ON THE BRODY HYPERBOLICITY OF MODULI SPACES FOR CANONICALLY POLARIZED MANIFOLDS

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Abstract

We show that the moduli stack $\mathscr{M}_h$ of canonically polarized complex manifolds with Hilbert polynomial h is Brody hyperbolic. Hence if M_h denotes the corresponding coarse moduli scheme, and if $U \to M_h$ is a quasi-finite morphism, induced by a family, then there are no nonconstant holomorphic maps $\mathbb{C} \to U$.

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0. Introduction

Given a polynomial h, let $\mathscr{M}_h$ be the moduli functor of canonically polarized complex manifolds with Hilbert polynomial h. By [31], there exists a coarse quasi-projective moduli scheme M_h for $\mathscr{M}_h$, but in general M_h will not carry a universal family. Except for curves, there are no known natural level structures that can be added to enforce the existence of fine moduli schemes. However, C. S. Seshadri and J. Kollár constructed finite coverings $Z \to M_h$ which are induced by a universal family in $\mathscr{M}_h(Z)$ (see [31, Th. 9.25]). Moreover, if a general element in $\mathscr{M}_h(\mathrm{Spec}(\mathbb{C}))$ has no nontrivial

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automorphism, then there exists an open subscheme $M_h^0 \subset M_h$ which carries a universal family. It is the aim of this article to show that both the coverings Z and the open subscheme M_h^0 are Brody hyperbolic. More generally, we show that the moduli stack $\mathscr{M}_h$ is Brody hyperbolic in the following sense.

THEOREM 0.1
Assume that for some quasi-projective variety U there exists a family $f : V \to U \in \mathscr{M}_h(U)$ for which the induced morphism $\varphi : U \to M_h$ is quasi-finite over its image. Then U is Brody hyperbolic; that is, there are no nonconstant holomorphic maps $\gamma : \mathbb{C} \to U$.

Assume that the variety U in Theorem 0.1 is an open subvariety of a projective r-dimensional manifold Y with $B = Y \setminus U$ a normal crossing divisor. We conjecture that the quasi-finiteness of φ implies that $\Omega^1_Y(\log B)$ is weakly positive over some open dense subset of U (see Def. 3.1) and that $\kappa(\Omega^r_Y(\log B)) = r$. Paper [33] gives an affirmative answer if, for all the fibres of $V \to U$, the local Torelli theorem holds true, and Theorem 0.1 adds some more evidence.

An algebraic version of Theorem 0.1 (saying that, for abelian varieties A or for $A = \mathbb{C}^*$, all algebraic morphisms $\gamma : A \to U$ have to be constant) has been shown by S. Kovács in [15] and [16] (see also [23]).

The nonexistence of abelian subvarieties of moduli stacks presumably can also be deduced from the bounds for the degree of curves in moduli spaces (see [3], [32], [17]) by following the arguments used to prove [7, Th. 2.1].

Our arguments do not imply that the variety U in Theorem 0.1 is hyperbolic in the sense of S. Kobayashi, except of course if U is a compact manifold and hence the Brody hyperbolicity equivalent to the Kobayashi hyperbolicity. We do not speculate about possible Diophantine properties of moduli schemes which conjecturally are related to hyperbolicity (see [18]).

A question similar to Theorem 0.1 can be asked for moduli of polarized manifolds, that is, for the moduli functor of pairs $(f : V \to U, \mathscr{H})$ where f is a smooth projective morphism with ω_F semiample for all fibres F of f, and where $\mathscr{H}$ is fibrewise ample with Hilbert polynomial h. Hence $\mathscr{P}_h(U)$ is the set of such pairs, up to isomorphisms and up to fibrewise numerical equivalence for $\mathscr{H}$. By [31, Sec. 7.6], there exists a coarse quasi-projective moduli scheme P_h for $\mathscr{P}_h$.

In [32] we have shown that, for U an elliptic curve or for $U = \mathbb{C}^*$, there are no nonisotrivial smooth families $V \to U$ with $\omega_{V/U}$ relative semiample. Being optimistic, one could ask the following.

QUESTION 0.2
Does the existence of some $(f : V \to U, \mathscr{H}) \in \mathscr{P}_h(U)$, *for which the induced morphism* $\varphi : U \to P_h$ *is quasi-finite over its image, imply that* U *is Brody hyperbolic?*

The methods used in this paper give an affirmative answer to Question 0.2 only under the additional assumption that, for some $\nu > 0$ and for all fibres F of f, the ν-canonical map $F \to \mathbb{P}(H^0(F, \omega_F^\nu))$ is smooth over its image. Except if $\omega_F^\nu = \mathscr{O}_F$, this additional assumption is by far too much to ask for, and we do not consider this case in our paper.

An outline of the content of this paper and a guide to the proof of Theorem 0.1 are given at the end of Section 1.

1. A reformulation

Theorem 0.1 follows immediately from the next proposition. In fact, if there is a holomorphic map $\gamma : \mathbb{C} \to U$, we can replace U by the Zariski closure of $\gamma(\mathbb{C})$, and the proposition tells us that the Zariski closure must be a point, and hence that γ is constant.

PROPOSITION 1.1
Assume that for some $f : V \to U \in \mathscr{M}_h(U)$ *the induced map* $\varphi : U \to M_h$ *satisfies*

$$\dim U = \dim \overline{\varphi(U)} > 0.$$

Then there exists no holomorphic map $\gamma : \mathbb{C} \to U$ *with Zariski-dense image.*

Proposition 1.1 is formulated in such a way that, given a proper birational morphism $U' \to U$, the assumptions allow us to replace $f : V \to U$ by the fibre product $f' : V' = V \times_U U' \to U'$. We call such a pullback family f' a smooth birational model for f.

