

Picard-Fuchs equations, Integrable Systems and higher algebraic K-theory

Pedro Luis del Angel

Pedro Luis del Angel, Cimat, Guanajuato, Mexico

Stefan Müller-Stach

Stefan Müller-Stach, McMaster University, Canada

Abstract. This paper continues the work done in [9] and is an attempt to establish a conceptual framework which generalizes the work of Manin [21] on the relation between non-linear second order ODE of type Painlevé VI and integrable systems. The principle behind everything is a strong interaction between K-theory and Picard-Fuchs type differential equations via Abel-Jacobi maps. Our main result is an extension of a theorem of Donagi and Markman [12].

Dedicated to Andrei Tyurin

Introduction

In [21], building up on his work on the functional Mordell conjecture, Y.I. Manin has found a framework in which the relation between Picard-Fuchs differential equations for the Legendre family of elliptic curves and the non-linear equations of type Painlevé VI due to Richard Fuchs can be connected to mathematical physics and the theory of integrable systems. One consequence of this is an approach for the understanding of mirror symmetry in the case of Fano manifolds. Inspired by his work and the related work of Griffiths [16] about differential equations satisfied by normal functions associated to classical

1991 *Mathematics Subject Classification.* 14C25, 19E20 .

We thank DFG, Schwerpunkt and Heisenberg program, Conacyt (grant 37557-E), CIMAT, Fields Institute, McMaster University and Universität Essen for supporting this project.

cycles, we have imitated the relation between periods and non-linear equations in the case of K3 surfaces in [9]. It turned out that the role of classical algebraic cycles, e.g. sums of points, has to be replaced by elements in higher algebraic K-groups [26], resp. motivic cohomology groups [2].

In this paper we attempt to push these ideas further in the case of families of Calabi-Yau manifolds $f : X \rightarrow B$ of relative dimension d . We study the differential equations obtained from algebraic cycles in higher Chow groups $CH^p(X, n)$ which have good intersection with each fiber of f when $2p - n - 1 = d$. These can be looked at as generalizations of non-linear ODE of order less than or equal to the rank of the local system associated to $R^d f_* \mathbb{C}$. We describe the connection to the theory of analytically completely integrable Hamiltonian systems (with non-complete fibers) and give several examples where the computations are partially worked out. This part builds up on work of Donagi and Markman [12]. The upshot is that algebraic K-theory brings a new flavour to the theory of Hamiltonian systems and conversely we get new insight into differential integrable systems via generalized Abel-Jacobi maps. There is a general understanding that mirror symmetry also plays a role in such investigations, but we are still far from understanding the connection to mathematical physics.

We want to emphasize that these notes have a certain survey character required by the publisher. However all new results come with a self-contained proof, except some announcements made in the last section.

1 Symbol map and Picard-Fuchs operators

Let $f : X \rightarrow B$ a smooth family of projective manifolds of dimension d over \mathbb{C} , where we assume that B is a smooth curve. Recall that the short exact sequence

$$0 \rightarrow \mathcal{T}_{X/B} \rightarrow \mathcal{T}_X \rightarrow f^* \mathcal{T}_B \rightarrow 0$$

induces the Kodaira-Spencer map:

$$\mathcal{T}_B = f_* f^* \mathcal{T}_B \rightarrow R^1 f_* \mathcal{T}_{X/B}$$

whose iterated d -fold cup product (a standard method in IVHS) produces a map

$$S^d \mathcal{T}_B \rightarrow S^d R^1 f_* \mathcal{T}_{X/B} \rightarrow \text{Hom}(f_* \Omega_{X/B}^d, R^d f_* \mathcal{O}) \cong (\mathcal{L}^\vee)^2$$

since $(R^d f_* \mathcal{O})^\vee = f_* \Omega_{X/B}^d =: \mathcal{L}$, a vector bundle over B , which is a line bundle in the case where all fibers have trivial canonical bundle, e.g.

in the case of Calabi-Yau manifolds. Summarizing, we obtain a symbol map

$$S^d \mathcal{T}_B \rightarrow (\mathcal{L}^\vee)^2,$$

or, what amounts to the same, a section of $S^d \Omega_B^1 \otimes (\mathcal{L}^\vee)^2$. It is also possible to write a version of this with logarithmic poles along the boundary divisor of a smooth compactification of B as follows:

$$S^d \mathcal{T}_B(-\log \Sigma) \rightarrow (\bar{\mathcal{L}}^\vee)^2,$$

where one has the following commutative diagram

$$\begin{array}{ccccc} \Delta & \subset & \bar{X} & \leftarrow & X \\ \downarrow & & \downarrow \bar{f} & & \downarrow f \\ \Sigma & \subset & \bar{B} & \leftarrow & B \end{array}$$

and $\bar{\mathcal{L}} := \bar{f}_* \Omega_{\bar{X}/\bar{B}}^d(\log \Delta)$. Here $\Sigma = \bar{B} \setminus B$ and $\Delta = \bar{f}^{-1}(\Sigma)$ are reduced normal crossing divisors. This situation can be obtained after semistable reduction.

