

# TEICHMÜLLER CURVES, TRIANGLE GROUPS, AND LYAPUNOV EXPONENTS

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ABSTRACT. We construct a Teichmüller curve uniformized by the Fuchsian triangle group  $\Delta(m, n, \infty)$  for every  $m < n \leq \infty$ . Our construction includes the Teichmüller curves constructed by Veech and Ward as special cases. The construction essentially relies on properties of hypergeometric differential operators. We interpret some of the so-called Lyapunov exponents of the Kontsevich–Zorich cocycle as normalized degrees of some natural line bundles on a Teichmüller curves. We determine the Lyapunov exponents for the Teichmüller curves we construct.

## INTRODUCTION

Let  $C$  be a smooth curve defined over  $\mathbb{C}$ . The curve  $C$  is a *Teichmüller curve* if there exists a generically injective, holomorphic map from  $C$  to the moduli space  $\mathcal{M}_g$  of curves of genus  $g$  which is geodesic for the Teichmüller metric. Consider a pair  $(X, \omega_X)$ , where  $X$  is a Riemann surface of genus  $g$  and  $\omega_X$  is a holomorphic 1-form on  $X$ . If the projective affine group  $\Gamma$  of  $(X, \omega_X)$  is a lattice in  $\mathrm{PSL}_2(\mathbb{R})$  then  $C := \mathbb{H}/\Gamma$  is a Teichmüller curve. Such a pair  $(X, \omega_X)$  is called a *Veech surface*. Moreover, the curve  $X$  is a fiber of the family of curves  $\mathcal{X}$  corresponding to the map  $C \rightarrow \mathcal{M}_g$ . We refer to Section 1 for precise definitions and more details.

Teichmüller curves naturally arise in the study of dynamics of billiard paths on a polygon in  $\mathbb{R}^2$ . Veech ([Ve89]) constructed a first class of Teichmüller curves  $C = C_n$  starting from a triangular billiard. The corresponding projective affine group is commensurable to the triangle group  $\Delta(2, n, \infty)$ . Ward ([Wa98]) found other triangles which generate Teichmüller curves, with projective affine group  $\Delta(3, n, \infty)$ . Several authors tried to find other triangles which generate Teichmüller curves, but only sporadic examples were found. Many types of triangles were disproven to be Veech surfaces ([Vo96], [KeSm00], [Pu01]).

The goal of this paper is to show that essentially all triangle groups  $\Delta(m, n, \infty)$  occur as the projective affine group of a Teichmüller curve. Since Teichmüller curves are never complete ([Ve89]), triangle groups  $\Delta(m, n, k)$  with  $k \neq \infty$  do not occur. We use a different construction from previous authors; we construct the family  $\mathcal{X}$  of curves defined by  $C$  rather than the individual Veech surface (which is a fiber of  $\mathcal{X}$ ). However, starting from our description it is possible to compute algebraic equations for the corresponding Veech surfaces, since the family  $f : \mathcal{X} \rightarrow C$  we consider is very explicit. It is given as the quotient of an abelian cover  $\mathcal{Y} \rightarrow \mathbb{P}^1$  by a finite group. The Teichmüller curves we construct (for  $m, n$  finite and odd) are naturally related to  $(m+3)/2$ -gons. For  $m = 5$  we actually give the corresponding billiard tables.

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Our approach to construct Teichmüller curves is based on a Hodge-theoretical criterion (Möller [Mö04a]). We translate this abstract criterion in concrete terms. Proposition 3.3 gives the flavor of our methods in a first case. From our construction we obtain new information even for the Teichmüller curves found by Veech and Ward. Namely, we determine the complete decomposition of the relative de Rham cohomology  $R^1 f_* \mathbb{C}_X$  and the Lyapunov exponents, see below.

There exist Teichmüller curves whose projective affine group is not a triangle group. McMullen ([McM03]) constructed a series of such examples in genus  $g = 2$ . It would be interesting to try and extend our method to other Fuchsian groups than triangle groups. This would probably be much more involved due to the appearance of so-called *accessary parameters*.

We now give a more detailed description of our results. Suppose that  $m \geq 4$  and  $m < n \leq \infty$  or that  $m \geq 2$  and  $3 \leq n < \infty$ . We consider a family of  $N$ -cyclic covers

$$\mathcal{Z}_t : z^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}$$

of the projective line branched at 4 points. Note that  $\mathcal{Z}$  defines a family over  $C = \mathbb{P}_t^1 - \{0, 1, \infty\}$ . It is easy to compute the differential equations corresponding to the eigenspaces  $\mathbb{L}_i$  of the action of  $\mathbb{Z}/N$  on the relative de Rham cohomology of  $\mathcal{Z}$  (Section 3). These eigenspaces are local systems of rank 2, and the corresponding differential equations are hypergeometric. Cohen and Wolfart ([CoWo90]) showed that we may choose  $N, a_i$  in terms of  $n$  and  $m$  such that the projective monodromy group of at least one of the eigenspaces  $\mathbb{L}_i$  is the triangle group  $\Delta(m, n, \infty)$ .

First consider the case that  $m$  and  $n$  are finite and relatively prime. Here we show that the particular choice of  $N$  and the  $a_i$  implies that, after replacing  $C$  by a finite unramified cover, the automorphism group of  $\mathcal{Z}$  contains a subgroup isomorphic to  $\mathbb{Z}/N \rtimes H$ , where  $H \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$ . If  $n$  is infinite the group  $H$  has order 2. This case corresponds to half of Veech's series of Teichmüller curves (Section 4). If  $m$  and  $n$  are not relatively prime instead of  $\mathcal{Z}$ , we consider a family  $\mathcal{Y}$  which is a  $G_0$ -Galois cover of the projective line. Here  $G_0$  is a suitable subgroup of  $\mathbb{Z}/N \times \mathbb{Z}/N$ , rather than a cyclic group of order  $N$ . The description of  $\mathcal{Y}$  in this case is just as explicit (Section 5).

**Theorem 4.1 and 5.1:** *The quotient family  $\mathcal{X} := \mathcal{Y}_C/H$  is the pullback of the universal family over the moduli space of curves to  $C$ . The curve  $C$  is an unramified cover of a Teichmüller curve.*

The proof of this result relies on the Hodge-theoretical characterization of Teichmüller curves ([Mö04a]). A key ingredient is the characterization of the vanishing of the Kodaira–Spencer map in terms of invariants of the hypergeometric differential equation corresponding to  $\mathbb{L}_i$  (Proposition 2.2).

**Theorem 5.9:** *Suppose that  $n$  is finite and  $m$  is different from  $n$ . Then the projective affine group of  $\mathcal{X}$  is the triangle group  $\Delta(m, n, \infty)$ .*

It is interesting to note that we determine the projective affine group of our Teichmüller curves directly from the construction of the family  $\mathcal{X}$ . We do not need to consider the corresponding Veech surfaces, as is done by Veech and Ward.

The following result shows that we recover the most important geometric invariant of the Veech surfaces corresponding to our Teichmüller curves.

**Theorem 5.11:** *Suppose that  $n$  is finite, and let  $\gamma = \gcd(m, n)$ . Let  $(X, \omega_X)$  be a Veech surface corresponding to  $\mathcal{X}$ . The differential  $\omega_X$  has  $\gamma/2$  zeros if  $m$  and  $n$  are both even, and has  $\gamma$  zeros otherwise.*

Our last main result concerns Lyapunov exponents. Let  $V$  be a flat normed vector bundle on a manifold with flow. The *Lyapunov exponents* measure the rate of growth of the length of vectors in  $V$  under parallel transport along the flow. We refer to Section 7 for precise definitions and a motivation of the concept. We express the Lyapunov exponents for an arbitrary Teichmüller curves in terms of the degree of certain local systems.

Let  $f : \mathcal{X} \rightarrow C$  be the universal family over an unramified cover  $C$  of an arbitrary Teichmüller curve. The relative de Rham cohomology  $R^1 f_* \mathbb{C}_{\mathcal{X}}$  has  $r$  local subsystems  $\mathbb{L}_i$  of rank two. The associated vector bundles carry a Hodge filtration (Theorem 1.1). The  $(1, 0)$ -parts of the Hodge filtration are line bundles  $\mathcal{L}_i$  and the ratios

$$\lambda_i := 2 \deg(\mathcal{L}_i) / (2g(C) - 2 + s), \quad s = \text{card}(\overline{C} \setminus C)$$

are unchanged if we pass to an unramified cover of  $C$ .

**Theorem 7.2:** *The ratios  $\lambda_i$  are  $r$  of  $g$  non-negative Lyapunov exponents of the Kontsevich–Zorich cocycle over the Teichmüller geodesic flow on the canonical lift of a Teichmüller curve to the one-form bundle over the moduli space.*

A sketch of the relation between the degree of  $f_* \omega_{\mathcal{X}/C}$  and the sum of all Lyapunov exponents already appears in [Ko97].

Now suppose that  $C$  is an unramified cover of one of the Teichmüller curves from Theorems 4.1 and 5.1, and let  $f : \mathcal{X} \rightarrow C$  be the corresponding family of curves. In Corollaries 4.2 (Veech’s series), 4.5 and 5.6 we give an explicit expression for all Lyapunov exponents of  $C$ . For Veech’s series of Teichmüller curves and for a series of square-tiled coverings the Lyapunov exponents were calculated independently by Kontsevich and Zorich (unpublished). They form an arithmetic progression in these cases. Example 5.7 shows that this does not hold in general.

It is well-known that the largest Lyapunov exponent  $\lambda_1 = 1$  occurs with multiplicity one. We interpret  $1 - \lambda_i$  as the number of zeros of the Kodaira–Spencer map of  $\mathbb{L}_i$ , counted with multiplicity (Section 1), up to a factor. For the Teichmüller curves constructed in Theorems 4.1 and 5.1 we determine the position of the zeros of the Kodaira–Spencer map. These zeros are related to elliptic fixed points of the projective affine group  $\Gamma$  (Propositions 2.2 and 3.2). For an arbitrary Teichmüller curve it is an interesting question to determine the position of the zeros of the Kodaira–Spencer map. Precise information on the zeros of the Kodaira–Spencer map might shed new light on the defects of the Lyapunov exponents  $1 - \lambda_i$ .

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## 1. TEICHMÜLLER CURVES

A *Teichmüller curve* is a generically injective, holomorphic map  $C \rightarrow M_g$  from a smooth algebraic curve  $C$  to the moduli space of curves of genus  $g$ , which is geodesic for the Teichmüller metric. A Teichmüller curve arises as quotient  $C = \mathbb{H}/\Gamma$ , where  $\mathbb{H} \rightarrow T_g$  is a complex Teichmüller geodesic in Teichmüller space  $T_g$ . Here  $\Gamma$  is the subgroup in the

Teichmüller modular group fixing  $\mathbb{H}$  as a subset of  $T_g$  (setwise, not pointwise) and where  $C$  is the normalization of the image  $\mathbb{H} \rightarrow T_g \rightarrow M_g$ .

Veech showed that a Teichmüller curve  $C$  is never complete ([Ve89] Prop. 2.4). We let  $\overline{C}$  be a smooth completion of  $C$  and  $S := \overline{C} \setminus C$ . In the sequel, it will be convenient to not consider Teichmüller curves themselves but finite unramified coverings of  $C$  that satisfy two conditions: The corresponding subgroup of  $\Gamma$  is torsion free and the moduli map factors through a fine moduli space of curves (e.g. with level structure  $M_g^{[n]}$ ). We nevertheless stick to the notation  $C$  for the base curve and let  $f : \mathcal{X} \rightarrow C$  be the pullback of the universal family over  $M_g^{[n]}$  to  $C$ . We will use  $\overline{f} : \overline{\mathcal{X}} \rightarrow \overline{C}$  for the family of stable curves extending  $f$ . See also [Mö04a] Section 1.3.

Teichmüller curves, or more generally geodesic discs in Teichmüller space, are generated by a pair  $(X, q)$  of a Riemann surface and a quadratic differential  $q \in \Gamma(X, (\Omega_X^1)^{\otimes 2})$ . If a pair  $(X, q)$  generates a Teichmüller curve, the pair is called a *Veech surface*. Any smooth fiber of  $f$  together with the suitable quadratic differential is a Veech surface. Theorem 1.1 below characterizes Teichmüller curves where  $q = \omega^2$  is the square of a holomorphic one-form  $\omega \in \Gamma(X, \Omega_X^1)$ . The examples we construct will have this property, too. Hence:

*From now on the notion ‘Teichmüller curve’ includes ‘generated by a one-form’.*

For a pair  $(X, \omega)$  we let  $\text{Aff}^+(X, \omega)$  be the group of orientation preserving diffeomorphism of  $X$  that are affine with respect to the charts provided by integrating  $\omega$ . Associating with an element of  $\text{Aff}^+(X, \omega)$  its matrix part gives a well-defined map to  $\text{SL}(2, \mathbb{R})$ , whose image  $\text{SL}(X, \omega)$  is called the *affine group* of  $(X, \omega)$ . The stabilizer group  $\Gamma$  of  $\mathbb{H} \hookrightarrow T_g$  coincides, up to conjugation with the affine group  $\text{SL}(X, \omega)$  (see [McM03]). We denote throughout by  $K = \mathbb{Q}(\text{tr}(\gamma), \gamma \in \Gamma)$  the trace field and let  $r := [K : \mathbb{Q}]$ . We call the image of  $\text{SL}(X, \omega)$  in  $\text{PSL}_2(\mathbb{R})$  the *projective affine group* and denote it by  $\text{PSL}(X, \omega)$ .

We refer to [KMS86] and [KeSm00] for the billiard origins of Teichmüller curves.

We recall from [Mö04a] Theorem 2.6 and Theorem 5.5 a description of the variation of Hodge structures (VHS) over a Teichmüller curve, and a characterization of Teichmüller curves in these terms.

Let  $\mathbb{L}$  be a rank two irreducible  $\mathbb{C}$ -local system on an affine curve  $C$ . Suppose that the Deligne extension  $\mathcal{E}$  of  $\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}$  ([De70] Proposition II.5.2) to  $\overline{C}$  carries a Hodge filtration of weight one  $\mathcal{L} := \mathcal{E}^{(1,0)} \subset \mathcal{E}$ . We denote by  $\nabla$  the corresponding logarithmic connection on  $\mathcal{E}$ . The *Kodaira–Spencer mapping* (also: Higgs field, or: second fundamental form) with respect to  $S$  is the composition map

$$(1) \quad \Theta : \mathcal{L} \rightarrow \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega_{\overline{C}}^1(\log S) \rightarrow (\mathcal{E}/\mathcal{L}) \otimes \Omega_{\overline{C}}^1(\log S).$$

A VHS of rank 2 and weight one whose Kodaira–Spencer map with respect to some  $S$  vanishes nowhere on  $\overline{C}$ , is called *maximal Higgs* in [ViZu04]. The corresponding vector bundle  $\mathcal{E}$  is called *indigenous bundle*. See [BoWe05] or [Mo99] for appearances of such bundles with more emphasis on char  $p > 0$ .

**Theorem 1.1.** *Let  $f : \mathcal{X} \rightarrow C$  be the universal family over a finite unramified cover of a Teichmüller curve. Then we have a decomposition of the VHS of  $f$  as*

$$(2) \quad R^1 f_* \mathbb{Q} = \mathbb{W} \oplus \mathbb{M} \quad \text{and} \quad \mathbb{W} \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{i=1}^r \mathbb{L}_i.$$

*In this decomposition the  $\mathbb{L}_i$  are Galois conjugate, irreducible, pairwise non-isomorphic,  $\mathbb{C}$ -local systems of rank two. The  $\mathbb{L}_i$  are in fact defined over some field  $F \subset \mathbb{R}$  that is*

*Galois over  $\mathbb{Q}$  and contains the trace field  $K$ . Moreover,  $\mathbb{L}_1$  is maximal Higgs. Conversely, suppose  $f : \mathcal{X} \rightarrow C$  is a family of smooth curves such that  $R^1 f_* \mathbb{C}$  contains a local system of rank two which is maximal Higgs with respect to the set  $S = \overline{C} \setminus C$ . Then  $f$  is the universal family over a finite unramified cover of a Teichmüller curve.*

Note that ‘maximal Higgs’ depends on  $S$ . We will encounter cases where  $\mathbb{L}$  extends over some points of  $S$  and becomes maximal Higgs with respect to a smaller set  $S_u \subset S$ , but it is not maximal Higgs with respect to all of  $S_u$ . See also Proposition 3.3 and Remark 3.4.