By the next lemma, the conclusion in Proposition 1.1 is compatible with replacing f by any smooth birational model.

LEMMA 1.2
Let $\tau : U' \to U$ *be a projective birational morphism between quasi-projective varieties. Then a holomorphic map* $\gamma : \mathbb{C} \to U$ *with Zariski-dense image lifts to a holomorphic map* $\gamma' : \mathbb{C} \to U'$.

Proof
Let $U_0 \subset U$ be an open set with $\tau|_{\tau^{-1}(U_0)}$ an isomorphism. The image $\gamma(\mathbb{C})$ meets U_0; hence γ' exists on the complement of a discrete subset $A \subset \mathbb{C}$. Let Δ be a small disk

in $\mathbb{C}$, centered at $a \in A$. The projective morphism τ factors through $U' \to U \times \mathbb{P}^M$ for some M, and the composite $\mathrm{pr}_2 \circ \gamma'|_{\Delta^*} : \Delta^* \to \mathbb{P}^M$ is given by meromorphic functions. Obviously, it extends to a holomorphic map on Δ, and the image of the induced map $\Delta \to U \times \mathbb{P}^M$ lies in U'. □

Using Lemma 1.2, we assume in the sequel that the quasi-projective variety U in Proposition 1.1 is nonsingular.

For the proof of Proposition 1.1, we first gather and generalize some methods of an algebraic nature, in particular, the weak semistable reduction theorem of D. Abramovich and K. Karu (see [2]) and the positivity results for direct images of certain sheaves (see [12], [14], [28], [29]). In Section 4 both are applied to certain product families, and the main result, Proposition 4.1, is quite similar to the one obtained by Abramovich in [1]. It allows us to replace the family $f : V \to U$ by a smooth birational model of the r-fold product $f^r : V^r \to U$ and to assume the stronger positivity properties stated in Corollary 4.3 and Proposition 4.4. Whereas the results of Section 2 hold true for arbitrary smooth projective morphisms, those of Section 3 and 4 use the semiampleness of ω_F for all fibres F of f.

Starting with Section 5, we assume that, contrary to Proposition 1.1 or Proposition 4.4, there exists a holomorphic map $\gamma : \mathbb{C} \to U$ with dense image. In order to use covering constructions, as we did in [32] for $\dim(U) = 1$, we choose a hyperplane H on V whose discriminant locus over U is in a general position with respect to $\gamma(\mathbb{C})$. At this point the ampleness of ω_F is needed.

In Section 6 we use the cyclic covering, obtained by taking a root out of H to compare and to study certain Higgs bundles and their pullback to $\mathbb{C}$. The main properties are gathered in Lemma 6.5. Finally, Section 7 contains some curvature estimates that show that the existence of γ, encoded in Lemma 6.5, contradicts the Ahlfors-Schwarz lemma. The content of this section is influenced by the work of J.-P. Demailly [7], S. S.-Y. Lu and S.-T. Yau [22], Lu [21], and Y.-T. Siu [27] on hyperbolicity.

2. Mild reduction

Let $f : X \to Y$ be a morphism between projective manifolds with connected general fibre. Abramovich and Karu constructed in [2] a generically finite proper morphism $Y' \to Y$ and a proper birational map $Z' \to (X \times_Y Y')\tilde{}$ such that the induced morphism $g' : Z' \to Y'$ is weakly semistable. Here $\tilde{}$ denotes the main component, that is, the component dominant over X. We do not recall the definition of weak semistability but just list the main properties needed later.

Definition 2.1

A morphism $g' : Z' \to Y'$ between projective varieties is called mild if

(a) g' is flat, Gorenstein with reduced fibres;

(b) Y' is nonsingular and Z' normal with at most rational singularities;

(c) given a dominant morphism $Y'_1 \to Y'$, where Y'_1 has at most rational Gorenstein singularities, $Z' \times_{Y'} Y'_1$ is normal with at most rational singularities;

(d) given Y'_0 an open subvariety of Y' with $g'^{-1}(Y'_0) \to Y'_0$ smooth, a nonsingular curve C', and a morphism $\pi : C' \to Y'$ whose image meets Y'_0, the fibred product $Z' \times_{Y'} C'$ is normal, Gorenstein with at most rational singularities.

In [2] the definition of a mild morphism uses just the first three conditions, and by [2, Lems. 6.1 and 6.2], those hold true for weakly semistable morphisms. As pointed out by Karu in [11, proof of Lem. 2.12], the proof of property (c) carries over word for word to show (d). Hence (d) holds true for weakly semistable morphisms as well.

Hence, starting with $f : X \to Y$, over some Y', generically finite over Y, one can find a mild model of the pullback family, that is, a mild morphism $g' : Z' \to Y'$ birational to $X \times_Y Y' \to Y'$. However, it might happen that one has to blow up the general fibre, and the smooth locus of g' would not be the pullback of the smooth locus of f. Nevertheless, the existence of g' will have strong consequences for direct images of powers of dualizing sheaves.

LEMMA 2.2

Let $g' : Z' \to Y'$ be mild.