For a smooth family as before $f : X \rightarrow B$, one has a canonical class of local systems on B , given by the vanishing of the Gauß-Manin connection ∇ on the higher direct images of \mathbb{C} ,

$$\nabla : R^k f_* \mathbb{C} \otimes \mathcal{O}_B \rightarrow R^k f_* \mathbb{C} \otimes \Omega_B^1.$$

Let \mathcal{D} be the sheaf of algebraic differential operators on B . Remember that \mathcal{D} admits a filtration by the order of the operator, in particular one has a short exact sequence

$$0 \rightarrow \mathcal{D}^{\leq m-1} \rightarrow \mathcal{D}^{\leq m} \rightarrow S^m \mathcal{T}_B \rightarrow 0$$

A homogenous element in the image of $\mathcal{D}^{\leq m}$ under the last map will be called an m -symbol. Differential operators act via the Lie derivative on relative holomorphic k -forms:

$$\mathcal{D} \otimes f_* \Omega_{X/B}^k \rightarrow R^k f_* \mathbb{C} \otimes \mathcal{O}_B, \quad \sigma \otimes \omega \mapsto \mathcal{L}_\sigma(\omega),$$

see [20]. If the rank of the (irreducible) local system $R^k f_* \mathbb{C}$ is m , to any m -symbol σ one can associate a *unique* differential operator D_σ , called the *Picard-Fuchs operator* of the symbol σ , which annihilates all sections of $R^k f_* \mathbb{C}$ ([4], Corollary III.2.3.2). The associated homogeneous differential equation $D_\sigma \varphi = 0$ is called the *Picard-Fuchs equation*. In particular every such D_σ is contained in the left kernel of $\mathcal{D} \otimes f_* \Omega_{X/B}^k \rightarrow R^k f_* \mathbb{C} \otimes \mathcal{O}_B$. The collection of all D_σ defines an ideal in the sheaf \mathcal{D} of algebraic differential operators. In other words, for every holomorphic relative k -form ω and every m -symbol σ there is a $(k-1)$ -form β such that

$$D_\sigma \omega = d_{\text{rel}} \beta,$$

which we call the *inhomogenous Picard-Fuchs equation*. More generally, if $\mathbb{R}^k f_* \mathbb{C}$ is not an irreducible representation of the monodromy group, let r be the rank of a maximal irreducible representation M contained in it. Then to any r -symbol σ there corresponds a unique differential operator D_σ annihilating all sections of M . Sometimes, if we do not want to specify σ we will write D_{PF} instead of D_σ . But the reader should be warned that these differential ideals are not always principal ideals. After integration over non-closed topological chains, the equation $D_\sigma \omega = d_{\text{rel}} \beta$ is mainly responsible for the appearance of new differential equations. We will use this construction in the case where $k = d$ and $f : X \rightarrow B$ is a family of Calabi-Yau manifolds.

2 Cycle class maps from motivic cohomology into Deligne cohomology

Recall the following definition:

Definition 2.1 *A Calabi-Yau variety X is a projective complex manifold of dimension d with the property that the canonical bundle $K_X = \Omega_X^d = \mathcal{O}_X$ is trivial and $h^0(\Omega_X^i) = 0$ for $1 \leq i \leq d - 1$.*

Why are Calabi-Yau manifolds interesting for our purpose? The point is that the motive $h^d(X)$ is interesting, as shown first by Griffiths [15] and later generalized by Voisin [29] in the case where $d = 3$. In fact, what we do in the following is equally interesting in the more general case where K_X is trivial, e.g. in the case of abelian varieties. There are cycle class maps defined by Bloch (see [3]) which can be realized by explicit Abel-Jacobi type integrals. There are formulas in [13] for real Deligne cohomology, general formulas have recently been given in [17] which we describe now. The maps

$$c_{p,n} : CH^p(X, n) \longrightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$$

were defined first by Bloch in [3] using Deligne cohomology with supports for classical cycles and a spectral sequence construction. The abelian groups $H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p))$ sit in exact sequences

$$0 \rightarrow J^{p,n}(X) \rightarrow H_{\mathcal{D}}^{2p-n}(X, \mathbb{Z}(p)) \rightarrow F^p H^{2p-n}(X, \mathbb{Z}(p)) \rightarrow 0,$$

where

$$J^{p,n} = \frac{H^{2p-n-1}(X, \mathbb{C})}{F^p + H^{2p-n-1}(X, \mathbb{Z}(p))}$$

are generalized intermediate Jacobians. Note that, if $n = 0$ we get the Griffiths intermediate Jacobians and for $p = 1, n - 1$ (resp.) the classical Jacobian, resp. Albanese tori, which are Abelian varieties. Griffiths intermediate Jacobians are not abelian varieties in general but they are

non-degenerate compact tori. The generalized intermediate Jacobians are no longer compact, but they are still complex manifolds and vary holomorphically in families. If we restrict to cycles homologous to zero, we obtain Abel-Jacobi type maps [3, 18, 17]

$$c_{p,n} : CH^p(X, n)_{\text{hom}} \longrightarrow J^{p,n}(X).$$

For those we have explicit formulas found by M. Kerr [18] (real versions can be found in [13]): consider an irreducible subvariety $W \subset X \times \square^n$, of codimension p , (using the cubical complex description), with coordinates (z_1, \dots, z_n) , together with projections $\pi_1 : X \times \square^n \rightarrow X$ and $\pi_2 : X \times \square^n \rightarrow \square^n$. Let $\alpha \in F^{d-p+1} H^{2d-2p+n+1}(X, \mathbb{C})$. One considers the current associated to W :

$$\begin{aligned} \alpha \mapsto & \frac{1}{(2\pi i)^{d-p+n}} \left[\int_{W \setminus (W \cap \pi_2^{-1}([0, \infty] \times \square^{n-1}))} \pi_2^*(\log z_1 d \log z_2 \wedge \dots \wedge d \log z_n) \wedge \pi_1^* \alpha \right. \\ & - (2\pi i) \int_{\{W \cap \pi_2^{-1}([0, \infty] \times \square^{n-1})\} \setminus \{W \cap \pi_2^{-1}([0, \infty]^2 \times \square^{n-2})\}} \pi_2^*(\log z_2 d \log z_3 \wedge \dots \wedge d \log z_n) \wedge \pi_1^* \alpha \\ & \left. + \dots + (-1)^{n-1} (2\pi i)^{n-1} \int_{W \cap \pi_2^{-1}([0, \infty]^{n-1} \times \square^1)} \pi_2^*(\log z_n) \wedge \pi_1^* \alpha + (-1)^n (2\pi i)^n \int_{\Gamma} \pi_1^* \alpha \right], \end{aligned}$$

where $\partial\Gamma = W \cap \pi_2^{-1}([0, \infty]^n)$ (the existence of Γ follows from W being homologous to zero, since the homology class of W is given by $W \cap \pi_2^{-1}([0, \infty]^n)$). If $2p - n - 1 = d$ (i.e., $n = 2p - d - 1$), then we can truncate these maps and get

$$\bar{c}_{p,n} : CH^p(X, n)_{\text{hom}} \longrightarrow \frac{H^0(X, \Omega_X^d)^*}{H_d(X, \mathbb{Z})}.$$