## 2. LOCAL EXPONENTS OF DIFFERENTIAL EQUATIONS AND ZEROS OF THE KODAIRA–SPENCER MAP

In this section we provide a dictionary between local systems plus a section on one side and differential equations on the other side. In particular, we translate local properties of a differential operator into vanishing statements of the Kodaira–Spencer map. Both in the Sections 4 and 5 we essentially start with a hypergeometric differential equation whose local properties are well-known. Via Proposition 2.2 the vanishing Kodaira–Spencer mapping of the corresponding local system is completely determined. This knowledge is then exploited in a criterion (Proposition 3.3) for  $f : \mathcal{X} \rightarrow C$  to be the universal family over a Teichmüller curve.

Let  $\mathbb{L}$  be a rank two irreducible  $\mathbb{C}$ -local system on an affine curve  $C$ , not necessarily a Teichmüller curve.

Suppose we are given a non-vanishing, holomorphic section  $s$  of  $\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_C$  and let  $t$  be a local coordinate of  $C$ . We denote  $D := \nabla(\frac{\partial}{\partial t})$ . Since  $\mathbb{L}$  is irreducible, the sections  $s$  and  $Ds$  are linearly independent. Hence  $s$  satisfies a differential equation  $Ls = 0$ , where

$$L = D^2 + p(t)D + q(t),$$

for some meromorphic functions  $p, q$  on  $\overline{C}$ .

Conversely, the set of solutions of a second order differential operator  $L : \mathcal{O}_C \rightarrow \mathcal{O}_C$  forms a local system  $\text{Sol} \subset \mathcal{O}_C$ . If  $L$  was obtained from  $\mathbb{L}$  then  $\text{Sol} \cong \mathbb{L}^\vee$ , see [De70] §1.4. The canonical map

$$\varphi : \text{Sol} \otimes_{\mathbb{C}} \mathcal{O}_C \rightarrow \mathcal{O}_C, \quad f \otimes g \mapsto fg$$

hence defines a section  $s = s_\varphi$  of  $\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_C$ .

A point  $c \in \overline{C}$  is a *singular point* of  $L$  if  $p$  or  $q$  has a pole at  $c$ . We will only need the case that  $L$  has regular singularities. By Fuchs’ theorem this is equivalent to  $(t - c)p$  and  $(t - c)^2 q$  being holomorphic for each point  $c \in \overline{C}$ , where  $t$  denotes a local coordinate at  $c$ .

The local exponents  $t_1, t_2$  of  $L$  at  $c$  are the roots of the characteristic equation

$$t(t - 1) + tp_{-1} + q_{-2} = 0,$$

where  $p = \sum_{i=-1}^{\infty} p_i(t - c)^i$  and  $q = \sum_{i=-2}^{\infty} q_i(t - c)^i$ . The table recording singularities and the local exponents is usually called *Riemann scheme*. See e.g. [Yo87] §2.5 for more details.

Note that  $L$  and the local exponents not only depend on  $\mathbb{L}$  but also on the section chosen: Replacing  $s$  by  $\alpha s$  will shift the local exponents at  $c$  by the order of the function  $\alpha$  at  $c$ . The following criterion is well-known (e.g. [Yo87] §2.6).

**Lemma 2.1.** *The monodromies of  $\text{Sol}$  are unipotent if and only if for each  $c \in \overline{C}$  both local exponents are integers.*

In the classical case  $\overline{C} \cong \mathbb{P}^1$ , the differential operator  $L$  is determined by the local exponents exactly if the number of singularities is three. We will exploit this fact in the next sections. If the number of singularities is greater than three one needs to know *accessary parameters* in addition to the local exponents to determine the differential operator ([Yo87] §3.2).

In the rest of this section we will only consider local systems such that for all  $c \in S_u := \overline{C} \setminus C$  the monodromy around  $c$  is unipotent. Suppose, moreover, that  $\mathbb{L}$  carries a polarized VHS of weight one and that  $s$  is a section of  $(\mathbb{L} \otimes_{\mathbb{C}} \mathcal{O}_C)^{(1,0)}$ . The following proposition expresses the order of vanishing of the Kodaira–Spencer mapping (1) at  $c$  in terms of the local exponents at  $c$ .

Let  $S$  be a subset of  $\overline{C}$  containing  $S_u$ . The reader should think of  $S$  as the set of singular fibers of a family of curves over  $\overline{C}$ .

- Proposition 2.2.** (a) *If  $c \in \overline{C} \setminus C$  then  $\Theta$  does not vanish at  $c$ .*  
(b) *If  $c \in C$  then the local exponents at  $c$  are  $(0, n)$  for some  $1 \leq n \in \mathbb{N}$ .*  
(c) *For  $n$  as in (b), if  $c \notin S \supset S_u$  then  $\Theta$  vanishes of order  $n - 1$  at  $c$ .*  
(d) *For  $n$  as in (b), if  $c \in S \setminus S_u$  then  $\Theta$  vanishes of order  $n$  at  $c$ .*

**Proof:** Suppose first that  $c \in C$  and choose a local parameter  $t$  at  $c$ . Since  $\mathcal{E}$  has rank two, there are two linearly independent, non-vanishing sections of  $\mathcal{E}$  in a neighborhood of  $c$ . Hence both local exponents are non-negative, one of them is zero. The local exponents at  $c$  are integral, since the monodromy around  $c$  is trivial by definition. This establishes (b).

If  $c \notin S$  the differential equation has solutions  $s_1, s_2$  with leading terms 1 and  $t^n$ , respectively ([Yo87] I, 2.5). We want to determine the vanishing order of  $D(s)$  in  $\mathcal{E}/(s \otimes_{\mathbb{C}} \mathcal{O}_C)$ . By the above correspondence between the local system and the differential equation we may as well calculate the vanishing order of  $D(\varphi)$  in  $(\text{Sol}^{\vee} \otimes_{\mathbb{C}} \mathcal{O}_C)/(\varphi \otimes_{\mathbb{C}} \mathcal{O}_C)$ . A basis of  $\text{Sol}^{\vee} \otimes_{\mathbb{C}} \mathcal{O}_C$  around  $c$  is

$$s_i^{\vee} : s_1 \otimes g_1 + s_2 \otimes g_2 \mapsto s_i g_i \quad (i = 1, 2).$$

By definition of the dual connection and the flatness of  $s_i$  one calculates that  $D(\varphi)$  is the class of

$$s_1 \otimes g_1 + s_2 \otimes g_2 \mapsto g_1 s_1' + g_2 s_2'$$

in  $(\text{Sol}^{\vee} \otimes_{\mathbb{C}} \mathcal{O}_C)/(\varphi \otimes_{\mathbb{C}} \mathcal{O}_C)$ . Since both  $\varphi$  and  $s_1$  are non-vanishing at  $c$ , we conclude that  $D(\varphi)$  vanishes of order  $n - 1$  at  $c$ . This proves (c).

In case  $c \in S$  we should consider the contraction against  $t \frac{\partial}{\partial t}$ , which increases the order of zero by one.

We now treat the case  $c \in \overline{C} \setminus C$ . Consider the residue map  $\text{Res}_c(\nabla) \in \text{End}(\mathcal{E}_c)$ . Suppose the Kodaira–Spencer map vanishes at  $c$ . This implies that  $\text{Res}_c(\nabla)$  is a diagonal matrix in a basis consisting of an element from  $\mathcal{L}_c$  and an element from its orthogonal complement. But  $\text{Res}_c(\nabla)$  is nilpotent ([De87] Proposition II.5.4 (iv)), hence zero. This implies that two linearly independent sections of  $\mathbb{L}$  extend over  $c$ . This contradicts the hypothesis on the monodromy around  $c$ .  $\square$

The ratios  $\lambda(\mathbb{L}, S) := 2 \deg(\mathcal{L})/\Omega_{\overline{C}}^1(\log S)$  will be of central interest in the sequel. The factor 2 is motivated from the normalization in Section 7, where we show that the  $\lambda(\mathbb{L}, S)$  are Lyapunov exponents for some flow with respect to some ergodic measure etc. Nevertheless we call the ratios  $\lambda(\mathbb{L}, S)$  from now on *Lyapunov exponents*. We will suppress  $S$  if it is clear from the context.

**Remark 2.3.** We will only be interested in  $\mathbb{C}$  local systems  $\mathbb{L}$  that arise as local subsystems of a  $R^1 f_* \mathbb{C}$  for a family of curves  $f : \mathcal{X} \rightarrow C$ . In this case a Hodge filtration exists on  $\mathbb{L}$  and is unique (see [De87] Prop. 1.13).

Due to this remark it suffices to keep track of the local systems e.g. in the covering constructions in Section 5. The right VHS comes for free. The following lemma is noted for future reference and easily checked.

**Lemma 2.4.** *The ratio  $\lambda(\mathbb{L}, S)$  does not change by taking unramified coverings.*

### 3. CYCLIC COVERS OF THE PROJECTIVE LINE BRANCHED AT 4 POINTS

Let  $N > 1$  be an integer, and suppose given a 4-tuple of integers  $(a_1, \dots, a_4)$  with  $0 < a_\mu < N$  and  $\sum_{\mu=1}^4 a_\mu = (k+1)N$ , for some integer  $k$ . We denote by  $\mathbb{P}^1$  the projective line with coordinate  $t$ , and put  $\mathbb{P}^* = \mathbb{P}^1 - \{0, 1, \infty\}$ . Let  $\mathcal{P} \simeq \mathbb{P}^1 \times \mathbb{P}^* \rightarrow \mathbb{P}^*$  be the trivial fibration with fiber coordinate  $x$ . Let  $x_1 = 0, x_2 = 1, x_3 = t, x_4 = \infty$  be sections of  $\mathcal{P} \rightarrow \mathbb{P}^*$ . We fix an injective character  $\chi : \mathbb{Z}/N \rightarrow \mathbb{C}^*$ . Let  $g : \mathcal{Z} \rightarrow \mathbb{P}^*$  be the  $N$ -cyclic cover of type  $(x_\mu, a_\mu)$  ([Bo04] Definition 2.1). This means that  $\mathcal{Z}$  is the family of projective curves with affine model

$$(3) \quad \mathcal{Z}_t : \quad z^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}.$$

We suppose, furthermore, that  $\gcd(a_1, a_2, a_3, a_4, N) = 1$ . This implies that the family is connected. The genus of  $\mathcal{Z}_t$  is  $N + 1 - (\sum_{\mu=1}^4 \gcd(a_\mu, N))/2$ .

In this section, we collect some well-known facts on such cyclic covers. We write

$$\sigma_\mu(i) = \langle ia_\mu/N \rangle = a_\mu(i)/N,$$

where  $\langle \cdot \rangle$  denotes the fractional part. Let  $k(i) + 1 = \sum_{\mu=1}^4 \sigma_\mu(i)$ . We fix an injective character  $\chi : \mathbb{Z}/N \rightarrow \mathbb{C}^*$  such that  $h \in \text{Gal}(\mathcal{Z}/\mathcal{P}) \cong \mathbb{Z}/N$  acts as  $h \cdot z = \chi(h)z$ .

**Lemma 3.1.** *For  $0 < i < N$ , we let  $s(i)$  be the number of  $a_\mu$  unequal to  $0 \pmod{N/\gcd(i, N)}$ . Then*

- (a)  $\dim_{\mathbb{C}} H_{\text{dR}}^1(\mathcal{Z}/\mathbb{P}^*) = s(i) - 2$ ,
- (b)  $\text{rank } g_*(\Omega_{\mathcal{Z}/\mathbb{P}^*}^1)_{\chi^i} = s(i) - 2 - k(i), \quad \text{rank}(R^1 g_* \mathcal{O}_{\mathcal{Z}})_{\chi^i} = k(i)$ ,
- (c) *If  $k(i) = 1$  then*

$$\omega_i = \frac{z^i dx}{x^{1+[i\sigma_1]}(x-1)^{1+[i\sigma_2]}(x-t)^{1+[i\sigma_3]}}$$

*is a non-vanishing section of  $g_*(\Omega_{\mathcal{Z}/\mathbb{P}^*}^1)_{\chi^i}$ . It is a solution of the hypergeometric differential operator*

$$L(i) := \nabla \left( \frac{\partial}{\partial t} \right)^2 + \frac{(A(i) + B(i) + 1)t - C(i)}{t(t-1)} \nabla \left( \frac{\partial}{\partial t} \right) + \frac{A(i)B(i)}{t(t-1)},$$

*where  $A(i) = 1 - \sigma_3(i)$ ,  $B(i) = 2 - (\sigma_1(i) + \sigma_2(i) + \sigma_3(i))$ ,  $C(i) = 2 - (\sigma_1(i) + \sigma_3(i))$ .*

**Proof:** The second statement of (b) is proved in [Bo01] Lemma 4.3. The first statement follows from Serre duality and [Bo01] Lemma 4.5. Part (a) follows immediately from (b). The statement that  $\omega_i$  is holomorphic and non-vanishing is a straightforward verification.

The statement that  $L(i)\omega_i = 0$  in  $H_{\text{dR}}^1(\mathcal{Z}/\mathbb{P}^*)_{\chi^i}$  is proved for example in [Bo05], Lemma 1.1.4.  $\square$

The differential operator  $L(i)$  corresponds to the local system  $\mathbb{L}(i) = H_{\text{dR}}^1(\mathcal{Z}/\mathbb{P}^*)_{\chi^i}$  together with the choice of a section  $\omega_i$  via the correspondence described at the beginning of Section 2. It has singularities precisely at 0, 1 and  $\infty$ . Its local exponents are summarized in the Riemann scheme

$$(4) \quad \begin{bmatrix} t=0 & t=1 & t=\infty \\ 0 & 0 & A(i) \\ 1-C(i) & C(i)-A(i)-B(i) & B(i) \end{bmatrix}.$$

A (Fuchsian)  $(m, n, p)$ -triangle group for  $m, n, p \in \mathbb{N} \cup \{\infty\}$  satisfying  $1/m + 1/n + 1/p < 1$  is a Fuchsian group in  $\text{PSL}_2(\mathbb{R})$  generated by matrices  $M_1, M_2, M_3$  satisfying  $M_1 M_2 M_3 = 1$  and

$$\text{tr}(M_1) = \pm 2 \cos(\pi/m), \quad \text{tr}(M_2) = \pm 2 \cos(\pi/n), \quad \text{tr}(M_3) = \pm 2 \cos(\pi/p).$$

A triangle group is determined, up to conjugation in  $\text{PSL}_2(\mathbb{R})$ , by the triple  $(m, n, p)$ . It is well-known that the projective monodromy groups of the hypergeometric differential operators  $L(i)$  are triangle groups under suitable conditions on  $A(i), B(i), C(i)$ . These conditions are met in the cases we consider in Section 4 and 5.

We are interested in determining the order of vanishing of the Kodaira–Spencer map. Note that if  $k(i) = 0$  or  $k(i) = 2$  then the Hodge filtration on the corresponding eigenspace will be trivial and hence the Kodaira–Spencer map will be zero.