(i) *If $Y'' \to Y'$ is a dominant morphism between manifolds, then*

$$\mathrm{pr}_2 : Z' \times_{Y'} Y'' \longrightarrow Y'' \quad \textit{is mild.}$$

(ii) *Let $g'' : Z'' \to Y'$ be a second mild morphism. Then*

$$(g', g'') : Z' \times_{Y'} Z'' \longrightarrow Y' \quad \textit{is mild.}$$

(iii) *For all $\nu \geq 1$ the sheaf $g'_* \omega^{\nu}_{Z'/Y'}$ is reflexive.*

Proof

(i) Property (a) in Definition 2.1 is compatible with base change, and in (c) one enforces the compatibility of (b) with base change as well.

(ii) Since Z'' has rational Gorenstein singularities, property (c) for Z' implies that $Z' \times_{Y'} Z''$ has at most rational Gorenstein singularities. The other properties asked for in (a) and (b) are obvious. For (c), remark that $Z'' \times_{Y'} Y'_1$ is normal with rational Gorenstein singularities, and hence

$$(Z'' \times_{Y'} Y'_1) \times_{Y'} Z' = (Z'' \times_{Y'} Z') \times_{Y'} Y'_1$$

has the same property. The same argument with Y_0' replaced by C' gives (d).

The sheaf $g'_*\omega^{\nu}_{Z'/Y'}$ is torsion free and hence locally free outside of a closed codimension-two subvariety T of Y'. Since Z' is normal and equidimensional over Y', for $U_0 \subset Y'$ open and for $V_0 = g'^{-1}(U_0)$ one has

$$H^0(V_0, \omega^{\nu}_{Z'/Y'}) = H^0\big(V_0 \setminus g'^{-1}(T), \omega^{\nu}_{Z'/Y'}\big),$$

and thereby

$$H^0(U_0, g'_*\omega^{\nu}_{Z'/Y'}) = H^0(U_0 \setminus T, g'_*\omega^{\nu}_{Z'/Y'}).$$

So $g'_*\omega^{\nu}_{Z'/Y'}$ coincides with the maximal extension of $g'_*\omega^{\nu}_{Z'/Y'}|_{Y'\setminus T}$ to Y'. □

Let $V \to U$ be any smooth projective morphism between quasi-projective manifolds. We choose for Y and X projective nonsingular compactifications, with $Y \setminus U$ and $X \setminus V$ normal crossing divisors, in such a way that $V \to U$ extends to a morphism $f : X \to Y$. If $g : Z' \to Y'$ denotes the weak semistable reduction, we choose a birational morphism $\epsilon : Y_1 \to Y$ such that the main component $Y_1' = (Y' \times_Y Y_1)\tilde{}$ is finite over Y_1. Let $\Delta(Y_1'/Y_1)$ denote the discriminant locus in Y_1 of $Y_1' \to Y_1$, and let $B_1 = Y_1 \setminus \epsilon^{-1}(U)$ be the boundary divisor. Blowing up a bit more, we can assume that Y_1 is nonsingular and that $\Delta(Y_1'/Y_1) + B_1$ is a normal crossing divisor.

By Y. Kawamata's covering construction (see [31, Cor. 2.6]), there exists a nonsingular projective manifold Y_2', finite over Y_1'. In particular, there is a morphism $Y_2' \to Y'$, and by Lemma 2.2(i) the pullback of $Z' \to Y'$ is again mild.

Let us choose a desingularization $\psi : X_1 \to X \times_Y Y_1$ such that

$$(\mathrm{pr}_2 \circ \psi)^*\big(B_1 + \Delta(Y_1'/Y_1)\big)$$

is a normal crossing divisor.

Changing the smooth birational model, we may replace U by its preimage in Y_1 and by abuse of notation rename $\mathrm{pr}_2 \circ \psi : X_1 \to Y_1$ as $f : X \to Y$. We also write Y' instead of Y_2' and Z' instead of $Z \times_{Y'} Y_2'$. Doing so, we reach the following situation.

LEMMA 2.3
Any smooth projective morphism with connected fibres has a smooth birational model $V \to U$ which fits into a diagram of morphisms of normal varieties

$$\begin{CD}
V @>\subset>> X @<\tau'<< X' @<\sigma<< Z @<\rho<< X'' @<\delta<< Z' \\
@VVV @VfVV @Vf'VV @VgVV @Vf''VV @Vg'VV \\
U @>\subset>> Y @<\tau<< Y' @<=<< Y' @<=<< Y' @>=>> Y'
\end{CD} \tag{2.3.1}$$

with the following properties:

(i) *Y, Y', X, Z, and X'' are nonsingular projective varieties;*

(ii) *τ is finite, and X' is the normalization of $X \times_Y Y'$;*

(iii) *ρ and δ are birational, and σ is a blowing up with center in the singular locus of X';*

(iv) *for $B = Y \setminus U$, the divisors $B + \Delta(Y'/Y)$ and $f^*(B + \Delta(Y'/Y))$ are normal crossing divisors;*

(v) *$g' : Z' \to Y'$ is mild.*

COROLLARY 2.4

Conditions (i) – (v) stated in Lemma 2.3 imply that

(vi) *X' has rational singularities;*

(vii) *for all $\nu \geq 1$ there exist isomorphisms*

$$g'_*\omega^{\nu}_{Z'/Y'} \xrightarrow{\simeq} f''_*\omega^{\nu}_{X''/Y'} \xleftarrow{\simeq} g_*\omega^{\nu}_{Z/Y'};$$

in particular, $g_*\omega^{\nu}_{Z/Y'}$ is a reflexive sheaf;