The group on the right hand side is no longer a manifold since we divide by a subgroup of rank bigger than the real dimension of $H^0(X, \Omega_X^d)$, which is 2 in our case. If we assume furthermore that $p \leq d$, then, given a cycle $W \in CH^p(X, 2p - d - 1)$, we have the formula

$$\bar{c}_{p,n}(W) = (-1)^n \frac{1}{(2\pi i)^{d-p}} \cdot \int_{\Gamma} \omega_X,$$

since the holomorphic d -form α vanishes on W , as $\dim(W) = p - 1 < d$. Γ is as above and satisfies

$$\partial\Gamma = \sum_j (f_1^{(j)}, \dots, f_n^{(j)})^{-1}([0, \infty]^n) = \sum_j \bigcap_{\ell=1}^n (f_\ell^{(j)})^{-1}([0, \infty]).$$

Note that $\dim_{\mathbb{R}}(\Gamma) = 2 \dim(W) - n = d$ in this situation. If $p > d$ then the truncated Abel-Jacobi map involves more integrals. An interesting example is $CH_{\text{hom}}^2(X, 2)$ for $d = 1$, i.e., where X is an elliptic curve. More generally such examples are given by the groups $CH^{d+1}(X, d+1)$ for arbitrary d .

3 Higher Chow groups of algebraic surfaces

Let X be a complex projective surface and $Z \subseteq X$ a normal crossing divisor with complement $U = X \setminus |Z|$. Then we have an exact localization sequence

$$\dots \rightarrow CH^{r+1}(U, r+1) \xrightarrow{\partial} CH^r(Z, r) \rightarrow CH^{r+1}(X, r) \rightarrow \dots$$

We are interested in the groups $CH^{r+1}(X, r)$ and, in view of this sequence, we are looking to have control over the group $CH^r(Z, r)$ which is of Milnor-type and hence combinatorially easy to describe. We also would like to give criteria when classes in $CH^r(Z, r)$ come from $CH^{r+1}(U, r+1)$ via the coboundary map ∂ . There is a spectral sequence

$$E_1^{a,b} = CH^{a+n}(Z^{[-a]}, 1-b) \implies CH^{n-1}(Z, -a-b)$$

computing higher Chow groups of Z from its smooth components Z_i . Let $Z^{[1]} := \coprod_i Z_i$ and $Z^{[2]} := \coprod_{i < j} Z_i \cap Z_j$. In particular for $r = 1, 2$, we have

$$CH^r(Z^{[1]}, r) \rightarrow CH^r(Z, r) \rightarrow CH^r(Z, r)^{\text{ind}} \rightarrow 0,$$

and

$$CH^r(Z, r)^{\text{ind}} = \text{Ker} (CH^{r-1}(Z^{[2]}, r-1) \rightarrow CH^r(Z^{[1]}, r-1)).$$

Hence

$$CH^1(Z, 1)^{\text{ind}} = \text{Ker} (\mathbb{Z}[Z^{[2]}] \rightarrow \text{Pic}(Z^{[1]})),$$

and

$$CH^2(Z, 2)^{\text{ind}} = \text{Ker} (CH^1(Z^{[2]}, 1) \rightarrow CH^2(Z^{[1]}, 1)).$$

For $r = 1$, this observation had been the motivation to find complex projective surfaces with large motivic cohomology groups $CH^2(X, 1)$ in [23]:

Theorem 3.1 [23] *Let $X \subseteq \mathbb{P}^3$ be a very general quartic hypersurface containing a line. Then $CH^2(X, 1)^{\text{ind}}$ is not a torsion group, i.e., it contains elements with no integer multiple which is decomposable.*

In [8], Collino has independently found indecomposable cycles in $CH^2(X, 1)$ on Jacobian surfaces. Related examples were found by Gordon and Lewis [14], the authors [9], Morihiko Saito [28], Voisin and Oliva (both unpublished). In [24] this method was further exploited in the case $r = 2$ using some new techniques of Asakura and Saito:

Theorem 3.2 [23, 24] *Let $X \subseteq \mathbb{P}^3$ be a very general hypersurface of degree d and let $Z = \cup Z_i$ be a NCD with Z_i very general hypersurface sections of degrees e_i . Then*

(1) *If $d \geq 5$, $CH^2(U, 2) \rightarrow CH^1(Z, 1)^{\text{ind}}$ is surjective.*

(2) Assume $d \geq 6$ and that all triples $(e_i, e_j, e_k) \neq (1, 1, 2)$. Then $CH^3(U, 3) \rightarrow CH^2(Z, 2)^{\text{ind}}$ is surjective as well.

Using the same methods one can also detect cycles in $CH^2(X, 1)$ and $CH^3(X, 2)$. Here is a typical application:

Example 3.3 Let K be a general cubic polynomial in $\mathbb{C}[x_0, \dots, x_3]$. On the family $X_{u,v} = \{F_{u,v} = x_0^5 + x_1x_2^4 + x_2x_1^4 + x_3^5 + ux_1^2x_2^3 + vx_0x_3K = 0\}$, $u, v \in \mathbb{C}$, of quintic surfaces, there exist elements $Z_{u,v}$ in $CH^3(X_{u,v}, 2)$ such that, for $u, v \in \mathbb{C}$ very general, every integer multiple of these elements is not contained in the image of $\text{Pic}(X_{u,v}) \otimes K_2(\mathbb{C})$.