Let  $\pi : \overline{C} \rightarrow \mathbb{P}^1$  a finite cover, unbranched outside  $\{0, 1, \infty\}$ , such that the monodromy of the pullback of  $\mathcal{Z}$  via  $\pi$  is unipotent for all  $c \in C = \pi^{-1}(\mathbb{P}^*) \subset \overline{C}$ . In the rest of this paper, we will only consider families  $g : \mathcal{Z} \rightarrow \mathbb{P}^*$  of curves which have infinite monodromy over one of the points  $t = 0, 1, \infty$ . It is no restriction to suppose that this happens for  $t = \infty$ . In terms of the invariants  $a_\mu$  this means that  $a_3 + a_4 \equiv 0 \pmod{N}$ . It follows that  $A(i) = B(i)$ . Let  $b_0$  (resp.  $b_1$ ) be the common denominator of the local exponents  $1 - C(i)$  (resp.  $C(i) - A(i) - B(i)$ ) for  $1 \leq i < N$ . Write  $|1 - C(i)| = n_0(i)/b_0$  and  $|C(i) - A(i) - B(i)| = n_1(i)/b_1$ . We may choose the cover  $g$  to be branched of order  $b_\mu$  over  $t = \mu \in \{0, 1\}$ . Note that the monodromy of  $g$  at  $t = \mu$  becomes trivial after pullback by a cover which is branched at  $t = \mu$  of order  $b$  if and only if  $b_\mu | b$ .

We let  $S_u$  be the points  $c \in \overline{C}$  whose monodromy has infinite order. Unless  $n_0(i) = n_1(i) = 0$ , the set  $S_u$  is a proper subset of  $g^{-1}(\{0, 1, \infty\})$ .

**Proposition 3.2.** *Let  $0 < i < N$  be an integer with  $k(i) = 1$ . Denote by  $\mathcal{L}_{\chi^i}$  the  $(1, 0)$ -part of the local system  $\mathbb{L}(i)$  over  $C$ . Let  $d$  be the degree of  $\pi : \overline{C} \rightarrow \mathbb{P}^1$ . Then*

$$\deg \mathcal{L}_{\chi^i} = \frac{\deg(\pi)}{2} \left( 1 - \frac{n_0(i)}{b_0} - \frac{n_1(i)}{b_1} \right)$$

with the convention that  $1/b_\mu = 0$  if  $n_\mu = 0$ . In particular, the Lyapunov exponent

$$\lambda(\mathbb{L}(i), S_u) = \left( 1 - \frac{n_0(i)}{b_0} - \frac{n_1(i)}{b_1} \right) / \left( 1 - \frac{1}{b_0} - \frac{1}{b_1} \right)$$

is independent of the choice of  $\overline{\pi}$ .

**Proof:** We only treat the case that both  $n_0(i)$  and  $n_1(i)$  are non-zero, leaving the few modifications in the other cases to the reader. One checks that

$$\deg \Omega_{\overline{C}}^1(\log S_u) = \deg(\overline{\pi}) \left(1 - \frac{1}{b_0} - \frac{1}{b_1}\right)$$

is independent of the ramification order of  $g$  over  $t = \infty$ . It follows from the definition (1) of the Kodaira–Spencer map  $\Theta$  that  $2 \deg \mathcal{L}_{\chi^i} - \deg \Omega_c^1(\log S_u)$  is the number of zeros of  $\Theta$ , counted with multiplicity. Therefore the proposition follows from Proposition 2.2.  $\square$

We now single out the basic form in which we will apply Proposition 3.2 and Theorem 1.1 to construct Teichmüller curves.

**Proposition 3.3.** *Consider a family of curves  $\mathcal{Z}$  as in (3) with  $a_1(i) = a_2(i) = 1$  for some  $i$ . Suppose the quotient family  $\mathcal{X} = \mathcal{Z}/H$  extends to a smooth family over  $\widetilde{C} := \overline{C} \setminus S_u$  and a local subsystem  $\mathbb{L} \subset R^1 g_* \mathbb{C}$  isomorphic to  $\mathbb{L}(i)$  descends to  $\mathcal{X}$ . Then the moduli map  $\widetilde{C} \rightarrow M_g$  is an unramified covering of a Teichmüller curve.*

**Proof:** Proposition 3.2 implies in particular that the local system  $\mathbb{L}(i)$  is maximal Higgs with respect to  $S_u$ . The condition on the singular fibers says precisely that  $S_u$  is precisely the subset of  $\overline{C}$  needed to apply Theorem 1.1.  $\square$

**Remark 3.4.** The structure of the stable model  $g_{\overline{C}}$  of the family  $g_C : \mathcal{Z}_C \rightarrow C$  is given in the next subsection. It implies that all fibres of preimages of  $\{0, 1, \infty\}$  are singular. Hence the pullback family  $g_{\overline{C}}$  is the universal family over a Teichmüller curve if and only if every  $c \in \overline{C} - C$  is a parabolic fixed point of the uniformizing group. This happens for example for the families

$$y^2 = x(x-1)(x-\lambda) \quad \text{and} \quad y^4 = x(x-1)(x-\lambda).$$

Here  $\overline{C} = \mathbb{P}^1$ , and the uniformizing group is the triangle group  $\Delta(\infty, \infty, \infty)$ . Clearly, this is a very special situation.

**3.1. Degenerations of cyclic covers.** We now describe the stable model of the degenerate fibers of  $\mathcal{Z}$ . For simplicity, we only describe the fiber  $\mathcal{Z}_0$  above  $t = 0$ . The other degenerate fibers may be described similarly, by permuting  $\{0, 1, t, \infty\}$ . A general reference for this is [We98] Section 4.3. However, since we consider the easy situation of cyclic covers of the projective line branched at 4 points, we may simplify the presentation.

As before, we let  $\mathcal{P} \rightarrow \mathbb{P}^*$  be the trivial fibration with fiber coordinate  $x$ . We consider the sections  $x_1 = 0, x_2 = 1, x_3 = t, x_4 = \infty$  of  $\mathcal{P} \rightarrow \mathbb{P}^*$  as marking on  $\mathcal{P}$ . We may extend  $\mathcal{P}$  to a family of stably marked curves over  $\mathbb{P}$ , which we still denote by  $\mathcal{P}$ . The fiber  $\mathcal{P}_0$  of  $\mathcal{P}$  at  $t = 0$  consists of two irreducible components which we denote by  $\mathcal{P}_0^1$  and  $\mathcal{P}_0^2$ . We assume that  $x_1$  and  $x_3$  (resp.  $x_2$  and  $x_4$ ) specialize to the smooth part of  $\mathcal{P}_0^1$  (resp.  $\mathcal{P}_0^2$ ). We denote the intersection point of  $\mathcal{P}_0^1$  and  $\mathcal{P}_0^2$  by  $\xi$ . It is well known that the family of curves  $f : \mathcal{Z} \rightarrow \mathcal{P}$  over  $\mathbb{P}^*$  extends to a family of *admissible covers* over  $\mathbb{P}^1$ . See for example [HaSt99] or [We99]. For a short overview we refer to [BoWe04] Section 2.1.

The definition of type ([Bo04] Definition 2.1) immediately implies that the restriction of the admissible cover  $f_0 : \mathcal{Z}_0 \rightarrow \mathcal{P}_0$  to  $\mathcal{P}_0^1$  (resp.  $\mathcal{P}_0^2$ ) has type  $(x_1, x_3, \xi; a_1, a_3, a_2 + a_4)$  (resp. type  $(x_2, x_4, \xi; a_2, a_4, a_1 + a_3)$ ). (Admissibility amounts in our situation to  $(a_1 + a_3) + (a_2 + a_4) \equiv 0 \pmod{N}$ ). Let  $Z_0^j$  be a connected component of the restriction of  $\mathcal{Z}_0$  to  $\mathcal{P}_0^j$ . Choosing

suitable coordinates,  $Z_0^1$  (resp.  $Z_0^2$ ) is a connected component of the smooth projective curve defined by the equation  $z^N = x^{a_1}(x-1)^{a_3}$  (resp. the equation  $z^N = x^{a_2}(x-1)^{a_4}$ ).

Denote by  $H^j = \text{Gal}(Z_0^j, P_0^j) \subset H \simeq \mathbb{Z}/N$  the subgroups obtained by restricting the Galois action. Then  $\mathcal{Z}_0$  is obtained by suitably identifying the points in the fiber above  $\xi$  of  $\text{Ind}_{H^1}^H Z_0^1$  and  $\text{Ind}_{H^2}^H Z_0^2$ .

The following proposition follows from the explicit description of the components of  $\mathcal{Z}_0$ . Put  $\beta_1 = \text{gcd}(a_1, a_3, N)$  and  $\beta_2 = \text{gcd}(a_2, a_4, N)$ .

**Proposition 3.5.** (a) *The degree of  $Z_0^1 \rightarrow P_0^1$  (resp.  $Z_0^2 \rightarrow P_0^2$ ) is  $N/\beta_1$  (resp.  $N/\beta_2$ ).*  
 (b) *The genus of  $Z_0^1$  (resp.  $Z_0^2$ ) is  $(N - \text{gcd}(a_1, N) - \text{gcd}(a_3, N) - \text{gcd}(a_1 + a_3, N))/2\beta_1$  (resp.  $(N - \text{gcd}(a_2, N) - \text{gcd}(a_4, N) - \text{gcd}(a_1 + a_3, N))/2\beta_2$ ).*  
 (c) *The number of singular points of  $\mathcal{Z}_0$  is  $\text{gcd}(a_1 + a_3, N)$ .*

#### 4. VEECH'S $n$ -GONS REVISITED

In this section we realize  $(n, \infty, \infty)$ -triangle groups as affine groups of Teichmüller curves. This result is due to Veech but our method is completely different. The advantage of our method is that we obtain the Lyapunov exponents in Corollary 4.2 with almost no extra effort.

The reader may take this section as a guideline to the more involved next section. Here the family of cyclic covers we consider has only one elliptic fixed point. A  $(\mathbb{Z}/2\mathbb{Z})$ -quotient of this family is shown to be a Teichmüller curve. In the next section there are two elliptic fixed points and we will need a  $(\mathbb{Z}/2\mathbb{Z})^2$ -quotient. Moreover common divisors of  $m$  and  $n$  in the next section make a fiber product construction necessary that does not show up here.

We specialize the results of Section 3 for  $n = 2k \geq 4$  to the family  $g : \mathcal{Z} \rightarrow \mathbb{P}^*$  of curves of genus  $n - 1$  given by the equation

$$\mathcal{Z}_t : z^n = x(x-1)^{n-1}(x-t),$$

i.e. we consider the case  $N = n$ ,  $a_1 = a_3 = 1$  and  $a_2 = a_4 = n - 1$ . The exponents are chosen such that the local systems  $\mathbb{L}(i)$  for  $i = (n - 2)/2$  and  $i = (n + 2)/2$  have as projective monodromy group the triangle group  $\Delta(n, \infty, \infty)$ . The Riemann scheme for  $\mathbb{L}(i)$  is

$$(5) \quad \begin{bmatrix} t = 0 & t = 1 & t = \infty \\ 0 & 0 & (n - i)/2 \\ 1 - i/k & 0 & (n - i)2 \end{bmatrix}.$$

see e.g. [CoWo90]. We let

$$\varphi(x, y) = (x, \zeta_n y)$$

for some primitive  $n$ -th root of unity  $\zeta_n$ . The geometric fibers of  $g$  admit an involution covering  $x \mapsto t/x$  on the quotient by  $\varphi$ . We choose this involution to be

$$\sigma(x, y) = \left( \frac{t}{x}, \frac{t^{2/n}(x-1)(x-t)}{xy} \right) \quad (k \text{ even}), \quad \sigma(x, y) = \left( \frac{t}{x}, \zeta_n \frac{t^{2/n}(x-1)(x-t)}{xy} \right) \quad (k \text{ odd}).$$

We will see below that  $\sigma$  was chosen such to have (in fact 4) fixed points.

After having chosen an  $n/2$ -th root of  $t$ , which can be done by an unramified base change  $\pi : C \rightarrow \mathbb{P}^*$  as considered in Section 3, the map  $\sigma$  extends to an automorphism of the family of curves  $g_C : \mathcal{Z}_C \rightarrow C$ . Recall that we let  $\bar{\pi} : \bar{C} \rightarrow \mathbb{P}^1$  be the extension to a smooth

completion. We replace  $\pi : C \rightarrow \mathbb{P}^*$  by a larger unramified covering, still denoted by  $C$  with the following properties: The ramification order at  $\bar{\pi}^{-1}(0)$  is still  $n/2$  and the family  $g_C$  has unipotent monodromies.

We let  $f : \mathcal{X}_C = \mathcal{Z}/\langle\sigma\rangle \rightarrow C$ . The stable model  $\bar{f} : \bar{\mathcal{X}} \rightarrow \bar{C}$  will be shown to have smooth fibers over  $\bar{C} = \bar{\pi}^{-1}(\mathbb{P}^1 \setminus \{1, \infty\})$ .

**Theorem 4.1.** *Via the natural map  $m : \bar{C} \rightarrow M_g$  induced from  $\bar{f}$  the curve  $\bar{C}$  is an unramified covering of a Teichmüller curve. Here  $g = (n - 2)/2$ .*

**Proof:** We first determine the degeneration of  $g_C$  at  $c \in \bar{C}$  with  $\bar{\pi} \in \{0, 1, \infty\}$ . Since the monodromies of  $g_C$  at  $c \in \bar{C}$  are unipotent, the curve is stable and we may apply Proposition 3.5. For  $\bar{\pi}(c) \in \{1, \infty\}$  the components of  $\mathcal{Z}_c^i$  of  $\mathcal{Z}_c$  have genus 0. In fact, the monodromy around the preimages of these points on  $C$  is unipotent of infinite order as can be read off from the Riemann scheme and the knowledge of the projective monodromy group of  $\mathbb{L}(i)$ . Similarly we see that the monodromy around  $c$  with  $\bar{\pi}(c) = 0$  is finite. The definition of  $C$  implies therefore that it is trivial. The set  $S_u \subset C$  (notation of Section 3) consists precisely of  $\bar{\pi}^{-1}\{1, \infty\}$ .

By Proposition 3.5 the degenerate fiber of  $\mathcal{Z}_c$  over  $c \in \pi^{-1}(0)$  consists of two components of genus  $(n/2 - 1)$ . Note that  $\sigma$  acts as the permutation  $(0\infty)(1t)$  on the branch points of  $g : Y \rightarrow P$ . Hence  $\sigma$  interchanges the two components of  $\mathcal{Z}_c$ . From the action of  $\sigma$  on holomorphic one-forms given by equation (6) below, we deduce that the generic fiber of  $\mathcal{X}$  is smooth of genus  $(n/2 - 1)$ . We conclude that  $\mathcal{Z}_0/\langle\sigma\rangle$  is smooth and hence the set of singular fibers of  $\mathcal{X}$  is not larger than  $S_u$ .

One checks that for  $\sigma$  acts as

$$(6) \quad \sigma^* \omega_i = (-1)^i d(i) \omega_{n-i} \quad \text{for } i \neq n/2, \quad \sigma^* \omega_{n/2} = -\omega_{n/2}$$

on one-forms, where  $d(i) = t^{2i/n-1}$  if  $k$  is odd and  $d(i) = t^{2i/n-1} \zeta_n^i$  if  $k$  is even.