(viii) *for all $\nu \geq 1$ there exists an inclusion*

$$\iota : g_*\omega^{\nu}_{Z/Y'} \longrightarrow \tau^* f_*\omega^{\nu}_{X/Y},$$

which is an isomorphism over U;

(ix) *for all $\nu \geq 1$ there exists some N_ν and an invertible sheaf λ_ν on Y with*

$$\tau^*\lambda_\nu \simeq \det(g_*\omega^{\nu}_{Z/Y'})^{N_\nu}.$$

In Corollary 2.4(ix) the determinant of $g_*\omega^{\nu}_{Z/Y'}$ is $i_* \det(g_*\omega^{\nu}_{Z/Y'}|_{Y\setminus T})$, where T is any codimension-two subvariety with $g_*\omega^{\nu}_{Z/Y'}|_{Y\setminus T}$ locally free and $i : Y \setminus T \to Y$ the inclusion.

Proof

Since $\Delta(X'/X) \subset f^*\Delta(Y'/Y)$ are both normal crossing divisors, one obtains (vi).

Z' is normal with rational Gorenstein singularities; hence $\delta^*\omega_{Z'/Y'} \subset \omega_{X''/Y'}$ and $\omega^{\nu}_{Z'/Y'} = \delta_*\delta^*\omega^{\nu}_{Z'/Y'} \subset \delta_*\omega^{\nu}_{X''/Y'}$. The sheaf on the left-hand side is invertible, the one on the right-hand side is torsion free, and both coincide outside of a codimension-two subvariety. Hence they are equal, and one obtains the first isomorphism in (vii). For the second one, one repeats the argument for ρ instead of δ. By Lemma 2.2(iii), all the three sheaves in (vii) are reflexive. Part (viii) has been shown in [28, Lem. 3.2] (see also [24, Th. 4.10]).

Let B_ν denote the zero divisor of $\det(\iota)$; hence

$$\det(g_*\omega^{\nu}_{Z/Y'}) \otimes \mathcal{O}_{Y'}(B_\nu) = \tau^* \det(f_*\omega^{\nu}_{X/Y}).$$

In order to show that B_ν is the pullback of a $\mathbb{Q}$-divisor on Y, we have to show that the multiplicities of two components of B_ν coincide whenever both have the same image in Y. To this end, given any component $\tilde{B}$ of $Y \setminus U$, consider a general curve C which intersects $\tilde{B}$ in some point q. Replacing C by a neighborhood of q, we assume that this is the only intersection point.

Let us write $T_C = T \times_Y C$, where T stands for any of the varieties in the diagram (2.3.1). Similarly, if $h : T \to T'$ is any of the morphisms in the diagram (2.3.1), h_C denotes the restriction of h to T_C.

By Definition 2.1(d), the variety Z'_C is again normal, Gorenstein with at most rational singularities, and, for C sufficiently general, X_C and X''_C will be nonsingular. Applying part (viii) with Y replaced by C, one obtains a natural inclusion

$$\iota_C : g'_{C*}\omega^\nu_{Z'_C/Y'_C} \longrightarrow \tau^*_C(f_{C*}\omega^\nu_{X_C/C}), \tag{2.4.1}$$

and the zero divisor of $\det(\iota_C)$ is the restriction of B_ν to Y'_C. In order to show (ix), we just have to verify that B_ν is the pullback of a $\mathbb{Q}$-divisor on C.

By [13], there exists a finite morphism $C' \to C$, totally ramified in q, such that $X_C \times_C C'$ has a semistable model $S \to C'$.

By Definition 2.1(d), the pullback of Z'_C to some nonsingular covering of C remains normal with rational Gorenstein singularities. By flat base change, (2.4.1) is compatible with further pullbacks. Hence we may as well assume for a moment that $Y'_C \to C$ factors through C'. Then

$$\mathrm{pr}_1 : S' = S \times_{C'} Y'_C \longrightarrow Y'_C \qquad \text{and} \qquad g'_C : Z'_C \longrightarrow Y'_C$$

are two flat Gorenstein morphisms, S' and Z'_C are birational, and both are normal with at most rational singularities. Therefore, repeating the argument used to prove (vii), one obtains

$$g'_{C*}\omega^\nu_{Z'_C/Y'_C} = \mathrm{pr}_{1*}\,\omega^\nu_{S'/Y'_C},$$

and the divisor $B_\nu|_{Y'_C}$ is the pullback of a divisor Π on C'. Since $C' \to C$ is totally ramified in q, the divisor Π is itself the pullback of a $\mathbb{Q}$-divisor on C. □

3. Positivity of direct image sheaves

As in [28] and [29], we use the following convention: If $\mathscr{F}$ is a coherent sheaf on a quasi-projective normal variety Y, we consider the largest open subscheme $i : Y_1 \to Y$ with $i^*\mathscr{F}$ locally free. For

$$\Phi = S^\mu, \qquad \Phi = \bigotimes^{\mu}, \qquad \text{or} \qquad \Phi = \det,$$

we define

$$\Phi(\mathscr{F}) = i_*\Phi(i^*\mathscr{F}).$$

Definition 3.1

Let $\mathscr{F}$ be a torsion free coherent sheaf on a quasi-projective normal variety Y, and let $\mathscr{H}$ be an ample invertible sheaf. Let $U \subset Y$ be an open subvariety.

(a) $\mathscr{F}$ is globally generated over U if the natural morphism

$$H^0(Y, \mathscr{F}) \otimes \mathscr{O}_Y \longrightarrow \mathscr{F}$$

is surjective over U.