Note that the parameter v is redundant since it is already contained in the coefficients of the cubic form K . However in the proof we fix $K = x_0^2x_1 + x_0x_1x_2 + x_0x_2^2 + x_2^3 + x_0x_1x_3$ and vary u and v in a 2-dimensional local parameter space to compute the infinitesimal data. In an appendix to [24], Collino has constructed the following very interesting examples on K3 surfaces:

Theorem 3.4 [24] On every (very) general quartic K3-surface S , there exists a 1-dimensional family of elements Z_t in $CH^3(S, 2)$ such that, for t very general, every integer multiple of these elements is not contained in the image of $\text{Pic}(S) \otimes K_2(\mathbb{C})$.

All these cycles originate from the existence of smooth bielliptic hyperplane sections C of genus 3 on S , which means that there exists a double cover $C \rightarrow E$ onto a smooth elliptic curve E . Their existence is guaranteed by the following fact: on every general quartic surface S there exists a 1-dimensional family of bielliptic curves C_t such that the underlying family E_t of elliptic curves has varying j -invariant.

We refer to section 5 for the construction of indecomposable elements in $CH^2(X, 1)$ of other K3-surfaces which can be detected using differential equations as in [9].

4 Poisson structures on generalized intermediate Jacobians

Here we define the structure of a completely integrable Hamiltonian system with in general non-compact leaves on the generalized intermediate Jacobians from section 2. The normal functions associated to cycles in higher Chow groups are shown to be isotropic subvarieties of the corresponding Poisson (i.e., degenerate symplectic) structure. In the case of classical intermediate Jacobians this is a theorem of Donagi and Markman [12]. See also Voisin's book [30, page 12-14] and compare

with [21] in the case of elliptic curves. Many famous classical examples of integrable systems can be found in [19].

Definition 4.1 (cf. [12]) *A holomorphic symplectic structure on a complex manifold M is given by a holomorphic $(2,0)$ -form $\sigma \in H^0(X, \Omega_X^2)$ such that the contraction operator*

$$\lrcorner\sigma : T_M \rightarrow \Omega_M^1$$

induces an isomorphism.

In particular, the complex dimension of M is even. Under this isomorphism, the differential df of every function f corresponds to a vector field X_f . The flow of X_f determines a dynamical system and one wants to know when it is integrable.

Definition 4.2 *A complex Poisson manifold is a complex manifold with a bracket $\{-, -\}$ on C^∞ -functions which is a derivation in each argument and satisfies a Jacobi identity (Lie algebra structure). Every symplectic structure induces a Poisson structure on M via $\{f, g\} := \sigma(X_f, X_g)$. On the other hand, any Poisson manifold determines a global tensor $\Psi \in H^0(M, \Lambda^2 T_M)$ with the property $\Psi(df, dg) = \{f, g\}$, which in the symplectic case is just $(\lrcorner\sigma)^{-1}$. The rank of a Poisson manifold M is given by the rank of Ψ . An analytically completely integrable Hamiltonian system is a set of pairwise Poisson commuting analytic functions F_1, \dots, F_m such that the analytic map $F = (F_1, \dots, F_m) : M \rightarrow B \subset \mathbb{C}^m$ is submersive with rank equal to $\dim M - \frac{1}{2}\text{rank}\Psi$ at every point.*

Each of the functions F_i is called a Hamiltonian function and the corresponding vector fields are called Hamiltonian vector fields. In this case Liouville's theorem guarantees that the fibers of F have a natural affine structure and the flow of the Hamiltonian fields X_{F_i} is linear. Hence they are generalized tori and compact if and only if F is proper. A generalized torus here is a quotient of \mathbb{C}^n by a discrete subgroup \mathbb{Z}^g where $2g \leq n$. We say that the Hamiltonian system is algebraically completely integrable, if $F : M \rightarrow B$ is a surjective algebraic morphism, where degenerate algebraic fibers are allowed. This property implies that such systems are integrable via generalized Theta-functions. Here is one more important definition:

Definition 4.3 *A subvariety Z of a symplectic manifold (M, σ) is Lagrangian, if the tangent space $T_{Z,z}$ to each generic point $z \in Z$ is isotropic and co-isotropic, i.e., maximally isotropic. If (M, Ψ) is only Poisson, then we replace the tangent bundle T_M by the image T of the*

induced map $\lfloor \Psi : \Omega_M^1 \rightarrow T_M$ in the definition of Lagrangian subspace, see [12, Def. 2.5].

Note that the rank of Ψ can vary, but there always exists a stratification of M into Poisson strata of constant rank r of Ψ , which are themselves foliated into symplectic leaves, i.e., on which there is an induced symplectic structure of rank r .