Now, consider the local system  $\mathbb{M} = \mathbb{L}((n-2)/2) \oplus \mathbb{L}((n+2)/2)$  in  $R^1(g_C)_* \mathbb{C}$  on  $C$ . It is invariant under  $\sigma$ . The part of  $\mathbb{M}$  on which  $\sigma$  acts trivially is a local subsystem  $\mathbb{L} \subset \mathbb{M}$ . This  $\mathbb{L}$  is necessarily of rank 2, since  $\omega_{(n-2)/2} + d(i)\omega_{(n+2)/2}$  is  $\sigma$ -invariant (resp. anti-invariant), if  $k$  is odd (resp. even) and  $\omega_{(n-2)/2} - d(i)\omega_{(n+2)/2}$  is  $\sigma$ -anti-invariant (resp. invariant) for  $k$  odd (resp. even). This also implies that the compositions

$$\mathbb{L} \rightarrow \mathbb{L}((n-2)/2) \oplus \mathbb{L}((n+2)/2) \rightarrow \mathbb{L}((n-2)/2)$$

and

$$\mathbb{L} \rightarrow \mathbb{L}((n-2)/2) \oplus \mathbb{L}((n+2)/2) \rightarrow \mathbb{L}((n+2)/2)$$

are non-trivial. Since the monodromy group  $\Gamma$  contains two non-commuting parabolic elements we conclude that  $\mathbb{L}((n-2)/2)$  is an irreducible local system and hence

$$\mathbb{L} \cong \mathbb{L}((n-2)/2) \cong \mathbb{L}((n+2)/2).$$

From Proposition 3.2 we deduce that  $\mathbb{L}((n-2)/2)$ , and hence  $\mathbb{L}$  as well, is maximal Higgs with respect to  $S_u$ . By Proposition 3.3 we conclude that  $\mathcal{X}$  is the universal family over an unramified cover of a Teichmüller curve as claimed.  $\square$

The last argument does not only apply to

$$\mathbb{L}((n-2)/2) \oplus \mathbb{L}((n+2)/2)$$

but to each local system  $\mathbb{L}((n-2i)/2) \oplus \mathbb{L}((n+2i)/2)$  for  $i = 1, \dots, n/2 - 1$ . The following now follows from Proposition 3.2:

**Corollary 4.2.** *The VHS of the family  $f : \mathcal{X} \rightarrow C$  splits as*

$$R^1 f_* \mathbb{C} \cong \bigoplus_{j=1}^{(n-2)/2} \mathbb{L}_j,$$

where  $\mathbb{L}_j$  is a rank 2 local system isomorphic to  $\mathbb{L}_{\mathcal{X}(n-2j)/2}$ . Moreover

$$\lambda(\mathbb{L}_j) = \frac{k-j}{k-1}.$$

Anton Zorich has communicated the authors that he (with Maxim Kontsevich) independently calculated these Lyapunov exponents.

**Remark 4.3.** The trace field of  $\Delta(n, \infty, \infty)$  is  $K = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$ , hence  $r = [K : \mathbb{Q}] \leq \phi(n/2)$ . The above corollary decomposes the VHS completely into rank two pieces. This is much finer than Theorem 1.1 that predicts only  $r$  pieces of rank two plus some rest.

Each fiber  $\mathcal{Z}_t$  admits an extra isomorphism, namely

$$\tau(x, y) = \left( \frac{x-t}{x-1}, y \frac{t-1}{(x-1)^2} \right)$$

It extends to an automorphism of the family  $g_{\tilde{C}} : \mathcal{Z}_{\tilde{C}} \rightarrow \tilde{C}$ . One checks that  $\tau$  and  $\sigma$  commute. Hence  $\tau$  descends to an automorphism of  $\mathcal{X}$ , which we also denote by  $\tau$ . Let  $p : \mathcal{U} = (\overline{\mathcal{X}}|_{\tilde{C}})/\langle \tau \rangle \rightarrow \tilde{C}$  the quotient family. One calculates that

$$\tau^* \omega_i = (-1)^{i+1} \omega_i.$$

From this we deduce that the fibers of  $\mathcal{X}$  are Veech surfaces that cover non-trivially Veech surfaces of smaller genus, the fibers of the fibers of  $p$ :

**Theorem 4.4.** *The moduli map  $\tilde{C} \rightarrow M_{g(\mathcal{U})}$  of the family of curves  $p : \mathcal{U} \rightarrow \tilde{C}$  is an unramified covering of Teichmüller curve. Its VHS decomposes as*

$$R^1 p_* \mathbb{C} \cong \bigoplus_{j=0}^{t(n)} \mathbb{L}_{1+2j},$$

where  $\mathbb{L}_j$  are the local systems appearing in the VHS of  $f$  and  $t(n) = (n-6)/4$  if  $k$  is odd, and  $t(n) = (n-4)/4$  if  $k$  is even.

In particular the genus of  $\mathcal{U}$  is  $t(n) + 1$  and

$$\lambda(\mathbb{L}_{1+2j}) = \frac{k - (1 + 2j)}{k - 1}$$

**Proof:** Both for  $k$  odd and  $k$  even the generating holomorphic one-form in  $\mathbb{L}_1$  is  $\tau$ -invariant. Hence this local system descends to  $\mathcal{U}$ . The property of being a Teichmüller curve now follows from Theorem 1.1. The remaining statements are easily deduced from Corollary 4.2.  $\square$

Let  $U$  be a fiber of  $\mathcal{U}$ . We denote by  $\omega_X \in \Gamma(X, \Omega_X^1)$  (resp.  $\omega_U \in \Gamma(U, \Omega_U^1)$ ) the differential that pulls back to  $\omega_{(n-2)/2} \pm d(i)\omega_{(n+2)/2}$  on  $\mathcal{Z}_c$ , where the sign depends on the parity of  $n$  and refer to it as the *generating differential* of the Teichmüller curve.

**Corollary 4.5.** *The Teichmüller curve  $\mathcal{X}$  is the one generated by the regular  $n$ -gon studied in [Ve89].*

**Proof:** The fiber  $\mathcal{Z}_0$  consists of two components isomorphic to

$$\mathcal{X}_0 : y^n = x(x-1)$$

which are interchanged by  $\sigma$ . The generating differential  $\omega_X$  specializes to the differential

$$\omega_0 = y^{(n-2)/2} dx/x(x-1)$$

on  $\mathcal{X}_0$ . There is an obvious isomorphism between the curve  $w^n - 1 = z^2$  and  $\mathcal{X}_0$  such that  $\omega_0$  pulls back to the differential  $dw/z$  considered by Veech ([Ve89] Theorem 1.1).  $\square$

Actually the family  $\mathcal{X}$  is isomorphic (after some base change) to

$$y^2 = p_t(z) = \prod_{i=1}^n (x - \zeta_n^i - t\zeta_n^{-i}).$$

This was shown by Lochak ([Lo05], see also [McM04]).

The following proposition is shown in [Ve89] Theorem 1.1. We give an alternative proof in our setting.

**Proposition 4.6.** *The projective affine group of a fiber of  $\mathcal{X}$  together with the generating differential contains the  $(n, \infty, \infty)$ -triangle group. The same holds for the fibers of  $\mathcal{U}$ .*

**Proof:** We first consider  $\mathcal{X}$ . We have to show that moduli map  $C \rightarrow M_g$  given by  $\mathcal{X}$  factors through  $\pi : C \rightarrow \mathbb{P}^*$ . That is, we have to show that two generic fibers  $\mathcal{X}_c$  and  $\mathcal{X}_{\tilde{c}}$  with  $c, \tilde{c} \in C$  such that  $\pi(c) = \pi(\tilde{c})$  are isomorphic. Equivalently, we have to show that for  $c, \tilde{c}$  as above there is an isomorphism  $i_0 : \mathcal{Z}_c \rightarrow \mathcal{Z}_{\tilde{c}}$  which is  $\sigma$ -equivariant.

It suffices show the existence of  $i_0$  after any base change  $\pi : C' \rightarrow \mathbb{P}^*$  such that  $\sigma$  is defined on  $\mathcal{Y}_{C'}$ . I.e., we may suppose that  $\pi : C' \cong \mathbb{P}_s^1 \rightarrow \mathbb{P}_t^1$  is given by  $t = s^n/2$ . The hypothesis  $\pi(c) = \pi(\tilde{c})$  implies that  $c = \zeta_n^{2j} \tilde{c}$  for some  $j$ . It follows that canonical isomorphism  $i : \mathcal{Z}_c \rightarrow \mathcal{Z}_{\tilde{c}}$ , given by  $(x, y) \mapsto (x, y)$ , satisfies

$$\sigma \circ i = \varphi^{2j} \circ i \circ \sigma.$$

Hence  $i_0 = \varphi^j \circ i$  is the isomorphism we were looking for.

The proof for the family  $\mathcal{U}$  is similar.  $\square$

We record for completeness:

**Corollary 4.7.** *All  $(n, \infty, \infty)$ -triangle groups for  $n \geq 4$  arise as projective affine groups.*

**Remark 4.8.** For  $n$  odd the same construction works with  $N$  and  $a_i$  chosen as above. The local exponents of  $(\mathbb{L}(i), \omega_i)$  at  $t = 0$  are then  $1 - 2i/n$ . The local system  $\mathbb{L}(i)$  becomes maximal Higgs for  $i = (n-1)/2$  and  $i = (n+1)/2$ , after a base change  $\pi : C \rightarrow \mathbb{P}^*$  whose extension to  $\overline{C} \rightarrow \mathbb{P}^1$  is now ramified of order  $n$  over 0. The quotient family  $f : \mathcal{X} = \mathcal{Z}/\langle \sigma \rangle \rightarrow C$  may be constructed in the same way as above. Its moduli map yields as above a Teichmüller curve  $\tilde{C} \rightarrow M_g$  where  $g = (n-1)/2$ . The corresponding translation surfaces are again the ones studied in [Ve89]. Veech also determines that the affine group is not  $\Delta(n, \infty, \infty)$  but the bigger group  $\Delta(2, n, \infty)$  containing  $\Delta(n, \infty, \infty)$  with index two. We obtain the same family of curves also as a special case of the construction in Section 5, by putting  $m = 2$ . For this family we calculate similarly, using Proposition 3.2,

$$\lambda(\mathbb{L}_i) = \frac{2i}{n-1}, \quad i = 1, \dots, (n-1)/2$$

5. REALIZATION OF  $\Delta(m, n, \infty)$  AS PROJECTIVE AFFINE GROUP

Let  $m, n > 1$  be integers with  $mn \geq 6$ . We start by constructing a family of cyclic covers of the projective line branched at 4 points whose Riemann scheme is

$$(7) \quad \left[ \begin{array}{ccc} t=0 & t=1 & t=\infty \\ 0 & 0 & A \\ 1/n & 1/m & A \end{array} \right],$$

where  $A$  satisfies  $2A + 1/n + 1/m = 1$ . By the results of Section 3 this is achieved by the following construction. We let

$$\sigma_1 = \frac{nm + m - n}{2mn}, \quad \sigma_2 = \frac{nm - m + n}{2mn}, \quad \sigma_3 = \frac{nm + m + n}{2mn}, \quad \sigma_4 = \frac{nm - m - n}{2mn}.$$

and we let  $N$  be the least common denominator of these fractions. We let  $a_i = N\sigma_i$  and consider the family of curves  $g : \mathcal{Z} \rightarrow \mathbb{P}^*$  given by

$$\mathcal{Z}_t : y^N = x^{a_1}(x-1)^{a_2}(x-t)^{a_3}.$$

The exponents  $a_i$  are chosen such that the local system  $\mathbb{L}_\chi$  has as projective monodromy group the triangle group  $\Delta(m, n, \infty)$ , see again e.g. [CoWo90]. The family  $g$  cyclically covers the constant family  $\mathcal{P} \cong \mathbb{P}^1 \times \mathbb{P}^* \rightarrow \mathbb{P}^*$ .

The plan of this section is as follows. We construct a cover  $\mathcal{Y} \rightarrow \mathcal{Z}$  such that the involutions

$$(8) \quad \begin{aligned} \sigma(x) &= (t(x-1)/(x-t)), \\ \tau(x) &= (t/x) \end{aligned}$$

of  $\mathcal{P} \rightarrow \mathbb{P}^*$  lift to involutions of the family  $\mathcal{Y}_C \rightarrow C$  obtained from  $\mathcal{Y} \rightarrow \mathbb{P}^*$  by a suitable unramified base change  $\pi : C \rightarrow \mathbb{P}^*$ . We denote these lifts again by  $\sigma$  and  $\tau$ . If  $m$  and  $n$  are relatively prime then in fact  $\mathcal{Y}$  equals  $\mathcal{Z}$ . We then modify  $\tau$  and  $\sigma$  by appropriate powers of a generator of  $\text{Aut}(\mathcal{Z}/\mathcal{P})$  such that the group  $H = \langle \tau, \sigma \rangle$  is still isomorphic to  $(\mathbb{Z}/2)^2$  and such that  $\sigma$  and  $\tau$  and  $\sigma\tau =: \rho$  have ‘as many fixed points as possible’. We then consider the quotient family  $f : \mathcal{X} = \mathcal{Y}/H \rightarrow C$ . Its stable model  $\bar{f} : \bar{\mathcal{X}} \rightarrow \bar{C}$  has smooth fibers over  $\bar{C} = \bar{\pi}^{-1}(\mathbb{P}^1 \setminus \{\infty\})$ , where  $\bar{\pi} : \bar{C} \rightarrow \mathbb{P}^1$  extends  $\pi$ .

Together with an analysis of the action of  $H$  on differentials we can apply Theorem 1.1 to produce Teichmüller curves.

**Theorem 5.1.** *Via the natural map  $m : \bar{C} \rightarrow M_g$  induced from  $\bar{f}$  the curve  $\bar{C}$  is an unramified covering of a Teichmüller curve. The genus  $g$  is given in Corollary 5.4.*

As corollaries to this result we calculate the precise VHS of  $f$ , the projective affine group of the translation surfaces corresponding to  $f$  and we compare these Teichmüller curves in the case  $m = 3$  to the curves obtained by Ward.

We start with some more notation. We write  $Z$  (resp.  $P, X, Y$ ) for the geometric generic fiber of  $\mathcal{Z}$  (resp.  $\mathcal{P}, \mathcal{X}, \mathcal{Y}$ ). We choose a primitive  $N$ th root of unity  $\zeta_N \in \mathbb{C}$  and define the automorphism  $\varphi_1 \in \text{Aut}(\mathcal{Y}/\mathcal{P})$  by

$$\varphi_1(x, y) = (x, \zeta_N y).$$

We need to determine the least common denominator of the  $\sigma_i$ ,  $i = 1, \dots, 4$ , precisely. Let  $m = 2^\mu m'$ ,  $n = 2^\nu n'$  with  $m', n'$  odd. We may suppose that  $\mu \geq \nu$ . Define

$$\gamma_1 = \text{gcd}(2mn, mn + m - n), \quad \gamma_2 = \text{gcd}(2mn, mn + m + n), \quad \gamma = \text{gcd}(m, n)$$

and write  $\gamma = 2^\nu \gamma'$ . We distinguish four cases and determine the denominator  $N = 2mn/\gcd(\gamma_1, \gamma_2)$  accordingly. We let  $\delta = \min\{\mu - \nu + 2, \mu + 1\}$ .

Case O: odd	$\mu = \nu = 0,$	$N = 2mn/\gamma,$	$\widehat{N} = N/\gamma = 2^\delta m' n' / \gamma'^2,$
Case OE: $m$ odd, $n$ even	$\mu > \nu = 0,$	$N = 2mn/\gamma,$	$\widehat{N} = N/\gamma = 2^\delta m' n' / \gamma'^2,$
Case DE: different 2-val., even	$\mu > \nu > 0,$	$N = 2mn/\gamma,$	$\widehat{N} = 2N/\gamma = 2^\delta m' n' / \gamma'^2,$
Case S: same 2-valuation, even	$\mu = \nu \neq 0,$	$N = mn/\gamma,$	$\widehat{N} = N/\gamma = mn/\gamma^2.$

It is useful to keep in mind that  $\gamma = \gcd(\gamma_1, \gamma_2)$ , except in case  $S$  where  $2\gamma = \gcd(\gamma_1, \gamma_2)$ . For convenience we let  $\delta := 0$  in case  $S$ .