(b) $\mathscr{F}$ is weakly positive over U if the restriction of $\mathscr{F}$ to U is locally free and if for all $\alpha > 0$ there exists some $\beta > 0$ such that

$$S^{\alpha \cdot \beta}(\mathscr{F}) \otimes \mathscr{H}^{\beta}$$

is globally generated over U.

(c) $\mathscr{F}$ is ample with respect to U if there exists some $\mu > 0$ such that

$$S^{\mu}(\mathscr{F}) \otimes \mathscr{H}^{-1}$$

is weakly positive over U.

The basic properties of weakly positive sheaves are listed in [31, Sec. 2.3]. In particular, the definition of *weak positivity over* U does not depend on the ample sheaf $\mathscr{H}$ (see [31, Lem. 2.14]), and, if $\mathscr{F}$ is weakly positive over U and $\mathscr{F} \to \mathscr{G}$ surjective over U, then $\mathscr{G}$ is weakly positive over U (see [31, Lem. 2.16]). Moreover, weak positivity is a local property. If for each point $u \in U$ there is a neighborhood U_0 with $\mathscr{F}$ weakly positive over U_0, then $\mathscr{F}$ is weakly positive over U.

By definition, most of the properties of weakly positive sheaves $\mathscr{F}$ carry over to sheaves that are ample over U.

LEMMA 3.2

Let $\mathscr{H}$ be an ample invertible sheaf on Y. Then $\mathscr{F}$ is ample with respect to U if its restriction to U is locally free and if and only if for some $\eta > 0$ there exists a morphism

$$\bigoplus \mathscr{H} \longrightarrow S^{\eta}(\mathscr{F}),$$

surjective over U.

Proof

If $\mathscr{F}$ is ample with respect to U, for all β sufficiently large and divisible,

$$S^{2\cdot\beta}\big(S^{\mu}(\mathscr{F})\big) \otimes \mathscr{H}^{-2\cdot\beta} \otimes \mathscr{H}^{\beta}$$

is globally generated over U, as is its quotient sheaf

$$S^{2\cdot\beta\cdot\mu}(\mathscr{F}) \otimes \mathscr{H}^{-\beta}.$$

We may assume that $\mathscr{H}^{\beta-1}$ is very ample, and we obtain the morphism asked for in Lemma 3.2. On the other hand, if there is a morphism

$$\bigoplus \mathscr{O}_Y \longrightarrow S^\eta(\mathscr{F}) \otimes \mathscr{H}^{-1},$$

surjective over U, the sheaf $S^\eta(\mathscr{F}) \otimes \mathscr{H}^{-1}$ as a quotient of a weakly positive sheaf is weakly positive over U. □

The basic methods for studying positivity properties of direct images are contained in [28], [29], [30], and [31]. Unfortunately, in [28] and [29] we used "weak positivity" without specifying the open set, whereas in [31] we mainly work with smooth families or families without nonnormal fibres. So we have to recall some definitions in this section, and we have to make the arguments carefully enough to keep track of the open set U.

Let $f : X \to Y$ be a surjective projective morphism of quasi-projective manifolds. We want to extend the constructions from [32, Sec. 2] to the case of $\dim(Y) > 1$.

For an effective $\mathbb{Q}$-divisor $D \in \mathrm{Div}(X)$, the integral part $[D]$ is the largest divisor with $[D] \leq D$. For an effective divisor Γ on X, and for $N \in \mathbb{N} - \{0\}$, the algebraic multiplier sheaf is

$$\omega_{X/Y}\left\{\frac{-\Gamma}{N}\right\} = \tau_*\left(\omega_{T/Y}\left(-\left[\frac{\Gamma'}{N}\right]\right)\right),$$

where $\tau : T \to X$ is any blow-up with $\Gamma' = \tau_*\Gamma$ a normal crossing divisor (see, e.g., [8, Def. 7.4], or [31, Sec. 5.3]).

Let F be a nonsingular fibre of f. Using the definition given above for F, instead of X, and for a divisor Π on F, one defines

$$e(\Pi) = \mathrm{Min}\left\{N \in \mathbb{N} \setminus \{0\};\ \omega_F\left\{\frac{-\Pi}{N}\right\} = \omega_F\right\}.$$

By [8] or [31, Sec. 5.4], $e(\Gamma|_F)$ is upper semicontinuous, and there exists a neighborhood V of F with $e(\Gamma|_V) \leq e(\Gamma|_F)$. If $\mathscr{L}$ is an invertible sheaf on F with $H^0(F, \mathscr{L}) \neq 0$, one defines

$$e(\mathscr{L}) = \mathrm{Max}\big\{e(\Pi);\ \Pi \text{ an effective divisor and } \mathscr{O}_F(\Pi) = \mathscr{L}\big\}.$$

PROPOSITION 3.3

Let $U \subset Y$ be an open subscheme, let $\mathscr{L}$ be an invertible sheaf, let Γ be a divisor on X, and let $\mathscr{F}$ be a coherent sheaf on Y. Assume that, for some $N > 0$, the following conditions hold true:

(a) *$V = f^{-1}(U) \to U$ is smooth with connected fibres;*

(b) *$\mathscr{F}$ is weakly positive over U (in particular, $\mathscr{F}|_U$ is locally free);*

(c) *there exists a morphism* $f^*\mathscr{F} \to \mathscr{L}^N(-\Gamma)$*, surjective over* V*;*

(d) *none of the fibres* F *of* $f : V \to U$ *is contained in* Γ*, and for all of them,*

$$e(\Gamma|_F) \leq N.$$

Then $f_*(\mathscr{L} \otimes \omega_{X/Y})$ *is weakly positive over* U*.*

Proof

By [31, Lem. 5.23], the restriction of the sheaf $\mathscr{E} = f_*(\mathscr{L} \otimes \omega_{X/Y})$ to U is locally free. The verification of the weak positivity is done in several steps. Let us first show the following.