Let $f : X \rightarrow B$ a smooth and locally universal family of d -dimensional Calabi-Yau manifolds with smooth base B and \mathcal{H}^d be the flat vector bundle on B associated to the d -th cohomology of the fibers. We also denote by $\mathcal{H}_{\mathbb{Z}}^d \subset \mathcal{H}^d$ the local system of integral cohomology classes. The set of Hodge bundles is given by holomorphic subbundles

$$\mathcal{F}^d \subseteq \dots \subseteq \mathcal{F}^1 \subseteq \mathcal{F}^0 = \mathcal{H}^d.$$

The bundle \mathcal{F}^d is the same as the relative dualizing sheaf $f_*\omega_{X/B}$ and is a line bundle since the fibers are Calabi-Yau manifolds. These vector bundles are equipped with the Hodge metric. In particular we have metric isomorphisms for all i ,

$$\mathcal{H}^d/\mathcal{F}^i \cong (\mathcal{F}^{d-i+1})^\vee,$$

induced by the duality pairing

$$\langle -, - \rangle : \mathcal{H}^d \otimes \mathcal{H}^d \rightarrow \mathcal{O}_B.$$

The Gauss-Manin connection on \mathcal{H}^d is given as

$$\nabla : \mathcal{H}^d \rightarrow \mathcal{H}^d \otimes \Omega_B^1.$$

Using base change, we now pass to \tilde{B} which is the total space of the line bundle $f_*\omega_{X/B}$ with the zero-section removed. All vector bundles on B can be pulled back to \tilde{B} along the natural projection map $p : \tilde{B} \rightarrow B$ and they are denoted by $\tilde{\mathcal{H}}^d$, $\tilde{\mathcal{H}}_{\mathbb{Z}}^d$ and $\tilde{\mathcal{F}}^i$ respectively. Assume now that $2p - n - 1 = d$, not necessarily with $p \leq d$. The family of generalized intermediate Jacobians $\mathcal{J}^{p,n} \rightarrow B$ pulls back to $\tilde{\mathcal{J}}^{p,n} \rightarrow \tilde{B}$. Note that $\tilde{\mathcal{F}}^d$ has a tautological section η , which in each point of \tilde{B} picks up the holomorphic d -form defined by this point. The Gauß-Manin connection pulled back to \tilde{B} and applied to η induces an isomorphism

$$\nabla_{\bullet} \eta : T_{\tilde{B}} \xrightarrow{\cong} \tilde{\mathcal{F}}^{d-1}$$

and hence, dually, an isomorphism between $\Omega_{\tilde{B}}^1$ and $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$, see [12, thm. 7.7]. The proof of this fact is easy and consists of identifying the exact sequences

$$0 \rightarrow T_{\tilde{B}/B} \rightarrow T_{\tilde{B}} \rightarrow p^*T_B \rightarrow 0,$$

and

$$0 \rightarrow \tilde{\mathcal{F}}^d \rightarrow \tilde{\mathcal{F}}^{d-1} \rightarrow \tilde{\mathcal{F}}^{d-1}/\tilde{\mathcal{F}}^d \rightarrow 0,$$

via the map $\nabla_{\bullet}\eta$ and the identification of $T_{B,b} = H^1(X_b, T_{X_b})$ with $H^1(X_b, \Omega_{X_b}^{d-1})$, using the natural isomorphism $T_{X_b} \cong \Omega_{X_b}^{d-1}$.

Since the cotangent bundle on any complex manifold carries a natural holomorphic symplectic structure, we conclude that $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$ carries a natural holomorphic symplectic structure, given by a 2-form $d\theta$, where θ is the tautological 1-form on $\Omega_{\tilde{B}}^1$. We need the following description of θ from [30]: at a point (τ, b) of $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$ over $b \in \tilde{B}$, we have that

$$\theta_{(\tau,b)} = \pi^* \langle \tau, \nabla \eta \rangle,$$

where $\pi : \tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2 \rightarrow \tilde{B}$ is the projection map. This fact follows immediately from the definition of the contact form on any cotangent bundle together with the isomorphism induced by $\nabla_{\bullet}\eta$. The resulting 2-form $d\theta$ is a closed and non-degenerate holomorphic 2-form on $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$ and defines therefore a symplectic holomorphic structure such that the fibers of π are easily seen to be Lagrangian as in [30, 12].

Now the universal covering space of $\tilde{\mathcal{J}}^{p,n}$ is given by $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^p$, which maps surjectively onto $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$ if $p \geq 2$. Therefore we can pull back $d\theta$ to $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^p$ to get a Poisson structure. Note that the rank does not change under pullback, so that we cannot expect to get a symplectic structure anymore if $p > 2$. However the fibers of the map to \tilde{B} are still maximally isotropic subspaces since they come from pullbacks of maximally isotropic subspaces.

Lemma 4.4 *$d\theta$ descends to a closed 2-form σ on $\tilde{\mathcal{J}}^{p,n}$. Furthermore all fibers of $\pi : \tilde{\mathcal{J}}^{p,n} \rightarrow \tilde{B}$ are maximally isotropic.*

Proof As we remarked above, θ is given by the inner product $\langle -, \nabla \eta \rangle$ induced by the Hodge pairing on \mathcal{H}^d and where η is as above. Now, if two sections τ, τ' of $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^p$ differ by a flat section $\lambda \in \tilde{\mathcal{H}}_{\mathbb{Z}}^d$, then θ differs by $\pi^* \langle \lambda, \nabla \eta \rangle$. But since λ is flat,

$$d\langle \lambda, \nabla \eta \rangle = \langle \nabla \lambda, \nabla \eta \rangle = 0$$

and $d\theta$ is therefore invariant. Let σ be the induced form. The intermediate Jacobian fibers are Lagrangian by construction, since they have this property for $p = 2$. \square

With slightly more work one can even show that σ remains an exact form, see [12]. For $p = 2$, this structure will descend to an honest symplectic structure on the total space $\tilde{\mathcal{J}}^{2,n} \rightarrow \tilde{B}$ if $d \leq 3$ where $n = 3 - d$. In this case the fibers are generalized Lagrangian tori. The reader may wonder about the Hamiltonian functions: they only come

up when we choose a level structure, i.e., a basis (a_i) for the homology group $H_d(X_b, \mathbb{Z})$, respecting the polarized structure, e.g. a symplectic basis for d odd. Then the fiber integrals

$$F(b) = \int_{a_i} \eta(b)$$

for $i = 0, \dots, b_d/2$ (assuming d odd here) define pairwise commuting Hamiltonian functions, called the canonical coordinates in mathematical physics, see [5].

