$	2	-1	$\le$	$\frac{7b+3j}{13}$	
	$(3,1)$	$\frac{3}{8}b$	$b-1$	1	0	$=$	$\frac{4b+j}{7}$	
8	$(6,3)$	$\frac{7}{9}b$	$b-1$	2	1	$\le$	$\frac{9b+3j}{17}$	$\sqrt{\frac{(b+j)b}{4}}$
	$(10,6)$	$\frac{9}{16}b$	$\frac{7}{9}b$	4	-1	$\le$	$\frac{16b+6j}{31}$	
	$(15,10)$	$\frac{11}{25}b$	$\frac{9}{16}b$	6	1	$\le$	$\frac{25b+10j}{49}$	
	$(21,15)$	$\frac{13}{36}b$	$\frac{11}{25}b$	9	-1	$\le$	$\frac{36b+15j}{71}$	
	$(28,21)$	$\frac{15}{49}b$	$\frac{13}{36}b$	12	1	$\le$	$\frac{49b+21j}{97}$	
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

(2) for $k \geq 2$

r	*Type of curve*	*The submaximality area* $\ldots \leq j \leq \ldots$		m	δ	$\varepsilon(L;$	$x_1, \ldots, x_r)$	$\sqrt{\frac{L^2}{r}}$
1	$(1,0)$	0	$b-1$	0	1	$=$	b	$\sqrt{2(kb+j)b}$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	
$2k$	$(1,0)$	0	$b-1$	0	1	$=$	b	$\sqrt{\frac{(kb+j)b}{k}}$
$2k+1$	$(k,1)$	0	$b-1$	1	0	$=$	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{2(kb+j)b}{2k+1}}$
$2k+2$	$(k,1)$	0	$b-1$	1	-1	$=$	$\frac{2kb+j}{2k+1}$	$\sqrt{\frac{(kb+j)b}{k+1}}$
$2k+3$	$(k+1,1)$	0	$b-1$	1	0	$=$	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{2(kb+j)b}{2k+3}}$
$2k+4$	$(k+1,1)$	$\frac{1}{k+2}b$	$b-1$	1	-1	$=$	$\frac{(2k+1)b+j}{2k+3}$	$\sqrt{\frac{(kb+j)b}{k+2}}$
	$(k^2+k,k+1)$	0	$\frac{1}{k+2}b$	k	1	$\leq$	$\frac{(k+1)(2kb+j)}{2k^2+4k+1}$	
$2k+5$	$(k+2,1)$	0	$b-1$	1	0	$=$	$\frac{2(k+1)b+j}{2k+5}$	$\sqrt{\frac{2(kb+j)b}{2k+5}}$
$2k+6$	$(k^2+3k+2,k+2)$	$\frac{k^2+3k+3}{k^2+4k+4}b$	$b-1$	$k+1$	1	$\leq$	$\frac{(k+2)(2kb+j+b)}{2k^2+8k+7}$	$\sqrt{\frac{(kb+j)b}{k+3}}$
	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$		
$2k+2n+7$	$(k+n+3,1)$	$(4+n-\sqrt{2k++2n+7})b$	$b-1$	1	0	$=$	$\frac{(2k+3+n)b+j}{2k+2n+7}$	$\sqrt{\frac{2(kb+j)b}{2k+2n+7}}$

5.3 Application in the problem of symplectic packing of $\mathbb{P}^1 \times \mathbb{P}^1$.

As an application of Theorem 5.20 we prove the following

Theorem 5.41 *Consider $X = \mathbb{P}^1 \times \mathbb{P}^1$ with the polarization L of type (a, b). For every $N \geq R_0$ the polarized surface (X, L) admits full symplectic packing by N equal balls.*

Proof. Fir r the number of points. Let $D \subset X$ of type (α, β) be a R-R submaximal curve. Let $M_D = (m, \ldots, m, m + \delta)$, where $\delta \in \{-1, 0, 1\}$ and $m \in \mathbb{Z}$, be its multiplicity vector. Since $h^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(\alpha, \beta)) = \alpha\beta + \alpha + \beta + 1$ and D is R-R expected, and by Proposition 2.8 there is no continuous family of submaximal curves, then we must have

$$2\alpha\beta + 2\alpha + 2\beta = rm^2 + rm + 2m\delta + \delta^2 + \delta. \tag{5.41.1}$$

Rearranging terms on the right side we obtain that

$$rm^2 + rm + 2m\delta + \delta^2 + \delta = \sum_{i=1}^{r} m_i^2 + \sum_{i=1}^{r} m_i$$

(by m_i we mean the multiplicity D at x_i). By Theorem 5.20 we have that for $r \geq R_0$ points there are no R-R submaximal curves. In particular it means that there are no curves such that (5.41.1) becomes true. If this is the case, then the system of Diophantine equations in Theorem 5.9 does not have solutions and by the same theorem for $N \geq R_0$ we have $v_N = 1$.