We want to determine the maximal intermediate covering of  $Z \rightarrow P$  to which  $\tau$  lifts. This motivates the definition of  $\widehat{N}$ . Let  $0 < \bar{\alpha} < \widehat{N}$  be the integer satisfying

$$\bar{\alpha} \equiv 1 \pmod{m'/\gamma'}, \quad \bar{\alpha} \equiv -1 \pmod{n'/\gamma'}, \quad \bar{\alpha} \equiv \begin{cases} 1 & \pmod{2^\delta} \text{ cases O, OE, S,} \\ \frac{n'+2^{\mu-\nu}m'}{n'-2^{\mu-\nu}m'} & \pmod{2^\delta} \text{ case DE.} \end{cases}$$

It will be convenient to lift  $\bar{\alpha}$  to an element  $\alpha$  in  $\mathbb{Z}/N\mathbb{Z}$  such that  $\alpha^2 = 1$ .

Recall that for a rational number  $\sigma$ , we write  $\sigma(i) := \langle i\sigma \rangle$  (fractional part). Similarly, for an integer  $a$  we write  $a(i) = a(i; \nu) = \nu \langle ia/\nu \rangle$ , where  $\nu$  is mostly clear from the context. For each integer  $0 < i < N$  which is prime to  $N$ , we write as in the previous section

$$z(i) = \frac{z^i}{x^{[i\sigma_1]}(x-1)^{[i\sigma_2]}(x-t)^{[i\sigma_3]}}, \quad \text{hence} \quad z(i)^N = x^{a_1(i)}(x-1)^{a_2(i)}(x-t)^{a_3(i)}.$$

**Lemma 5.2.** (a) *In the cases O, OE and DE the covering  $Z \rightarrow P$  has ramification order  $\gamma N/\gamma_1$  (resp.  $\gamma N/\gamma_2$ ) in points of  $\mathcal{Z}$  over  $x = 0, 1$  (resp.  $x = t, \infty$ ). In case S the ramification orders are  $\gamma N/2\gamma_1$  (resp.  $\gamma N/2\gamma_2$ ). Therefore*

$$g(Z) = \begin{cases} 1 + N - \frac{\gamma_1 + \gamma_2}{2\gamma} & \text{case S,} \\ 1 + N - \frac{\gamma_1 + \gamma_2}{\gamma} & \text{(other cases).} \end{cases}$$

- (b) *The automorphism  $\sigma$  lifts to an automorphism  $\sigma$  of order 2 of  $Z$ .*
- (c) *The automorphism  $\tau$  of  $P$  lifts to an automorphism  $\tau$  of order 2 of  $\widehat{Z} := Z/\langle \varphi_1^{\widehat{N}} \rangle$ . Moreover, we may choose the lifts such that  $\sigma, \tau$  commute as elements of  $\text{Aut}(\widehat{Z})$ .*
- (d) *We may choose the lifts  $\sigma, \tau$  such that, moreover,  $\tau$  has  $4m/\gamma$  fixed points (resp.  $2m/\gamma$  in case S) and  $\rho := \sigma\tau$  has  $4n/\gamma$  fixed points on  $\widehat{Z}$  (resp.  $2n/\gamma$  in case S).*
- (e) *With  $\sigma$  and  $\tau$  chosen as in (c) the automorphism  $\sigma$  has no (2 in case S) fixed points both on  $Z$  and on  $\widehat{Z}$ .*

**Proof:** The statements in (a) are immediate from the definitions. For (b) and (c) we choose once and for all elements  $t^{1/n}, (t-1)^{1/m} \in \overline{\mathbb{C}(t)}$ . Define

$$(9) \quad c = (t-1)^{\sigma_2 + \sigma_3}, \quad d = t^{\sigma_1 + \sigma_3}.$$

Then

$$\sigma(z) = cd \frac{x(x-1)}{z(x-t)} = cd \frac{z(-1)}{(x-t)^2}$$

defines a lift of  $\sigma$  to  $Z$ , since  $\sigma_1 + \sigma_2 = \sigma_3 + \sigma_4 = 1$ . Moreover, this lift has order 2. We denote it again by  $\sigma$ . The quotient curve  $\widehat{Z}$  is defined by the equation

$$\bar{z}^{\widehat{N}} = x^{\bar{a}_1}(x-1)^{\bar{a}_2}(x-t)^{\bar{a}_3},$$

where  $\bar{a}_i$  denotes  $a_i \bmod \widehat{N}$ . One computes that  $\alpha$  has the property

$$(10) \quad (\bar{a}_1(\alpha), \bar{a}_2(\alpha), \bar{a}_3(\alpha), \bar{a}_4(\alpha)) = (\bar{a}_4, \bar{a}_3, \bar{a}_2, \bar{a}_1).$$

This implies that

$$\tau(\bar{z}) = d^\gamma \frac{\bar{z}(\alpha)}{x^{2\gamma}}$$

defines a lift of  $\tau$  to  $\widehat{Z}$  which has order 2. It is easy to check that  $\tau$  commutes with the image of  $\sigma$  on  $\widehat{Z}$ . This proves (b). Furthermore, one checks that  $\sigma$  is an involution and that

$$\tau\varphi_1\tau = \varphi_1^\alpha \in \text{Aut}(\widehat{Z}) \quad \text{and} \quad \sigma\varphi_1\sigma = \sigma^{-1} \in \text{Aut}(Z).$$

We start with the proof of (d). Let  $x_1 = \sqrt{t}$  be one of the fixed points of  $\tau$  on  $P$  and let  $R$  be a point in the fiber of  $\widehat{Z} \rightarrow P$  over  $x_1$ . We may describe the whole fiber by  $R_a := \varphi_1^a R$  for  $a = 0, \dots, \widehat{N} - 1$ . Suppose that  $\tau R = R_{a_0}$ , hence  $\tau R_a = R_{a_0 + \alpha a}$ . Since  $\tau$  is an involution,  $a_0$  satisfies necessarily  $a_0 \equiv 0 \pmod{m'/\gamma'}$  and  $2a_0 \equiv 0 \pmod{2^\delta}$ . Furthermore,  $R_a$  is a fixed point of  $\tau$  if and only if

$$(11) \quad a_0 \equiv 2a \pmod{n'/\gamma'} \quad \text{and} \quad a_0 \equiv 2^{\mu-\nu+1} a \frac{-m'}{n' - 2^{\mu-\nu}m'} \pmod{2^\delta}.$$

Hence if  $\tau$  has a fixed point in this fiber it has precisely  $2^{(\mu-\nu+1)}m'/\gamma'$  fixed points in this fiber ( $m'/\gamma' = m/\gamma$  in case S). Since  $\tau$  and  $\sigma$  commute,  $\sigma$  bijectively maps fixed points of  $\tau$  over  $x_1$  to fixed points of  $\tau$  over  $x_2 = -\sqrt{t}$ . Hence, if  $\tau$  has a fixed point, then the number of fixed points is as stated in (d).

Similarly, let  $x_3 = 1 + \sqrt{1+t}$  be one of the fixed points of  $\rho$  on  $P$  and let  $S$  be a point in the fiber over  $x_3$ . We may suppose that  $\rho S = S_{b_0}$  and as above we deduce  $b_0 \equiv 0 \pmod{m'/\gamma'}$  and  $2^{\mu-\nu+1}b_0 \equiv 0 \pmod{2^\delta}$ . The automorphism  $\rho$  has the fixed point  $S_b$  if

$$(12) \quad b_0 \equiv 2b \pmod{m'/\gamma'} \quad \text{and} \quad b_0 \equiv 2b \frac{n'}{n' - 2^{\mu-\nu}m'} \pmod{2^\delta}.$$

Analogously to the argument for  $\tau$ , one checks that if  $\rho$  has a fixed point then it has as many fixed points as claimed in (d).

Note that we may replace  $\sigma$  by  $\varphi^i\sigma$  and  $\tau$  by  $\varphi^j\tau$  without changing the orders of these elements and such that they still commute if the following conditions are satisfied:

$$(13) \quad j \equiv 0 \pmod{m'/\gamma'}, \quad j \equiv i \pmod{n'/\gamma'} \quad \text{and} \quad 2j \equiv 2^{\mu-\nu+1}i \pmod{2^\delta}.$$

The only obstruction for  $\tau$  and  $\rho$  to have fixed points consists in the condition modulo  $2^\delta$ . We check in each case that we can modify  $\tau$  and  $\rho$  respecting (13) such that this obstruction vanishes.

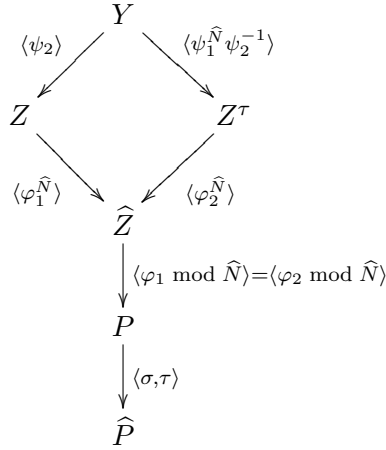
In case S there is nothing to do, since  $\delta = 0$ . In case O we might have to change the parity of  $a_0$  and  $b_0$  or both, since  $\delta = 1$ . This is possible since (13) imposes no parity condition in this case: we replace  $\sigma$  by  $\varphi^i\sigma$  and  $\tau$  by  $\varphi^j\tau$  such that  $j \equiv a_0 \pmod{2}$  and  $i + j \equiv b_0 \pmod{2}$ . In case OE the conditions for  $\tau$  to have fixed points are satisfied. We might have to change the parity of  $b_0$  which can be achieved since (13) imposes no parity conditions on  $i$  in this case. In case DE we can solve equations (11) resp. (12) for  $a$  resp.  $b$  using the conditions imposed on  $a_0$  and  $b_0$  from  $\tau$  and  $\rho$  being involutions.

For (e) we check with the same argument as above that  $\sigma$  has 0 or 4 (resp. 0 or 2 in case S) fixed points. Checking case by case one finds that  $\widehat{Z} \rightarrow P$  is totally ramified over  $\{0, 1, t, \infty\}$ . Hence  $g(\widehat{Z}) = \widehat{N} - 1$ . The Riemann–Hurwitz formula implies that there are

no fixed points on  $\widehat{Z}$  hence none on  $Z$  in case O, D and DE. The number of fixed points of  $\sigma$  in case S may be checked directly using the technique to count fixed points of  $\tau$  on  $\widehat{Z}$ .  $\square$

Let  $Z^\tau$  be the conjugate of  $Z$  under  $\tau$ . Define  $Y$  as the normalization of  $Z \times_{\widehat{Z}} Z^\tau$ . As remarked above, the definition of  $\widehat{N}$  implies that  $\widehat{Z} \rightarrow P$  is the largest subcover of  $Z \rightarrow P$  such that  $\tau$  lifts to  $\widehat{Z}$ . In other words,  $Y \rightarrow \widehat{P} := P/\langle\sigma, \tau\rangle$  is the Galois closure of  $Z \rightarrow \widehat{P}$ . This implies that  $Y$  is in fact connected. I.e., the particular choice of  $\widehat{N}$  is used precisely to guarantee that the Veech surfaces constructed in Theorem 5.1 are connected.

By construction,  $\sigma$  lifts to  $Z$  acting on both  $Z$  and  $Z^\tau$  and  $\tau$  lifts to  $Z$  by exchanging the two factors of the fiber product. These two involutions commute and  $\rho := \sigma\tau$  also has order 2. We have defined the following coverings. The labels indicate the Galois group of the morphism with the notation introduced in the following lemma.



**Lemma 5.3.** (a) We may choose a generator  $\varphi_2$  of  $\text{Aut}(Z^\tau/P)$  such that the Galois group,  $G_0$ , of  $Y/P$  is

$$G_0 \cong \{(\varphi_1^i, \varphi_2^j), i, j, \in \mathbb{Z}/N\mathbb{Z}, i \equiv j \bmod \widehat{N}\} \subset \langle\varphi_1\rangle \times \langle\varphi_2\rangle \cong (\mathbb{Z}/N\mathbb{Z})^2.$$

We fix generators  $\psi_1 = (\varphi_1, \varphi_2)$  and  $\psi_2 = (0, \varphi_2^{\widehat{N}})$  of  $G_0$ . The Galois group,  $G$ , of the covering  $Y/\widehat{P}$  is generated by  $\psi_1, \psi_2, \sigma, \tau$ , satisfying

$$\psi_1^N = \psi_2^\beta = \sigma^2 = \tau^2 = 1, \quad [\psi_1, \psi_2] = [\sigma, \tau] = 1,$$

$$\sigma\psi_i\sigma = \psi_i^{-1} \quad (i = 1, 2), \quad \tau\psi_1\tau = \psi_1^\alpha, \quad \tau\psi_2\tau = \psi_1^{\alpha N}\psi_2^{-\alpha} (= (\varphi_1^{\alpha N}, 0))$$

- (b) The genus of  $Y$  is  $g(Y) = 1 + N\beta - 2\beta$ , where  $\beta = \gamma/2$  in case DE and  $\beta = \gamma$  in the other cases.
- (c) The number of fixed points of  $\tau$  on  $Y$  is  $4m\beta/\gamma$  (resp.  $2m$  in case S).
- (d) The number of fixed points of  $\rho$  on  $Y$  is  $4n\beta/\gamma$  (resp.  $2n$  in case S).
- (e) The involution  $\sigma$  has no fixed points on  $Y$ .

**Proof:** The presentation in (a) follows from the above construction. To prove (b), we remark that  $Z^\tau$  is given by the equation

$$\tilde{z}^N = x^{a_4}(x-1)^{a_3}(x-t)^{a_2},$$

compare to (10). Recall that  $\widehat{Z} \rightarrow P$  is totally ramified over  $\{0, 1, t, \infty\}$ . Hence at each of the  $\gamma_1/\gamma$  points (resp.  $\gamma_1/2\gamma$  in case S) over 0 and 1 in  $Z$  the map  $Z \rightarrow \widehat{Z}$  is branched of

order  $\gamma^2/\gamma_1$  (resp.  $2\gamma^2/\gamma_1$  in case S and  $\gamma^2/2\gamma_1$  in case DE). The other covering  $Z^\tau \rightarrow \widehat{Z}$  is branched at the corresponding  $\gamma_1/\gamma$  (resp.  $\gamma_1/2\gamma$  in case S) points of order  $\gamma^2/\gamma_2$  (resp.  $2\gamma^2/\gamma_2$  in case S and  $\gamma^2/2\gamma_2$  in case DE). Over  $t$  and  $\infty$  instead of 0 and 1 the roles of  $\gamma_1$  and  $\gamma_2$  are interchanged.

It follows from Abhyankar's Lemma that  $Y \rightarrow \widehat{Z}$  is ramified in all cases at each point over  $0, 1, t, \infty$  of order  $\beta$ . Hence these fibers of  $Y \rightarrow P$  consist of  $\beta$  points in each case.

For (c), (d) and (e) note that  $Z \rightarrow P$  is unramified over the fixed points of  $\tau, \sigma$  and  $\rho$ . Hence  $Y$  is indeed the fiber product in neighborhoods of these points. Since  $\tau$  interchanges the two factors, exactly  $\beta$  of the  $\beta^2$  preimages in  $Y$  of a fixed point of  $\tau$  on  $Z$  will be fixed by the lift of  $\tau$  to  $Y$ . This completes the proof of (c).

For (d) note that  $\text{id} \times \sigma : Z \times_{\widehat{Z}} Z^\tau \rightarrow Z \times_{\widehat{Z}} Z^\sigma$  is an isomorphism and we may now argue as in (c).