On the other hand, if $p = 1$, then by our assumption $2p - n - 1 = d$ we only get the possibility $d = 1$ and $n = 0$, which leads essentially to the consideration of $CH^1(X)$ for an elliptic curve X .

So far we have proved the first part of the theorem of Donagi and Markman. Their second result says that (for $d = 2p - 1$ and $n = 0$) all quasi-horizontal normal functions (e.g., arising from cycles in higher Chow groups which are defined over B and hence over \tilde{B}) have Lagrangian subspaces as images under the Abel-Jacobi map.

A similar result holds for generalized intermediate Jacobians, but note that if $p \geq 3$, then we only have a surjection $\tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^p \rightarrow \tilde{\mathcal{H}}^d/\tilde{\mathcal{F}}^2$ and therefore we can only expect the image of the normal functions to be isotropic. We also refine the result to the case where the normal function is defined on an algebraic sublocus $B' \subset B$:

Theorem 4.5 *Assume that $p \leq d = 2p - n - 1$ and $p \geq 2$. Let $\nu : B' \rightarrow \mathcal{J}^{p,n}$ any quasi-horizontal normal function defined on a (locally) closed subset of B , i.e., one that satisfies $\nabla\nu \in \mathcal{F}^{p-1} \otimes \Omega_B^1$. Then the image of $\tilde{\nu}$ in $\tilde{\mathcal{J}}^{p,n}$ is an isotropic subvariety.*

Proof The essential point is that $\tilde{\nu}$ is isotropic if and only if $\tilde{\nu}^*\theta$ is a closed form, because then $d\theta$ will be zero along the image of $\tilde{\nu}$. To prove this, we express again θ using $\langle -, \nabla\eta \rangle$ and we get

$$d(\tilde{\nu}^*\theta) = d\langle \tilde{\nu}, \nabla\eta \rangle = d\langle \nabla\tilde{\nu}, \eta \rangle = 0,$$

since, by quasi-horizontality, we have $\nabla\tilde{\nu} \in \tilde{\mathcal{F}}^{p-1} \otimes \Omega_B^1$ and the pairing $\langle \tilde{\mathcal{F}}^{p-1}, \eta \rangle$ already gives zero by type reasons, since $\eta \in \tilde{\mathcal{F}}^d$ and $p-1 \geq 1$, i.e., η pairs to zero with $\tilde{\mathcal{F}}^{p-1}$. \square

Note that a normal function may be multivalued. The theorem shows that such normal functions defined on the whole of B give rise to Lagrangian (maximally isotropic) subvarieties. We would also like to say that the projection of $\tilde{\nu}$ in $\tilde{\mathcal{J}}^{p,n}/\tilde{\mathcal{F}}^1$ is an isotropic subvariety, but this is not in general a manifold anymore.

Question 4.6 *Do exotic symplectic structures like $\mathcal{J}^{p,n}/\mathcal{F}^1$ have a rigorous mathematical meaning in another context?*

5 More examples

As we remarked in section 4, the most natural and interesting *time-independent* Hamiltonian systems occur in dimensions $d \leq 3$ when $p = 2$. The case $d = 1$ was treated in the paper of Manin [21]: for the family of elliptic curves given by the equation $y^2 = x(x-1)(x-t)$, one has the inhomogeneous equation

$$\left[t(1-t) \frac{\partial^2}{\partial t^2} + (1-2t) \frac{\partial}{\partial t} - \frac{1}{4} \right] \frac{dx}{y} = \frac{1}{2} d_{E/\mathbb{P}^1} \frac{y}{(x-t)^2},$$

where the derivatives are understood in the covariant sense, i.e., $\frac{\partial x}{\partial t} = 0$ since x is a flat coordinate and $d_{E/\mathbb{P}^1} t = 0$ by definition of the relative differential. In uniformized coordinates the non-linear equations take the equivalent form

$$\frac{d^2}{d\tau^2} z(\tau) = \frac{1}{(2\pi i)^2} \sum_{2e=0} \alpha_e \mathfrak{p}_z(z+e, \tau),$$

where e runs through all 2-torsion points and \mathfrak{p}_z is the derivative of the Weierstrass \mathfrak{p} -function. The Hamiltonian function governing the dynamical flow of the *time-dependent* Hamiltonian system in [21] is given by

$$H = \frac{y^2}{2} - \frac{1}{(2\pi i)^2} \sum_{2e=0} \alpha_e \mathfrak{p}(z+e, \tau)$$

in (y, z, τ) -space. It is very interesting to compare this to our approach and formulate a time-dependent version, see [10].

In dimension 3 we have the work of Griffiths [16]. This case is only partly understood and we just mention that there is a possibility that families of cycles in $CH_{\text{hom}}^2(X)$ on Calabi-Yau 3-folds (e.g. differences of pairs of lines on quintic 3-folds) can be characterized by their inhomogeneous Picard-Fuchs differential equations. The family of quintic 3-folds is discussed in [6, 7] in connection with arithmetic questions.

It remains to treat the case $d = 2$: here we have started to give some examples related to Kummer surfaces in [9]. Their equations were given by

$$X_b = \{(x : y : z : w) \in \mathbb{P}^3 \mid xyz(x+y+z+bw) + w^4 = 0\}.$$

This is a one-dimensional family of K3 surfaces (quartic) with generic Picard number 19 (see [25] for this and other properties). Its mirror

family is the family of all quartic K3 surfaces

$$f = \sum_{i=1}^4 x_i^4 - t \prod_{i=1}^4 x_i = 0.$$

let us work out the inhomogenous Picard-Fuchs equation in this example: the algorithm of [22] produces the inhomogeneous equation

$$((t^4 - 256)(t \frac{\partial}{\partial t})^3 w + 2t^4 (t \frac{\partial}{\partial t})^2 + \frac{7}{6}t^4 (t \frac{\partial}{\partial t}) - \frac{1}{6}t^4)\omega = \frac{1}{3}dF[3] + \frac{1}{6}dF[2] + \frac{1}{6}dF[1],$$