CLAIM 3.4

In order to prove Proposition 3.3, we are allowed to assume that $\mathscr{F}$ *is ample with respect to* U*.*

Proof

Let $\mathscr{H}$ be a very ample sheaf on Y, and let $\rho : Y \to \mathbb{P}^M$ be an embedding. For a general choice of the coordinate planes $H_0, \ldots, H_M$, the intersection $H_i \cap (Y \setminus U)$ is of codimension two in Y. We choose a codimension-two subscheme T with $T \supset H_i \cap (Y \setminus U)$ for $i = 0, \ldots, M$. By definition, in order to show that $f_*(\mathscr{L} \otimes \omega_{X/Y})$ is weakly positive over U, we may replace Y by $Y \setminus T$ and assume that $H_i \cap (Y \setminus U) = \emptyset$. Moreover, for T large enough, f will be flat. By the local nature of weak positivity, it is sufficient to show that $f_*(\mathscr{L} \otimes \omega_{X/Y})$ is weakly positive over

$$U_0 = U \setminus \bigcup_{i=0}^{M} H_i.$$

In fact, one can cover U by such open sets for different choices of the coordinate planes.

Given $\alpha > 0$, we choose $d = 1 + 2 \cdot \alpha$ and consider the dth power map

$$\theta : \mathbb{P}^M \longrightarrow \mathbb{P}^M \quad \text{with } \theta(x_0, \ldots, x_M) = (x_0^d, \ldots, x_M^d).$$

Let Y' be the normalization of $\theta^{-1}(Y)$, and let $\tau : Y' \to Y$ be the induced map. For the pullback $\mathscr{H}'$ of $\mathscr{O}_{\mathbb{P}^1}(1)$ to Y', one obtains $\tau^*\mathscr{H} = \mathscr{H}'^d$.

Leaving out codimension-two subschemes in Y not meeting U_0, we may assume that Y' is nonsingular. Then $X' = X \times_Y Y'$ is nonsingular. In fact, $f' : X' \to Y'$ is smooth over $\tau^{-1}(U)$, and $\tau' : X' \to X$ is smooth over

$$X \setminus \bigcup_{i=0}^{M} f^{-1}(H_i).$$

Let us choose $\mathscr{F}' = \tau^*\mathscr{F} \otimes \mathscr{H}'^N$ and $\mathscr{L}' = \tau'^*\mathscr{L} \otimes f'^*\mathscr{H}'$. The sheaf $\mathscr{F}'$ is ample with respect to $U_0' = \tau^{-1}(U_0)$. Applying Proposition 3.3 to $\mathscr{F}'$ instead of $\mathscr{F}$, one finds

$$f'_*(\mathscr{L}' \otimes \omega_{X'/Y'}) = f'_*(\tau'^*\mathscr{L} \otimes \omega_{X'/Y'}) \otimes \mathscr{H}'$$

to be weakly positive over U_0'. By flat base change, this sheaf is isomorphic to

$$\tau^* f_*(\mathscr{L} \otimes \omega_{X/Y}) \otimes \mathscr{H}' = \tau^*(\mathscr{E}) \otimes \mathscr{H}'.$$

Hence for all β sufficiently large and divisible, the sheaf

$$S^{(2\cdot\alpha)\cdot\beta}\big(\tau^*(\mathscr{E})\otimes\mathscr{H}'\big)\otimes\mathscr{H}'^{\beta} = \tau^*\big(S^{2\cdot\alpha\cdot\beta}(\mathscr{E})\big)\otimes\mathscr{H}'^{(2\cdot\alpha+1)\cdot\beta} = \tau^*\big(S^{2\cdot\alpha\cdot\beta}(\mathscr{E})\otimes\mathscr{H}^{\beta}\big)$$

is globally generated over U_0'. We obtain morphisms

$$\bigoplus \mathscr{O}_{Y'} \longrightarrow \tau^*(S^{2\cdot\alpha\cdot\beta}(\mathscr{E}) \otimes \mathscr{H}^{\beta}),$$

surjective over U_0', and

$$\bigoplus \tau_*\mathscr{O}_{Y'} \longrightarrow S^{2\cdot\alpha\cdot\beta}(\mathscr{E}) \otimes \mathscr{H}^{\beta},$$

surjective over U_0. For β large enough, $\tau_*\mathscr{O}_{Y'} \otimes \mathscr{H}^{\beta}$ is generated by global sections, and hence $S^{\alpha\cdot(2\cdot\beta)}(\mathscr{E}) \otimes \mathscr{H}^{2\cdot\beta}$ is globally generated over U_0. □

Claim 3.4 allows us to assume that $\mathscr{F}$ is ample with respect to U. Then the sheaf $\mathscr{L}^{N\cdot\eta}(-\eta\cdot\Gamma)$ will be globally generated over V for some $\eta \gg 0$. Replacing N by $N\cdot\eta$ and Γ by $\eta\cdot\Gamma$, we may assume that $\mathscr{L}^N(-\Gamma)$ itself has this property as well. From now on, this assumption replaces conditions (b) and (c) in Proposition 3.3.