□

5.4 Conjecture.

As we remarked before, the Seshadri constant is known only in few examples and in every such case, computing curve was R-R expected. We observe also that on $\mathbb{P}^1 \times \mathbb{P}^1$ in that cases when there exists the full filling by N equal balls, there is no R-R submaximal curves at N points. This facts give us reason to formulate the following

Conjecture 5.42 *In the case $\mathbb{P}^1 \times \mathbb{P}^1$ the number N_0 in the Nagata-Biran Conjecture can be replaced by R_0.*

We observe that

Remark 5.43 *For a polarization of type (pa, pb) the number N_0, with respect to p, grows like the quadratic function. For the constant R_0 this is not the case. If we look at the Definition 5.11 then we can see, that R_0 is a rational function of a and b of degree 0 so the value of R_0 does not depend on p.*

In particular it means that

Remark 5.44 *The Biran number N_0 can be optimal applied only for polarizations of type (a, b) with a and b relative prime.*

References

[Ba1] Th. Bauer *Seshadri constants on algebraic surfaces*, Math. Ann. 313 (1999), 547-583.

[Bi1] P. Biran *Symplecic packing in dimension 4*, Geom. Funct. Anal. 7 (1997) 420-437.

[Bi2] P. Biran *A stability property of symplectic packing*, Invent. Math. 136 (1999) 123-155.

[CP] F. Campana, Th. Peternell *Algebraicity of the ample cone of projective varieties*, J. reine angew. Math. 407 (1990) 160-166

[De] J.-P. Demailly *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), Lect. Notes. Math. 1507 Springer-Verlag (1992) 87-104.

[EL] L. Ein and R. Lazarsfeld *Seshadri constants on smooth surfaces*, Astérisque 218 (1993) 177-186.

[F] L. Fuentes *Seshadri constants on ruled surfaces: the rational and the elliptic cases*, arXiv: math. AG/0503253 v1.

[FP] L. Fuentes and M. Pedreira *The projective theory of ruled surfaces*, arXiv: math.AG/0006204 v3.

[Har] B. Harbourne *Seshadri constants and very ample divisors on algebraic surfaces*, J. Reine Angew. Math, 559 (2003) 115-122.

[Ha] R. Hartshorne *Algebraic geometry*, Springer-Verlag, 1977.

[HK] J.-M. Hwang and J. Keum *Seshadri-exceptional foliations*, Math. Ann. 325 (2003) 287-297.

[Ku1] O. Küchle *Ample line bundles on blown up surfaces*, Math. Ann. 304 (1996) 151-155.

[La] R. Lazarsfeld *Positivity in Algebraic Geometry*, Springer-Verlag, 2004.

[MP] D. McDuff and L. Polterovich *Symplectic packings and algebraic geometry*, Invent. Math. 115 (1994) 405-429.

[Nak] M. Nakamaye *Seshadri constants and the geometry of surfaces*, J. Reine Angew. Math. 564 (2003) 205-214.

[Og] K. Oguiso *Seshadri constants in a family of surfaces*, Math. Ann. 323 (2002) 625-631.

[Sy1] W. Syzdek *Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$*, Ann. Polon. Math. 80 (2003), 223-230.

[Sy2] W. Syzdek *Seshadri constants of unisecant line bundles on ruled surfaces*, Annales Academiae Paedagogicae Cracoviensis. Studia Mathematica 23 (2004), 211-224.

[Sz1] T. Szemberg *Global and local positivity of line bundles*, Habilitationsschrift, Essen 2001.

[SzT-G] T. Szemberg and H. Tutaj-Gasińska *Seshadri fibrations on algebraic surfaces*, Annales Academiae Paedagogicae Cracoviensis. Studia Mathematica 23 (2004), 225-230.

[Xu1] G. Xu *Ample line bundles on smooth surfaces*, J. Reine. Angew. Math. 469 (1995) 199-209.