where

$$\begin{aligned} F[3] = & \frac{1}{f^3}((-16x_1t^5x_2x_3^4x_4^4 + 64x_2^4t^4x_3^3x_4^3)dx_3dx_4 \\ & + (-x_1^3t^7x_2^2x_3^3x_4^2 + 64x_3^4t^4x_2^3x_4^3)dx_2dx_4 \\ & + (-4x_1^2t^6x_2x_3^5x_4^2 + 64x_4^4t^4x_2^3x_3^3)dx_2dx_3 \\ & + (-x_2^3t^7x_1^2x_3^3x_4^2 + 16x_3^5t^5x_2x_4^4)dx_1dx_4 \\ & + (-4x_2^2t^6x_1x_3^5x_4^2 + 16x_4^5t^5x_2x_3^4)dx_1dx_3 \\ & + (-4x_3^6t^6x_1x_2x_4^2 + x_4^3t^7x_1^2x_2^2x_3^3)dx_1dx_2), \end{aligned}$$

$$\begin{aligned} F[2] = & \frac{1}{f^2}(-x_2(8t^5x_1x_3^4 - 2t^6x_1^2x_2x_3x_4)dx_3dx_4 \\ & + (x_1(-4t^5x_3x_4^4 - 8t^5x_1^4x_3) - x_3(8t^5x_1x_3^4 - 2t^6x_1^2x_2x_3x_4))dx_2dx_4 \\ & + (-x_1^2t^6x_2x_3x_4^2 - x_4(8t^5x_1x_3^4 - 2t^6x_1^2x_2x_3x_4))dx_2dx_3 \\ & + x_2(-4t^5x_3x_4^4 - 8t^5x_1^4x_3)dx_1dx_4 - x_2^2x_1t^6x_3x_4^2dx_1dx_3 \\ & + (-x_3^2x_1t^6x_2x_4^2 - x_4(-4t^5x_3x_4^4 - 8t^5x_1^4x_3))dx_1dx_2), \end{aligned}$$

and

$$\begin{aligned} F[1] = & \frac{1}{f}(-2x_2t^5x_1dx_3dx_4 - 4x_1t^5x_3dx_2dx_4 \\ & - x_1t^5x_4dx_2dx_3 - 2x_2t^5x_3dx_1dx_4 \\ & + x_2t^5x_4dx_1dx_3 + 3x_3t^5x_4dx_1dx_2). \end{aligned}$$

6 Outlook

In this section we want to give an outlook for the reader and present some ideas which came up after the conference in Toronto. Assume again that $2p - n - 1 = d$ and $p \leq d$. For every family $f : X \rightarrow B$ and every cycle $\mathcal{Z} \in CH^p(X, n)$ which is defined after restriction to every fiber, we get a complex function \bar{v} ,

$$\bar{v}(b) = \bar{c}_{p,n}(\mathcal{Z}|X_b) = \int_{\Gamma_b} \omega_{X_b},$$

induced by the truncated Abel-Jacobi map $\bar{c}_{p,n}$ from section 2 on each member of the family. The function $\bar{v}(b)$ is holomorphic on B but has singularities along the boundary $\bar{B} \setminus B$ of a compactification \bar{B} . Using the notations from section 1, we have:

Theorem 6.1 [10] *For all symbols σ , $D_\sigma \bar{v}(b)$ has at most poles in the points in $\bar{B} \setminus B$, i.e., becomes a rational function on the compactification \bar{B} .*

Proof See [10]. □

The proof is a clarification and generalization of the result in [9] for the case of K3 surfaces. It uses Clemens' extension theory of normal functions together with the results of [27]. This result can be used to prove several consequences. Assume that $f : X \rightarrow B$ is as above with B and the cycle \mathcal{Z} being defined over $\bar{\mathbb{Q}}$. Such a situation can for example be achieved by spreading out a cycle on a generic fiber X_η over the field obtained by the compositum of its field of definition and the function field of η . In other words all transcendental elements in the equations of X_η and \mathcal{Z} occur in the coordinates of B . Then there is a unique choice of relative canonical d -form by spreading out also ω from X_η .

Theorem 6.2 *In this situation, the rational function g has coefficients in $\bar{\mathbb{Q}}$.*

Proof See [10]. □

Such questions can be used to reprove the countability result [23, Cor. 3.3].

In the paper of Manin [21], the general formalism of μ -equations in the context of families of algebraic curves was established. This was used by him already in the solution of the functional Mordell conjecture. Let us explain how to generalize this and assume that we can solve $D_\sigma \omega_X = d\beta$ for some $(d-1)$ -form β , where D_σ is one of the generators of the differential ideal of Picard-Fuchs equations, see section 1. If we have a fibration $\Gamma \rightarrow B$ where Γ is a relative singular chain of real dimension d , then we are looking at the equation

$$D_\sigma \int_\Gamma \omega = \int_\gamma i^* \beta + T,$$

for some relative $(d-1)$ -form β and where $i : \gamma \hookrightarrow X$ is the inclusion of $\gamma = \partial\Gamma$ in the total space X . The term T arises from differentiating the boundary of Γ . This formalism is a direct generalization of [21, formula (1.5)]. The connection between non-linear equations and inhomogenous Picard-Fuchs equations is a direct consequence of these μ -equations. In [10] we give more examples of non-linear equations arising from higher dimensional examples.