Leaving out a codimension-two subset of Y not meeting U, we continue to assume that f is flat. Let us fix some nonsingular compactification $\bar{Y}$ of Y and a very ample invertible sheaf $\bar{\mathscr{A}}$ on $\bar{Y}$ such that $\bar{\mathscr{A}}^{\dim Y+1}\otimes\omega_{\bar{Y}}$ is ample. We write $\mathscr{A} = \bar{\mathscr{A}}|_Y$ and $\mathscr{H} = \mathscr{A}^{\dim Y+1} \otimes \omega_Y$.

CLAIM 3.5
$\mathscr{E} \otimes \mathscr{A}^{\dim Y+1} \otimes \omega_Y$ is globally generated over U.

Proof
Let us choose a compactification $\bar{X}$ of X such that f extends to a morphism $\bar{f} : \bar{X} \to \bar{Y}$. Moreover, we choose $\bar{\mathscr{L}}$ and $\bar{\Gamma}$ such that $\bar{\mathscr{L}}^N(-\bar{\Gamma})$ is again globally generated over V. Let $\tau : X' \to \bar{X}$ be a blow-up such that $\tau^*\bar{\Gamma} = \Gamma'$ is a normal crossing divisor, and let $f' = \bar{f} \circ \tau$. Assumption (d) in Proposition 3.3 implies that

$$\mathscr{E}' = f'_*\Big(\tau^*\bar{\mathscr{L}} \otimes \mathscr{O}_{X'}\Big(-\Big[\frac{\Gamma'}{N}\Big]\Big) \otimes \omega_{X'/\bar{Y}}\Big) \longrightarrow \bar{f}_*(\bar{\mathscr{L}} \otimes \omega_{\bar{X}/\bar{Y}})$$

is an isomorphism over U. Hence it is sufficient to show that

$$\mathcal{E}' \otimes \bar{\mathcal{A}}^{\dim Y+1} \otimes \omega_{\bar{Y}}$$

is globally generated over U. Blowing up a bit more, and enlarging Γ' by adding components supported in $X' \setminus \tau^{-1}(V)$, we can assume that $\tau^*(\bar{\mathcal{L}})^N \otimes \mathcal{O}_{X'}(-\Gamma')$ is globally generated over X' as well. Under this assumption, Claim 3.5 has been shown in [31, Cor. 2.37, part 2]). □

To finish the proof, we consider for any $\alpha > 0$ the α-fold product

$$X^\alpha = X \times_Y \cdots \times_Y X \quad (\alpha\text{-times})$$

and $f^\alpha : X^\alpha \to Y$. Let $\sigma : X^{(\alpha)} \to X^\alpha$ be a desingularization, and let $f^{(\alpha)} = f^\alpha \circ \sigma$,

$$\mathcal{L}^{(\alpha)} = \sigma^*\Big(\bigotimes_{i=1}^{\alpha} \mathrm{pr}_i^* \mathcal{L}\Big), \qquad \text{and} \qquad \Gamma^{(\alpha)} = \sigma^*\Big(\sum_{i=1}^{\alpha} \mathrm{pr}_i^* \Gamma\Big).$$

The morphism $f^{(\alpha)} : X^{(\alpha)} \to Y$ and the sheaf $\mathcal{L}^{(\alpha)}$ again satisfy assumption (a) in Proposition 3.3. Moreover, we assumed $\mathcal{L}^N(-\Gamma)$ to be globally generated over V; hence $\mathcal{L}^{(\alpha)^N}(-\Gamma^{(\alpha)})$ is globally generated over $V^r = V \times_U \cdots \times_U V$. Assumption (d) holds true for $\Gamma^{(\alpha)}$ by the following.

CLAIM 3.6
We have $e(\Gamma^{(\alpha)}|_{F^r}) = e(\Gamma|_F)$.

Proof
The proof, similar to the one of [31, Cor. 5.21], is by induction on r. Obviously, $e(\Gamma^{(\alpha)}) \geq e = e(\Gamma)$. Let C be the support of the cokernel of the inclusion

$$\omega_{F^r}\left\{\frac{-\Gamma^{(\alpha)}|_{F^r}}{e}\right\} \longrightarrow \omega_{F^r}.$$

Applying [31, Prop. 5.19], to the ith projection $\mathrm{pr}_i : F^r \to F$, one finds subschemes C_i of F with $C = \mathrm{pr}_i^{-1}(C_i)$. Since this holds true for $i = 1, \ldots, r$, C must be empty. □

By Claim 3.5, the sheaf $f_*^{(\alpha)}(\mathcal{L}^{(\alpha)} \otimes \omega_{X^{(\alpha)}/Y}) \otimes \mathcal{H}$ is globally generated over U. Hence Proposition 3.3 follows from the next claim.

CLAIM 3.7
There exists a morphism

$$f_*^{(\alpha)}(\mathcal{L}^{(\alpha)} \otimes \omega_{X^{(\alpha)}/Y}) \longrightarrow S^\alpha\big(f_*(\mathcal{L} \otimes \omega_{X/Y})\big),$$

surjective over U.