If  $\sigma$  has a fixed point on  $Y$  it has a fixed point on  $Z$ . This implies (e) for cases O, OE and DE. In case S we argue as in the previous lemma, and conclude that  $\sigma$  has 0 or two fixed points in  $Y$  above each fixed point in  $\widehat{Z}$ . We deduce the claim from the Riemann–Hurwitz formula applied to  $Y \rightarrow Y/H$ .  $\square$

**Corollary 5.4.** *The genus of  $X = Y/H$  is  $g(X) = (mn - m - n - \gamma)\beta/2\gamma + 1$  in case O, OE and D and  $g(X) = (mn - m - n - \gamma)/4 + 1$  in case S.*

Until now we have been working on a geometric fiber of  $g : \mathcal{Y} \rightarrow \mathbb{P}^*$  etc. Everything works fine in families if we pass to an unramified cover  $\pi : C \rightarrow \mathbb{P}^*$  obtained by adjoining the elements  $c, d$  defined in (9) to  $\mathbb{C}(t)$ . Passing to a further unramified cover, if necessary, we may suppose that the VHS of the pullback family  $h_C : \mathcal{Y}_C \rightarrow C$  is unipotent and that this family admits a stable model  $h_C : \overline{\mathcal{Y}}_C \rightarrow \overline{C}$  over this base curve.

The following lemma describes the action of  $H$  on the degenerate fibers of  $h_C$ .

**Lemma 5.5.** *Let  $c \in C$  be a point with  $\pi(c) \in \{0, 1\}$ . The quotient  $\mathcal{X}_c := (\mathcal{Y}_C)_c/H$  is smooth and*

$$g(\mathcal{X}_c) = \begin{cases} (mn - m - n - \gamma)\beta/2\gamma + 1 & \text{cases O, OE and DE,} \\ (mn - m - n - \gamma)/4 + 1 & \text{case S.} \end{cases}$$

**Proof:** Choose  $c \in \pi^{-1}(0)$ . The case that  $c \in \pi^{-1}(1)$  is similar, and left to the reader. By Proposition 3.5 the fiber  $(\mathcal{Z}_C)_c$  consists of two irreducible components which we call  $Z_0^1$  and  $Z_0^2$ ; we make the convention that the fixed points  $x = 0, t$  of  $\varphi_1$  on  $\mathcal{Z}^C$  specialize to  $Z_0^1$ . Choosing suitable coordinates, the curve  $Z_0^1$  is given by

$$(14) \quad z_0^N = x_0^{a_1}(x_0 - 1)^{a_3}.$$

The components  $Z_0^1$  and  $Z_0^2$  intersect in  $2m/\gamma$  points (resp.  $m/\gamma$  in case S). We write  $P_0^j$  for the quotient of  $Z_0^j$  by  $\langle \varphi_1 \rangle \cong \mathbb{Z}/N$ .

We claim that the fiber  $(\mathcal{Y}_C)_c$  consists of 2 irreducible components  $Y_0^1, Y_0^2$ , as well. Let  $\mathcal{N}$  be the normalization of the fiber product  $(\mathcal{Z}_C)_c \times_{(\overline{\mathcal{Z}}_C)_c} (\mathcal{Z}_C)_c^\tau$ . By Abhyankar's Lemma again  $\mathcal{N} \rightarrow (\mathcal{Z}_C)_c$  is étale at the preimages of the intersection point of the two components of  $(\mathcal{P}_C)_c$ . Hence  $\mathcal{N}$  consists of two curves: the fiber products over  $Z_0^j/\langle \varphi_1^{\widehat{N}} \rangle$  of  $Z_0^j$  with its  $\tau$ -conjugate for  $j = 1, 2$ . These two curves intersect transversally in  $2m\beta/\gamma$  points. This implies that  $\mathcal{N}$  is a stable curve and indeed the fiber  $(\mathcal{Y}_C)_c$ .

One computes that  $g(Y_0^j) = 1 + mn - m\beta/\gamma - \beta$  in cases O, OE and DE and  $g(Y_0^j) = mn - m/2 + 1 - \gamma$  in case S. Since  $\rho$  acts on the points  $\{0, 1, t, \infty\}$  as the permutation  $(0t)(1\infty)$  we conclude that  $\rho$  fixes the components  $Y_0^j$  while  $\sigma$  and  $\tau$  interchange them. Clearly, for a coordinate  $x_0$  as in (14) we have that  $\rho(x_0) = 1 - x_0$ , i.e.  $\rho$  fixes the points  $1/2$ . This is a specialization of one of the two fixed points  $1 \pm \sqrt{1-t} \in P$ . Since by Lemma 5.3 the automorphism  $\rho$  fixes  $2n$  ( $n$  in case S) points in  $Y$  above each of these points of  $P$  it follows that  $\rho$  fixes  $2n$  (resp.  $n$ ) points of  $Y_0^j$  with  $x_0 = 1/2$ . It remains to compute the number  $r_\infty$  of fixed points of  $\rho$  over  $x_0 = \infty$ .

Suppose we are not in case S. Then by the Riemann–Hurwitz formula

$$g(\mathcal{X}_c) = g(Y_0^j/\langle\rho\rangle) = (mn - m - n - \gamma)\beta/2\gamma + 1 - r_\infty/4.$$

On the other hand we may apply the Riemann–Hurwitz formula to the quotient map  $Z_0^j \rightarrow Z_0^j/\langle\rho\rangle$ . We conclude  $r_\infty \equiv 0 \pmod{4}$ . But representing the fiber in  $Z_0^j$  over  $\infty$  as  $\varphi_1^b R$  for  $b = 1, \dots, 2m/\gamma$  we conclude as in the proof of Lemma 5.2 that  $r_\infty$  equals zero or two. Together, it follows that  $r_\infty = 0$ .

In case S we have

$$g(\mathcal{X}_c) = (mn - m - n - \gamma)/4 + 1 - r_\infty/4.$$

and we conclude as above that  $r_\infty = 0$ .

Smoothness of the special fiber follows by comparing its genus to the genus of the generic fiber.  $\square$

**Proof of Theorem 5.1:** We have shown in Lemma 5.5 that the set of singular fibers of  $f : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{C}}$  is not larger than  $S_u = g^{-1}(\infty)$ . We have to show that the VHS of  $f$  contains a local subsystem of rank 2 which is maximal Higgs.

We decompose the VHS of  $g$  into the characters

$$\chi(i, j) : \begin{cases} G_0 & \rightarrow \mathbb{C} \\ \psi_1 & \mapsto \zeta_N^i \\ \psi_2 & \mapsto (\zeta_N^{\widehat{N}})^j. \end{cases}$$

We let  $\mathbb{L}(i, j) \subset R^1 h_* \mathbb{C}$  be the local system on which  $G$  acts via  $\chi(i, j)$ . Local systems with  $j = 0$  arise as pullbacks from  $\mathcal{Z}$ . By Lemma 3.1 the local systems  $\mathbb{L}(i, 0)$  are of rank two if  $i$  does not divide  $N$ . Using the presentation of  $G$  one checks that  $\sigma^* \mathbb{L}(i, j) = \mathbb{L}(-i, -j)$  and  $\tau^* \mathbb{L}(i, j) = \mathbb{L}(-\alpha i, \alpha(i - j))$ .

By construction of  $\mathcal{Z}$  at the beginning of this section and Proposition 3.2 we deduce that the 4 summands of

$$\mathbb{M} := \mathbb{L}(1, 0) \oplus \mathbb{L}(-1, 0) \oplus \mathbb{L}(-\alpha, \alpha) \oplus \mathbb{L}(\alpha, -\alpha)$$

are of rank two and maximal Higgs. Since  $H$  permutes these factors transitively, we conclude that for each character  $\xi$  of  $H$  there is a rank two local subsystem of  $\mathbb{M}$  on which  $H$  acts via  $\xi$ . Moreover the projection of the subsystem  $\mathbb{L} := \mathbb{M}^H$  to each summand is non-trivial. Since the 4 summands of  $\mathbb{M}$  are irreducible by construction, this implies that

$$\mathbb{L} \cong \mathbb{L}(1, 0) \cong \mathbb{L}(-1, 0) \cong \mathbb{L}(-\alpha, \alpha) \cong \mathbb{L}(\alpha, -\alpha).$$

Hence  $\mathbb{L}$  is maximal Higgs with respect to  $S_u$ . We conclude using Theorem 1.1 that the extension of  $f$  to  $\pi^{-1}(\mathbb{P}^1 \setminus \{\infty\})$  is the pullback of universal family of curves to an unramified cover of a Teichmüller curve.  $\square$

The proof of Theorem 5.1 contains more information on the VHS of  $f$  and on the Lyapunov exponents  $\lambda(\mathbb{L}_i)$ . We work out the details in the most transparent case  $m, n$  relatively

prime and both odd. The interested reader can easily work out the Lyapunov exponents in the remaining cases, too. In the case  $m, n$  relatively prime and odd, the curves  $\mathcal{Z}$  and  $\mathcal{Y}$  coincide and the local system  $\mathbb{L}(i, j)$  is  $\mathbb{L}(i)$  with the notation as in Lemma 3.1 and forgetting  $j$ .

We deduce from the arguments of the above proof that, for each  $i$  not divisible by  $m$  or  $n$ , there is an  $H$ -invariant local system  $\mathbb{L}_i$  with

$$\mathbb{L}_i \cong \mathbb{L}(i) \cong \mathbb{L}(\alpha i) \cong \mathbb{L}(-\alpha i) \cong \mathbb{L}(-i).$$

Since those  $i$  fall into  $(m-1)(n-1)/2$  orbits under  $\langle \pm 1, \pm \alpha \rangle$ , we have the complete description of the VHS of  $h$ . Let  $c_j(i) = \sigma_j(i) + \sigma_3(i) - 1$ .

**Corollary 5.6.** *In case  $m, n$  relatively prime and odd the VHS of  $f$  splits as*

$$R^1 f_* \mathbb{C} \cong \bigoplus_{j \in J} \mathbb{L}(j),$$

where  $\mathbb{L}(j)$  is an irreducible rank two local system and  $j$  runs through a set of representatives of

$$J = \{0 < i < N, m \nmid i, n \nmid i\} / \sim, \quad \text{where } i \sim -i \sim \alpha i \sim -\alpha i.$$

Moreover, the Lyapunov exponents are

$$\lambda(\mathbb{L}(i)) = \frac{mn - e_1(i)m - e_2(i)m}{mn - m - n}, \quad \text{where } e_1(i) = n|c_1(i)|; \quad e_2(i) = m|c_2(i)|.$$

**Proof:** This follows immediately from specializing the results of Section 3 to the  $a_i$  considered here.  $\square$

**Example 5.7.** We calculate the Lyapunov exponents explicitly for  $m = 3$  and  $n = 5$ . Then  $N = 2nm = 30$  and hence  $\alpha = 19$ . We need to calculate the  $\lambda(\mathbb{L}(i))$  only up to the relation ' $\sim$ ' and hence expect at most 4 different values. One checks:

$$\lambda(\mathbb{L}(i)) = \begin{cases} 7/7 & \text{if } i \sim 1, \\ 4/7 & \text{if } i \sim 2, \\ 2/7 & \text{if } i \sim 4, \\ 1/7 & \text{if } i \sim 7. \end{cases}$$

In particular, we see that the  $\lambda(\mathbb{L}(i))$  do in general not form an arithmetic progression as one might have guessed from studying Veech's  $n$ -gons.

**Remark 5.8.** Note that  $K := \mathbb{Q}(\cos(\pi/n), \cos(\pi/m))$  is the trace field of the  $\Delta(m, n, \infty)$ -triangle group. Hence  $r = [K : \mathbb{Q}] \leq \phi(mn)/4 \leq (m-1)(n-1)/4$ . Here again the decomposition of the VHS is finer than predicted by Theorem 1.1, compare the remark after Corollary 4.2.

Let  $X$  be any fiber of  $f$ . We denote by  $\omega_X \in \Gamma(X, \Omega_X^1)$  a generating differential, i.e. a holomorphic differential that generates  $(1, 0)$ -part of the maximal Higgs local system when restricted to the fiber  $X$ . This condition determines  $\omega_X$  uniquely up to scalar multiples.

**Theorem 5.9.** *The projective affine group of the translation surface  $(X, \omega_X)$  is*

- (a) *the  $(m, n, \infty)$ -triangle group, if  $m \neq n$ .*
- (b) *the  $(m, m, \infty)$ -triangle group or the  $(2, m, \infty)$ -triangle group, if  $m = n$ .*

**Proof:** We first show that the triangle group  $\Delta(m, n, \infty)$  is contained in the projective affine group of  $(X, \omega_X)$ . As in the proof of Proposition 4.6, we take two fibers  $\mathcal{Y}_c$  and  $\mathcal{Y}_{\tilde{c}}$  with  $\pi(c) = \pi(\tilde{c})$ . We need to show the existence of an isomorphism  $i_0 : \mathcal{Y}_c \rightarrow \mathcal{Y}_{\tilde{c}}$  which is equivariant with respect to  $H$ . By construction of  $\sigma$  and  $\tau$  it is sufficient to find  $i_0 : \mathcal{Z}_c \rightarrow \mathcal{Z}_{\tilde{c}}$  equivariant with respect to  $\sigma$  and  $\varphi_1$  and such that the quotient isomorphism  $\widehat{i}_0 : \widehat{\mathcal{Z}}_c \rightarrow \widehat{\mathcal{Z}}_{\tilde{c}}$  is equivariant with respect to  $\tau$ .

We denote by  $i : \mathcal{Y}_c \rightarrow \mathcal{Y}_{\tilde{c}}$  the canonical isomorphism and we will try  $i_0 := \varphi^j \circ i$  for a suitably chosen  $j$ . This  $i_0$  is automatically  $\varphi_1$ -equivariant. Let  $\pi_1$  (resp.  $\pi_2$ ) denote the maps from  $C$  to the intermediate cover given by  $s^n = t$  (resp. by  $s^m = (t-1)$ .) By hypothesis we have  $\pi_1(c) = \zeta_n^{e_1} \pi_1(\tilde{c})$  and  $\pi_2(c) = \zeta_m^{e_2} \pi_2(\tilde{c})$ , where  $\zeta_m$  resp.  $\zeta_n$  is an  $m$ -th (resp.  $n$ -th) root of unity. We have

$$\tau \circ \widehat{i} = \varphi^{2me_1} \circ \widehat{i} \circ \tau, \quad \sigma \circ i = \varphi^{2ne_2 + 2me_1} \circ i \circ \sigma.$$

In order to satisfy the equivariance properties for  $i_0 = \varphi^j \circ i$ , the exponent  $j$  must satisfy

$$(\alpha - 1)j + 2me_1 \equiv 0 \pmod{N/\gamma}, \quad \text{and} \quad -2j + 2ne_2 + 2me_1 \equiv 0 \pmod{N}.$$

This conditions are equivalent to

$$-2j + 2me_1 \equiv 0 \pmod{2n/\gamma} \quad \text{and} \quad -2j + 2ne_2 \equiv 0 \pmod{2m}.$$

We can solve this for  $j$  since  $\gcd(m, n/\gamma) = 1$ .

To see that the projective affine group is not bigger than  $\Delta(m, n, \infty)$  for  $m \neq n$  we note that a bigger projective affine group must be again a triangle group. Singerman ([Si72]) shows that any inclusion of triangle groups is a composition of inclusions in a finite list. The case  $\Delta(m, m, \infty) \subset \Delta(2, m, \infty)$  is the only one case that might occur here.  $\square$

**Corollary 5.10.** *All  $(m, n, \infty)$ -triangle groups for  $m, n > 1$  and  $mn \geq 6$  arise as projective affine groups of translation surfaces with  $\Delta(m, m, \infty)$  as possible exception.*

We determine the basic geometric invariant of the Teichmüller curves constructed in Theorem 5.1.