Acknowledgements: We would like to thank the organizers of the workshop on “Arithmetic, Geometry and Physics around Calabi-Yau Varieties and Mirror Symmetry” and the staff of the Fields Institute for the opportunity to present our research and visit the Fields Institute. We are grateful for the patience of the editors and would like to thank Xavier Gomez-Mont, Mark Green, Phillip Griffiths, James Lewis, Matthew Kerr and Noriko Yui for several discussions. We thank Jan Nagel and Morihiro Saito for help on normal functions. This paper is dedicated to Andrei Tyurin who has encouraged us to work in this direction and explained to us his ideas about mirror symmetry.

References

- [1] Bloch, S.: *Higher regulators, algebraic K-theory and zeta functions of elliptic curves (Irvine lectures 1971)*, CRM monograph series (2000).
- [2] Bloch, S.: *Algebraic cycles and higher K-theory*, Adv. in Math. **61**, (1986), 267-304.
- [3] Bloch, S.: *Algebraic cycles and the Beilinson conjectures*, Algebraic geometry, Proc. Lefschetz Centennial Conf., Mexico City/Mex. 1984, part I, Contemp. Math. **58**, 65-79 (1986).
- [4] Borel, A. et al.: *Algebraic D-Modules*, Perspectives in Mathematics, Vol. 2, Academic Press, (1987) 355 pp.
- [5] Candelas, Ph. and de la Ossa, X.: *Moduli space of Calabi-Yau manifolds*, Nuclear Physics **B355**, 455-481 (1991).
- [6] Candelas, Ph. and de la Ossa, X.: *Calabi-Yau manifolds over finite fields I*, hep-th/0012233.
- [7] Candelas, Ph. and de la Ossa, X.: *Calabi-Yau manifolds over finite fields II*, this volume.
- [8] Alberto Collino: *Griffiths’ infinitesimal invariant and higher K-theory of hyperelliptic Jacobians*, Journ. of Alg. Geom. **6**, (1997), 393-415.
- [9] del Angel, P. and Müller-Stach, S.: *The transcendental part of the regulator map for K_1 on a family of K3 surfaces*, Duke Math. Journal **112** No.3, 581-598 (2002).
- [10] del Angel, P. and Müller-Stach, S.: *Integrable Systems associated to families of algebraic cycles*, Preprint Feb 2003.
- [11] Dolgachev, I.: *Mirror symmetry for lattice polarized K3 surfaces*, J. Math. Sci. **81**, (1996), 2599-2630.
- [12] Donagi, R. and Markman, E.: *Spectral covers, algebraically completely integrable, Hamiltonian systems, and moduli of bundles*, Francaviglia, M. (ed.) et al., Integrable systems and quantum groups. Lectures given at the 1st session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, June 14-22, 1993, Berlin: Springer-Verlag. Lect. Notes Math. **1620**, 1-119 (1996).
- [13] Goncharov, A.: *Chow polylogarithms and regulators*, Math. Research Letters **2**, 95-112 (1995).
- [14] Gordon, B. and Lewis, J.: *Indecomposable higher Chow cycles on products of elliptic curves*, J. Algebr. Geom. **8**, No.3, 543-567 (1999).

- [15] Griffiths, Ph.: *Periods of integrals on algebraic manifolds: summary of main results and discussion of open problems*, Bull. Am. Math. Soc. **76**, 228-296 (1970).
- [16] Griffiths, Ph.: *A theorem concerning the differential equations satisfied by normal functions associated to algebraic cycles*, Amer. J. Math. **101** 94-131 (1979).
- [17] Kerr, M., Lewis, J. and Müller-Stach, S.: *The Abel-Jacobi map for higher Chow groups*, in preparation (Feb. 2003).
- [18] Kerr M.: *Geometric construction of regulator currents, with applications to algebraic cycles*, Thesis, Princeton (2002).
- [19] Horst Knörrer: *Hamiltonsche Systeme und Algebraische Geometrie*, Jahresberichte der DMV **88**, 82-103 (1986).
- [20] Lang, S.: *Number theory III: Diophantine geometry*, Encyclopaedia of Mathematical Sciences **60**, Springer-Verlag, (1991).
- [21] Manin Y.I.: *Sixth Painlevé equation, universal elliptic curve and mirror of \mathbb{P}^2* , Amer. Math. Soc. Translations (2), Vol. **186**, (1998), 131-151.
- [22] Morrison, D.: *Picard-Fuchs equations and mirror maps for hypersurface*, Yau, Shing-Tung (ed.), Mirror symmetry I, American Mathematical Society, AMS/IP Stud. Adv. Math. **9**, 185-199 (1998).
- [23] Müller-Stach, S.: *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geom. **6**, (1997), 513-543.
- [24] Müller-Stach, S. and Saito, S.: (Appendix by A. Collino): *On K_1 and K_2 of algebraic surfaces*, math.AG/0202173.
- [25] Narumiya, N. and Shiga, H.: *The mirror map for a family of $K3$ surfaces induced from the simplest 3 dimensional reflexive polyhedron*, Proceedings on Moonshine and related topics (Montreal, QC, 1999), 139-161, CRM Proc. Lecture Notes, **30**, Amer. Math. Soc., Providence, RI, 2001.
- [26] Quillen, D.: *Higher algebraic K-theory I*, Proc. Conf. Battelle Inst. 1972, Lecture Notes Math. **341**, (1973), 85-147.
- [27] Saito, Morihiko: *Admissible normal functions*, J. Algebraic Geom. **5**, 235-276 (1996).
- [28] Saito, Morihiko: *Bloch's conjecture, Deligne cohomology and higher Chow groups*, RIMS preprint July (2000).
- [29] Voisin, C.: *Variations of Hodge structure and algebraic cycles*, Proceedings of the ICM, Zürich, 1994, (S.D. Chatterji ed.), Birkhäuser, Basel, (1995), 706-715.
- [30] Voisin, C.: *Variations of Hodge Structure of Calabi-Yau Threefolds*, Lezioni Lagrange 1996 Roma, Pubblicazioni della classe di scienze, Scuola Normale Superiore 1997.