**Theorem 5.11.** *In case S and DE the generating differential  $\omega_X$  has  $\gamma/2$  zeros and in the cases O and OE the generating differential  $\omega_X$  has  $\gamma$  zeros.*

**Proof:** We only treat case O and OE. Cases S and DE are similar. We calculate the zeros of the pullback  $\omega_Y$  of  $\omega_X$  to the corresponding fiber of  $g$ . The differential  $\omega_i$  has on  $Z$  zeros of order  $\frac{a_1(i)\gamma}{\gamma_1} - 1$  (resp.  $\frac{a_2(i)\gamma}{\gamma_1} - 1$ ) at the  $\gamma_1/\gamma$  points over 0 (resp. 1). It has zeros of order  $\frac{a_3(i)\gamma}{\gamma_2} - 1$  (resp.  $\frac{a_4(i)\gamma}{\gamma_2} - 1$ ) at the  $\gamma_2/\gamma$  points over  $t$  (resp.  $\infty$ ). Therefore its pullback to  $Z$  has zeros of order  $a_\mu(i) - 1$  at the  $\gamma$  preimages of 0 (resp.  $1, t, \infty$ ). The differential  $\omega_Y$  is a linear combination with non-zero coefficients of  $\omega_1, \omega_{-1}$  and two differentials that are pulled back from  $Z^\tau$ . The vanishing orders of these differentials are obtained from those of  $\omega_1$  and  $\omega_{-1}$  in  $Z$  by replacing  $a_1$  by  $a_4, a_2$  by  $a_4$  and vice versa. Since the  $a_\mu$  are pairwise different, we conclude that  $\omega_Y$  vanishes at the (in total)  $4\gamma$  preimages of  $\{0, 1, t, \infty\}$  of order  $\min\{a_1, a_2, a_3, a_4\} - 1 = a_4 - 1$ . Since  $\omega$  vanishes also at the  $4m + 4n$  ramification points of  $Y \rightarrow X$  we deduce that it vanishes there to first order and nowhere else. The  $4\gamma$  zeros at the non-ramification points yield the  $\gamma$  zeros of  $X$ .  $\square$

**5.1. Comparison with Ward's results.** We now show that for  $m = 3$  the family  $f : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{C}}$  is the Teichmüller curve associated to the Veech surfaces found by Ward ([Wa98]). Furthermore what follows gives an explanation why the examination of triangular billiards in [Wa98], [Vo96] and [KeSm00] did not lead to the discovery of the family  $f : \mathcal{X} \rightarrow \mathcal{C}$  except for  $m = 3$ . In fact it was shown that most triangular billiards (e.g. for acute triangles, see [Pu01], or 'sharp' triangles, see [Wa98]) do not give Veech surfaces, except for the Ward and Veech series and few sporadic examples.

The assumptions on  $m$  and  $n$  in the following theorem are not necessary. We include them only to make the proof more transparent. The reader can easily work out the corresponding statement for the other cases as well.

**Theorem 5.12.** *Suppose that  $m$  and  $n$  are relatively prime. Then a fiber of  $f : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{C}}$  over a point in  $\pi^{-1}(0)$  is a  $2n$ -cyclic cover of the projective line branched over  $(m + 3)/2$  points if  $m$  is odd and  $(m + 4)/2$  points if  $m$  is even.*

For  $m = 3$  this cover is given by the equation

$$y^{2n} = u(u - 1)^2.$$

The generating differential of the Teichmüller curve is

$$\omega = \frac{y du}{u(u - 1)},$$

the one studied by Ward.

**Proof:** Our assumptions imply that  $\gamma = 1$  and  $\mathcal{Z} \cong \mathcal{Y}$ . Recall that each component of the special fiber  $Y_c^j$  over some point  $c$  with  $\overline{\pi}(c) = 0$  has affine model  $z_0^N = x_0^{a_1}(x_0 - 1)^{a_3}$ . The fiber  $X_c$  of  $f$  is  $X_c = Y_c^1 / \langle \rho \rangle$ . From the presentation of  $G$  (Lemma 5.2) we deduce that  $\varphi^k$  commutes with  $\rho$  if and only if  $k$  is a multiple of  $m$ . The situation is as follows, where the left diamond  $Y_0^1 \rightarrow P_u^1$  is an abelian covering. The indices denote coordinates on the projective spaces that will be introduced below.

$$\begin{array}{ccccc}
 & & Y_0^1 & & \\
 & \swarrow \langle \rho \rangle & & \searrow \langle \varphi_1^m \rangle & \\
 & p_1 & & p_2 & \\
 X_0 & & & & \overline{Y}_0^1 \\
 & \searrow q_1 & & \swarrow q_2 & \searrow \langle \varphi_1 \bmod m \rangle \\
 & & \mathbb{P}_u^1 & & \mathbb{P}_{x_0}^1
 \end{array}$$

On  $\mathbb{P}_{x_0}^1$  the involution  $\rho$  is given by  $\rho(x_0) = 1 - x_0$ . Therefore the ramification locus of the  $q_1$  is contained in the images in  $X_0$  of the possibly ramified over the points in  $Y_0^1$  with  $x_0 \in \{0, 1, 1/2, \infty\}$ . These points are the fixed points of  $\rho$  and the branch locus of  $Y_0^1 \rightarrow \mathbb{P}_{x_0}^1$ . Since  $p_2$  is ramified at  $x_0 = 0$  (and  $x_0 = 1$ ) the (single) image point in  $\mathbb{P}_u^1$  is a branch point of  $q_1$ . Since  $p_2$  is ramified of order two at each of the  $2m$  points  $\infty_1, \dots, \infty_{2m}$  in  $Y_0^1$  with  $x_0 = \infty$ , the map  $q_1$  is ramified over each of the points  $q_2(\infty_i)$  ( $i = 1, \dots, 2m$ ) in  $\mathbb{P}_u^1$ . Using the action of  $\rho$  on  $\mathbb{P}_u^1$  one checks that the set  $\{q_2(\infty_i), i = 1, \dots, 2m\}$  consists of  $(m + 1)/2$  points if  $m$  is odd and it consists of  $(m + 2)/2$  points if  $m$  is even.

To prove the first statement, it remains to check that  $q_1$  is unramified at the images in  $X_0$  of points with  $x_0 = 1/2$ . We can represent these images in  $X_0$  by  $R_i = \varphi_1^i R$ , where  $R$  is

a fixed points of  $\rho$  and  $R_i$  is identified with  $R_{-\alpha i}$ . Now it is straightforward (cf. proof of Lemma 5.2) to check that  $\varphi^m$  fixes none of the  $R_i$ .

From now on let  $m = 3$ . For the second statement we only need to compute the type of the cover. In this case  $\bar{Y}_0^1$  is the curve of genus zero given by  $z_0^3 = x_0(x_0 - 1)^2$ . It admits the coordinate  $w = z_0/(x_0 - 1)$ . In this coordinate,  $\rho$  is given by  $\rho(w) = 1/w$ . The points over  $x_0 = \infty$  now become  $w^3 = 1$ . If we choose  $u = w + 1/w$  the covering  $Y_0^1 \rightarrow \mathbb{P}_u^1$  is unramified outside  $\{2, -1, -2, \infty\}$ . We claim that the covering  $Y_0 \rightarrow \mathbb{P}_u^1$  is given by

$$\begin{aligned} \pi_1(\mathbb{P}_u^*) &\rightarrow \langle \rho \rangle \times \langle \varphi^m \rangle \cong \mathbb{Z}/2 \times \mathbb{Z}/2n \\ \ell_2 &\mapsto (1, (a_1 + a_3)/m) = (1, 2) \\ \ell_{-1} &\mapsto (0, (a_1 + a_3)/2m) = (0, 1) \\ \ell_{-2} &\mapsto (1, 0) \\ \ell_\infty &\mapsto (0, -a_3) = (0, 2n - 3) \end{aligned}$$

where  $\mathbb{P}_u^*$  is  $\mathbb{P}_u^1$  minus the ramification points and  $\ell_P$  denotes a loop around  $P$ . The product of these loops is one. The claim follows from the ramification behavior of  $q_2$  and  $p_1$ , i.e. from the knowledge of  $\pi_1(\mathbb{P}_w^*)$  as a subgroup of  $\pi_1(\mathbb{P}_u^*)$ . Composing  $\pi_1(\mathbb{P}_u^*) \rightarrow \langle \rho \rangle \times \langle \varphi^m \rangle$  with the projection onto the second factor determines  $q_1$ . We can now read off the type of  $q_1$ .

For the last statement, one checks that the differential  $\omega = dx/x(x-1)$  has only one zero, at the point with  $u = \infty$  in  $X_0$ . This is the image under  $p_1$  of the point in  $Y_0^1$  with  $x_0 = 0$  (and also the image of the point with  $x_0 = 1$ ). By Theorem 5.11 the generating differential has the same zeros as  $\omega$ .  $\square$

**Remark 5.13.** The last argument may be applied for general  $m$ , and we continue under the hypothesis  $m, n$  odd and relatively prime. We deduce that  $X_0$  is given by

$$y^{2n} = (u - 2) \prod_{k=1}^{(m-1)/2} (u - \cos(2k\pi/m))^2,$$

As above, the differential

$$\omega = \frac{y du}{(u - 2) \prod_{i=1}^{(m-1)/2} (u - \cos(2k\pi/m))}$$

has only one zero over  $u = \infty$  and coincides with the generating differential. Consider the Schwarz-Christoffel mapping

$$SC(w) = \int_0^w (u - 2)^{\frac{1}{2n} - 1} \prod_{k=1}^{(m-1)/2} (u - \cos(2k\pi/m))^{\frac{1}{n} - 1} du,$$

It maps the real axis to a  $(m + 3)/2$ -gon. If this  $(m + 3)/2$ -gon has no self-crossings and connected interior then  $SC$  maps the upper half-plane bijectively to the interior of the  $(m + 3)/2$ -gon. In this case the Veech surface  $(X_0, \omega)$  arises from billiards in the  $(m + 3)/2$ -gon by [Wa98] Theorem C'. The condition on the image of the real axis is met for  $m = 3$ , as studied by Ward, and for  $m = 5$ . In case  $m = 5$  the angles of the 4-gon are  $\{\pi/n, \pi/n, \pi/2n, (4n - 5)\pi/2n\}$ . Figure 1 shows a billiard that gives rise to a  $(5, 9, \infty)$  triangle group.

Of course, the way of representing a translation surface by a rational polygon is by no means unique. It may thus happen that the Teichmüller curves of Theorem 5.1 for  $m > 3$  and  $n > 3$  are nevertheless generated by triangles. For example, also other fibres of

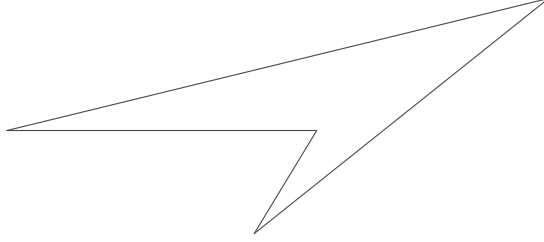


FIGURE 1. Billiard table with projective affine group  $\Delta(5, 9, \infty)$

$f : \mathcal{X} \rightarrow C$  than the ones over  $\pi^{-1}(0)$  might be translation surfaces of triangles. It seems quite likely that (some of) the sporadic examples arise in this way.

## 6. PRIMITIVITY

A *translation covering* is a covering  $q : X \rightarrow Y$  between translation surfaces  $(X, \omega_X)$  and  $(Y, \omega_Y)$  such that  $\omega_X = q^*\omega_Y$ . A translation surface  $(X, \omega_X)$  is called *geometrically primitive* if it does not admit a translation covering to a surface  $Y$  with  $g(Y) < g(X)$ .

A Veech surface  $(X, \omega)$  is called *algebraically primitive* if the trace field extension degree  $r$  equals  $g(X)$ . Algebraically primitive implies geometrically primitive but the converse does not hold. See also [Mö04b]. In loc. cit. Theorem 2.6 it is shown that a translation surface of genus greater than one covers a unique primitive translation surface.

Obviously the Veech examples ( $p : \mathcal{U} \rightarrow \tilde{C}$  in the notation of Theorem 4.4) for  $n = 2\ell$  and  $\ell$  prime and those for  $(2, n, \infty)$  (compare Remark 4.8) are algebraically primitive. We will not give a complete case by case discussion of primitivity of the  $(m, n, \infty)$ -Teichmüller curves but restrict to the case  $m, n$  odd and relatively prime. Comparing  $[\mathbb{Q}(\zeta_m + \zeta_m^{-1}, \zeta_n + \zeta_n^{-1}) : \mathbb{Q}] = r \leq \phi(m)\phi(n)/4$  with the genera given in Corollary 5.4, we deduce that these curves  $\mathcal{X} \rightarrow C$  are never algebraically primitive. We will show that there are infinitely many geometrically primitive ones:

**Theorem 6.1.** *Let  $m, n$  distinct odd primes. Then the Veech surfaces arising from the  $(m, n, \infty)$ -Teichmüller curve  $f : \mathcal{X} \rightarrow C$  of Theorem 5.1 are geometrically primitive.*

**Proof:** Let  $(X, \omega_X)$  be such a Veech surface and suppose there is a translation covering  $q : X \rightarrow Y$ . Then  $g(Y) \geq r$  by [Mö04b] Theorem 2.6. Since by Theorem 5.11 the generating differential has only one zero on  $\mathcal{X}_c$ , the covering  $q$  is totally ramified at this zero and nowhere else. This contradicts the Riemann–Hurwitz formula: A degree two covering cannot have this ramification type and higher degree contradicts  $g(Y) \geq r$ .  $\square$

**Remark 6.2.** At the time of writing the authors are aware of the following series of examples of Teichmüller curves: The triangle constructions in [Ve89] and [Wa98] and the Weierstrass eigenform or Prym eigenform constructions in [McM03] and [McM05c]. Besides them there is a finite number of sporadic examples.

**Corollary 6.3.** *The Veech surfaces arising from the case  $(m, n, \infty)$  with  $m, n$  sufficiently large distinct primes are not translation covered by any of the Veech surfaces listed in Remark 6.2.*

**Proof:** Recall that translation coverings between Veech surfaces preserve the affine group up to commensurability. In particular, they preserve the trace field.

Choose  $m$  and  $n$  sufficiently large such that the trace field  $K$  of the  $(m, n, \infty)$ -triangle group is none of the trace fields occurring in the sporadic examples and such that the genus of the Veech surface is larger than 5. This implies that the surface cannot be one of examples in [McM03] and [McM05c]. There is only a finite list of arithmetic triangle groups ([Ta77]). We choose  $m > 3$  and  $n > 5$  such that  $K$  is not one of the trace fields in this finite list. Non-arithmetic lattices have a unique maximal element ([Ma91]) in its commensurability class and the  $(m, n, \infty)$ -triangle groups are the maximal elements in their classes. Since the  $(2, n, \infty)$ - and  $(3, n, \infty)$ -triangle groups are the maximal elements in the commensurability classes of the examples of [Ve89] and [Wa98], these examples cannot be a translation cover of the examples given by Theorem 5.1 for  $(m, n)$  chosen as above.  $\square$

**Remark 6.4.** Even in the cases that the Veech surfaces with affine group  $\Delta(m, n, \infty)$  are geometrically primitive, Theorem 2.6 of [Mö04b] does not exclude that there are other primitive Veech surfaces with the same affine group. Nevertheless, by Remark 2.3 we know a rank  $2r$  subvariation of Hodge structures of the family of curves generated by such a Veech surface. In particular, we know the  $r$  of the Lyapunov exponents  $\lambda(L_i)$ .

## 7. LYAPUNOV EXPONENTS

Roughly speaking, a flat normed vector bundle on a manifold with a flow, i.e. an action of  $\mathbb{R}^+$ , can sometimes be stratified according to the growth rate of the length of vectors under parallel transport along the flow. The growth rates are then called Lyapunov exponents. In this section we will relate Lyapunov exponents to degrees of some line bundles in case that the underlying manifold is a Teichmüller curve.

For the convenience of the reader we reproduce Oseledec's theorem ([Os68]) that proves the existence of such exponents. We give a restatement due to [Ko97] in a language closer to our setting.

**7.1. Multiplicative ergodic theorem.** We start with some definitions. A measurable vector bundle is a bundle that can be trivialized by functions which only need to be measurable. If  $(V, \|\cdot\|)$  and  $(V', \|\cdot\|)$  are a normed vector bundles and  $T : V \rightarrow V'$ , then we let  $\|T\| := \sup_{\|v\|=1} \|T(v)\|$ . A reference for notions in ergodic theory is [CFS82].

**Theorem 7.1** (Oseledec). *Let  $T_t : (M, \nu) \rightarrow (M, \nu)$  be an ergodic flow on a space  $M$  with finite measure  $\nu$ . Suppose that the action of  $t \in \mathbb{R}^+$  lifts equivariantly to a flow  $S_t$  on some measurable real bundle  $V$  on  $M$ . Suppose there exists a (not equivariant) norm  $\|\cdot\|$  on  $V$  such that for all  $t \in \mathbb{R}^+$*

$$\int_M \log(1 + \|S_t\|) \nu < \infty.$$

*Then there exist real constants  $\lambda_1 \geq \dots \geq \lambda_k$  and a filtration*

$$V = V_{\lambda_1} \supset \dots \supset V_{\lambda_k} \supset 0$$

*by measurable vector subbundles such that, for almost all  $m \in M$  and all  $v \in V_m \setminus \{0\}$ , one has*

$$\|S_t(v)\| = \exp(\lambda_i t + o(t)),$$

*where  $i$  is the maximal value such that  $v \in (V_i)_m$ .*

*The  $V_{\lambda_i}$  do not change if  $\|\cdot\|$  is replaced by another norm of 'comparable' size (e.g. if one is a scalar multiple of the other).*

The numbers  $\lambda_i$  for  $i = 1, \dots, k \leq \text{rank}(V)$  are called the *Lyapunov exponents of  $S_t$* . Note that these exponents are unchanged if we replace  $M$  by a finite unramified covering with a lift of the flow and the pullback of  $V$ . We adopt the convention to repeat the exponents according to the rank of  $V_i/V_{i+1}$  such that we will always have  $2g$  of them, possibly some of them equal. A reference for elementary properties of Lyapunov exponents is e.g. [BGGSS80].

If the bundle  $V$  comes with a symplectic structure the Lyapunov exponents are symmetric with respect to 0, i.e. they are ([BGGSS80] Prop. 5.1)

$$1 = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq 0 \geq -\lambda_g \geq \dots \geq -\lambda_1 = -1.$$

We specialize these concepts to the situation we are interested in. Let  $\Omega M_g^*$  be the bundle of non-zero holomorphic one-forms over the moduli space of curves. Its points are translation surfaces. The one-forms define a flat metric on the underlying Riemann surface and we let  $\Omega_1 M_g \subset \Omega M_g^*$  be the hypersurface consisting of translation surfaces of area one. As usual we replace  $M_g$  by an appropriate fine moduli space adding a level structure, but we do not indicate this in the notation. This allows us to use a universal family  $f : \mathcal{X} \rightarrow M_g$ . Over  $\Omega_1 M_g$ , we have the local system  $\mathbb{V}_{\mathbb{R}} = R^1 f_* \mathbb{R}$ , whose fiber over  $(X, \omega)$  is  $H^1(X, \mathbb{R})$ . We denote the corresponding real  $C^\infty$ -bundle by  $V$ . This bundle naturally carries the Hodge metric

$$H(\alpha, \beta) = \int_X \alpha \wedge * \beta,$$

where classes in  $H^1(X, \mathbb{R})$  are represented by  $\mathbb{R}$ -valued one-forms, and where  $*$  is the Hodge star operator. We denote by  $\|\cdot\| := \|\cdot\|_T$  the associated metric on  $V$ .

There is a natural  $\text{SL}_2(\mathbb{R})$ -action on  $\Omega_1 M_g$  obtained by post-composing the charts given by integrating the one-form with the  $\mathbb{R}$ -linear map given by  $A \in \text{SL}_2(\mathbb{R})$  to obtain a new complex structure and new holomorphic one-form (see e.g. [McM03] and the reference there). The geodesic flow  $T_t$  on  $\Omega_1 M_g$  is the restriction of the  $\text{SL}_2(\mathbb{R})$ -action to the subgroup  $\text{diag}(e^t, e^{-t})$ . Since  $V$  carries a flat structure, we can lift  $T_t$  by parallel transport to a flow  $S_t$  on  $V$ . This is the *Kontsevich–Zorich cocycle*. The notion ‘cocycle’ is motivated by writing the flow on a vector bundle in terms of transition matrices.

Lyapunov exponents can be studied for any finite measure  $\nu$  on a subspace  $M$  of  $\Omega_1 M_g$  such that  $T_t$  is ergodic with respect to  $\nu$ . Starting with the work of Zorich ([Zo96]), Lyapunov exponents have been studied for connected components of the stratification of  $\Omega_1 M_g$  by the order of zeros of the one-form. The integral structure of  $\Omega M_g^*$  as an affine manifold can be used to construct a finite ergodic measure  $\mu$ . Lyapunov exponents for  $(\Omega_1 M_g, \mu)$  may be interpreted as deviations from ergodic averages of typical leaves of measured foliations on surfaces of genus  $g$ . The reader is referred to [Ko97], [Fo02] and the surveys [Kr03] and [Fo05] for further motivation and results.

**7.2. Lyapunov exponents for Teichmüller curves.** We want to study Lyapunov exponents in case of an arbitrary Teichmüller curve  $C$  or rather its canonical lift  $M$  to  $\Omega_1 M_g$  given by providing the Riemann surfaces parameterized by  $C$  with the normalized generating differential. The lift  $\pi : M \rightarrow C$  is an  $S^1$ -bundle. We equip  $M$  with the measure  $\nu$  which is induced by the Haar measure on  $\text{SL}_2(\mathbb{R})$ , normalized such that  $\nu(M) = 1$ . Locally,  $\nu$  is the product of the measure  $\nu_C$  coming from the Poincaré volume form and the uniform measure on  $S^1$ , both normalized to have total volume one.

We can apply Oseledec's theorem since  $\nu_M$  is ergodic for the geodesic flow ([CFS82] Theorem 4.2.1).

We start from the observation that the decomposition (2) of the VHS in Theorem 1.1 is  $\mathrm{SL}_2(\mathbb{R})$ -equivariant and orthogonal with respect to Hodge metric. This implies that the Lyapunov exponents of  $\mathbb{V}$  are the union of the Lyapunov exponents of the  $\mathbb{L}_i$  with those of  $\mathbb{M}$ .

Let  $\mathcal{L}_i := (\mathbb{L}_i)^{1,0}$  be the  $(1,0)$ -part of the Hodge filtration of the Deligne extension of  $\mathbb{L}_i$  to  $\overline{C}$ . Denote by  $d_i := \deg(\mathcal{L}_i)$  the corresponding degrees. Recall from Theorem 1.1 that precisely one of the  $\mathbb{L}_i$ , say the first one  $\mathbb{L}_1$  is maximal Higgs. Recall that  $S = \overline{C} \setminus C$  is the set of singular fibers.

**Theorem 7.2.** *Let  $\nu_M$  be the finite  $\mathrm{SL}_2(\mathbb{R})$ -invariant measure with support in the canonical lift  $M$  of a Teichmüller curve to  $\Omega_1 M_g$ . Then  $r$  of the Lyapunov exponents  $\lambda_i$  satisfy*

$$\lambda_i = d_i/d_1 = \lambda(\mathbb{L}_i, S).$$

*In particular, these exponents are rational, non-zero and their denominator is bounded by  $2g - 2 + s$ , where  $s = |S|$ .*

**Proof:** We write  $(\mathbb{L}_i)_{\mathbb{R}}$  for the local subsystem of  $R^1 f_* \mathbb{R}$  such that  $(\mathbb{L}_i)_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{L}_i$  and let  $L_i$  be the  $C^\infty$ -bundle attached to  $(\mathbb{L}_i)_{\mathbb{R}}$ . We apply Oseledec's theorem to  $L_i$ . Then

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|S_t(v_i)\|,$$

for  $v_i \in L_i \setminus (L_i)_{-\lambda_i}$ . By averaging, we have

$$\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{G(L_i)} \log \|S_t(v_i)\| d\nu_{G(L_i)}(v_i),$$

where  $\tau : G(L_i) \rightarrow M$  is the (Grassmann) bundle of norm one vectors in  $L_i$ . This bundle is locally isomorphic to  $S^1 \times M$ . The measure  $\nu_{G(L_i)}$  is locally the product measure of  $\nu$  with the uniform measure on  $S^1$ .

Following the idea of Kontsevich ([Ko97]) also exploited in Forni ([Fo02]), we estimate the growth of the length of  $v_i$  not only as a function on the  $T_t$ -ray through  $\tau(v_i)$  (given as the parallel transport of the corresponding vector) but as a function on the whole (quotient by a discrete group of a) Poincaré disc  $D_{\tau(v_i)}$  in  $M$ . For this purpose we write  $z = e^{i\theta} r$  ( $\theta \in [0, 2\pi)$ ) for  $z$  in the unit disc  $D$  and lift it to  $\rho_\theta \mathrm{diag}(e^t, e^{-t}) \in \mathrm{SL}_2(\mathbb{R})$ , where  $\rho_\theta$  is the rotation matrix by  $\Theta$  and  $t = (1/2) \log((1+r)/(1-r))$ . Using this lift  $D \rightarrow \mathrm{SL}_2(\mathbb{R})$  we obtain our disc  $D_{\tau(v_i)}$  in  $M$  using the (left)  $\mathrm{SL}_2(\mathbb{R})$ -action on  $M$ .

Consider the following functions

$$f_D := f_{D,i} : \begin{cases} (\pi^* L_i \setminus \{0\}) \times D & \longrightarrow \mathbb{R} \\ (v_i, z) & \longmapsto \log \|z \cdot v_i\|, \end{cases}$$

where  $z \cdot v_i$  is the parallel transport of  $v_i$  over the disc  $D_{\tau(v_i)}$ . This is well-defined since the monodromy of  $L_i$  acts by matrices in  $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Sp}_2(\mathbb{Z})$  and symplectic transformations do not affect the Hodge length. Note that by definition

$$(15) \quad f_D(v_i, z) = f_D(z \cdot v_i, 0).$$

On the discs  $D_{\tau(v_i)}$  we may apply the (hyperbolic) Laplacian  $\Delta_h$  to the functions  $f_{D_{\tau(v_i)}}$  with respect to the second variable, i.e. consider

$$h_D := h_{D,i} : \begin{cases} (\pi^*L_i \setminus \{0\}) \times D & \longrightarrow \mathbb{R} \\ (v_i, z) & \longmapsto (\Delta_h f_D(v_i, \cdot))(z). \end{cases}$$

Using (15) and the invariance of  $\Delta_h$  under isometries one deduces that there is a function  $h : \pi^*L_i \setminus \{0\} \rightarrow \mathbb{R}$ , such that

$$(16) \quad h_D(v_i, z) = h(z \cdot v_i).$$

Since obviously  $\int_{G(L_i)} h(S_t v_i) d\nu_{G(L_i)}(v_i) = \int_{G(L_i)} h(v_i) d\nu_{G(L_i)}(v_i)$  for any  $t$ , we can apply [Kr03] Equation (3) (see also [Fo02] Lemma 3.1) to obtain

$$(17) \quad \lambda_i = \int_{G(L_i)} h(v_i) \nu_{G(L_i)}(v_i).$$

We want to relate this expression to the degree  $d_i$  of the line bundles  $\mathcal{L}_i$ . Suppose  $s_i(u)$  is a holomorphic section of  $\mathcal{L}_i$  over some open  $U \subset C$ . Recall that  $L_i$  has unipotent monodromies, by assumption. Therefore [Pe84] Proposition 3.4 implies that the Hodge metric grows not too fast near the punctures and we have

$$(18) \quad d_i = \frac{1}{2\pi i} \int_{\bar{C}} \partial \bar{\partial} \log(\|s_i\|).$$

Here as usual, if there is no global section of  $\mathcal{L}_i$  the contributions of local holomorphic sections are added up using a partition of unity.

Instead of considering a holomorphic section  $s_i$ , we now consider a flat section  $v_i(u)$  of  $L_i$  over  $U$ . Then, in  $(\wedge^2(\mathbb{L}_i)_{\mathbb{C}})^{\otimes 2}(U)$  one checks the identity

$$(19) \quad (v_i \wedge *v_i) \otimes (s_i \wedge \bar{s}_i) = \frac{1}{2}(v_i \wedge s_i) \otimes (v_i \wedge \bar{s}_i).$$

We integrate this identity over the fibers  $\mathcal{X}_c$  of  $f : \mathcal{X} \rightarrow C$ , take logarithms and the Laplacian  $\frac{1}{2\pi i} \partial \bar{\partial}$ . Note that

$$(20) \quad \frac{1}{2\pi i} \partial \bar{\partial} \log \frac{1}{2}(v_i \wedge s_i) \otimes (v_i \wedge \bar{s}_i) = 0.$$

Let  $F$  be a fundamental domain for the action of the affine group  $\Gamma$  in a Poincaré disc  $D \hookrightarrow M$ . Then (18) and (20) implies that for any flat section  $v_i$  of  $L_i$  we have

$$d_i = \frac{-1}{2\pi i} \int_F \partial \bar{\partial} \log(\|v_i\|).$$

The differential operator  $\partial \bar{\partial}$  coincides, up to a scalar, with  $\Delta_h(\cdot)\omega_P$ , where  $\omega_P$  is the Poincaré area form. Therefore we obtain for each  $v_i \in (\pi \circ \tau)^*(L_i \setminus \{0\})$  that

$$d_i = \frac{1}{4\pi} \int_F \Delta_h \log \|v_i(z)\| \omega_P(z),$$

where  $v_i(z)$  is obtained from  $v_i$  via parallel transport. Hence by integrating over all  $G(L_i)$  and taking care of the normalization of  $\nu_{G(L_i)}$  we find that

$$(21) \quad d_i = \frac{1}{4\pi} \text{vol}(C) \int_{G(L_i)} \Delta_h \log \|v_i\| \nu_{G(L_i)}(v_i)$$

The statement of the theorem now follows by comparing (21) with (17).  $\square$

**Corollary 7.3.** *At least  $r$  of the Lyapunov exponents are non-zero.*

**Proof:** By Theorem 7.2, it is sufficient to show that for  $\mathcal{L}_i := (\mathbb{L}_i)^{(1,0)}$  the degree  $\deg(\mathcal{L}_i) \neq 0$ . If  $\mathcal{L}_i = 0$  then, by Simpson's correspondence ([ViZu04] Theorem 1.1),  $\mathbb{L}_i$  would be a reducible local system. But since  $\mathbb{L}_i$  is Galois conjugate to  $\mathbb{L}_1$ , this is a contradiction.  $\square$ .

**Remark 7.4.** If  $r \geq g - 1$  all the Lyapunov exponents are known. In fact in this case we can identify the remaining Lyapunov exponent by the formula ([Ko97], [Fo02] Lemma 5.3)

$$\sum_{i=1}^g \lambda_i = \frac{\deg(f_*\omega_{X/C})}{2g - 2 + s}$$

In the case of Teichmüller curves associated with triangle groups constructed in Section 4 and Section 5, the proof of Theorem 7.2 yields more. Since for these curves the VHS decomposes completely into subsystems of rank two (Remark 5.8) we can determine all the Lyapunov exponents.

**Proposition 7.5.** *Suppose the local system  $\mathbb{M}$  as in Theorem 1.1 contains a rank two local subsystem  $\mathbb{F}_i$ , whose  $(1,0)$ -part is a line bundle, which denote by  $\mathcal{F}_i$ . Then the Lyapunov spectrum contains (in addition to the  $d_i/d_1$ ) the exponents*

$$\deg(\mathcal{F}_i)/d_1.$$

By Theorem 7.2 and Proposition 7.5 it is justified to call  $\lambda(\mathbb{L}_i)$  Lyapunov exponents.

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