Proof
The natural morphism $\sigma_*\omega_{X^{(\alpha)}} \to \omega_{X^\alpha}$ induces a morphism

$$f_*^{(\alpha)}(\mathscr{L}^{(\alpha)} \otimes \omega_{X^{(\alpha)}/Y}) \longrightarrow f_*^\alpha\Big(\Big(\bigotimes_{i=1}^{\alpha} \mathrm{pr}_i^* \mathscr{L}\Big) \otimes \omega_{X^\alpha/Y}\Big),$$

which is an isomorphism over U. By flat base change, the right-hand side is nothing but

$$\bigotimes^{\alpha} f_*(\mathscr{L} \otimes \omega_{X/Y}). \qquad \square$$

Therefore Proposition 3.3 is proved. □

COROLLARY 3.8
Let $f : X \to Y$ be a projective surjective morphism between quasi-projective manifolds with connected general fibre. Assume that, for some open subscheme $U \subset Y$,

$$V = f^{-1}(U) \longrightarrow U$$

is smooth and that ω_{F_u} is semiample for all fibres $F_u = f^{-1}(u)$ with $u \in U$. Then $f_*\omega^{\nu}_{X/Y}$ is weakly positive over U.

Proof
Using Proposition 3.3 (with $\Gamma|_V = 0$), one can copy the arguments presented in [31, proof of Cor. 2.45] to obtain Corollary 3.8 as a corollary to Proposition 3.3. We leave this as an exercise since Corollary 3.8 has been shown under less restrictive assumptions in [30, Th. 3.7], using different (and more complicated) arguments. □

Remark 3.9
By [19], the assumption "ω_{F_u} is semiample for all fibres F_u with $u \in U$" is equivalent to the f-semiampleness of $\omega_{V/U}$. Hence for all ν sufficiently large and divisible, the natural morphism

$$f^* f_*\omega^{\nu}_{X/Y} \longrightarrow \omega^{\nu}_{X/Y}$$

is surjective over V. In particular, Corollary 3.8 implies that $\omega_{X/Y}$ is weakly positive over V.

Let us end this section by stating a stronger positivity result. Although it holds true by [14] for arbitrary families of manifolds of general type, we formulate it just for families with a semiample canonical sheaf. Recall that in [28], for a projective surjective morphism $f : X \to Y$ with connected general fibre, we defined $\mathrm{Var}(f)$ to be the smallest integer η for which there exists a finitely generated subfield K of $\overline{\mathbb{C}(Y)}$ of transcendence degree η over $\mathbb{C}$, a variety F' defined over K, and a birational equivalence

$$X \times_Y \operatorname{Spec}\left(\overline{\mathbb{C}(Y)}\right) \sim F' \times_{\operatorname{Spec}(K)} \operatorname{Spec}\left(\overline{\mathbb{C}(Y)}\right).$$

THEOREM 3.10
Under the assumptions made in Corollary 3.8, for all ν sufficiently large and divisible,

$$\kappa\left(\det(f_*\omega^{\nu}_{X/Y})\right) = \operatorname{Var}(f).$$

Proof
This has been shown in [29] if the general fibres of f are of general type, and in [12] in general (see also [14] or [30]). □

Remark 3.11
Let $f : V \to U$ be the morphism considered in Proposition 1.1. Since ω_F is ample on the fibres of f, we can replace the variety F' in the definition of $\operatorname{Var}(f)$ by its image under a multicanonical map and hence assume that it is also canonically polarized. One obtains a morphism $\varphi' : \operatorname{Spec}(K) \to M_h$, and K must contain the function field of $\overline{\varphi(U)}_{\text{red}}$. In particular, the assumption $\dim(\overline{\varphi(U)}) = \dim(U)$ implies $\operatorname{Var}(f) = \dim(U)$.

4. Products of families of canonically polarized manifolds

Again let $f : X \to Y$ be a surjective projective morphism between quasi-projective manifolds with connected fibres, and let $U \subset Y$ be a nonempty open subvariety such that

$$f : V = f^{-1}(U) \longrightarrow U$$

is smooth and such that $\omega_{V/U}$ is f-semiample.

In [32, Prop. 2.7], we showed that, for curves Y, the ampleness of $\det(f_*\omega^{\nu}_{X/Y})$ implies the ampleness of $f_*\omega^{\nu}_{X/Y}$ for $\nu \geq 2$. In [31, Th. 6.22] one finds a similar statement over U. In order to extend the latter to Y, one would like to control the nonlocal free locus of $f_*\omega^{\nu}_{X/Y}$. This could be done by using natural compactifications of moduli spaces, but those exist only for curves, for surfaces of general type, or by [11] under strong assumptions on the existence of minimal models.

Fortunately, the mild reduction of Abramovich and Karu can serve as a substitute, using, in particular, the reflexivity of the sheaves in Corollary 2.4(vii).

We assume in the sequel that $\dim(U) = \operatorname{Var}(f)$ and that $V \to U$ fits into the diagram considered in Lemma 2.3. Since Y' is finite over Y, one finds $\operatorname{Var}(g) = \operatorname{Var}(f) = \dim(Y')$, and Theorem 3.10 implies that $\det(g_*\omega^{\nu}_{Z/Y'})$ is big for all ν sufficiently large and divisible. We choose such $\nu \geq 3$, and we assume, in addition, that

$$f^* f_*\omega^\nu_{V/U} \longrightarrow \omega^\nu_{V/U}$$

and the multiplication morphisms

$$S^\beta(f_*\omega^\nu_{V/U}) \longrightarrow f_*\omega^{\beta\cdot\nu}_{V/U}$$

are surjective for all β. Define

$$e = \mathrm{Max}\big\{e(\omega^\nu_F);\ F \text{ a fibre of } V \to U\big\}.$$

By Corollary 2.4(ix), there is an invertible sheaf λ_ν on Y and some $N_\nu \in \mathbb{N}$ with

$$\tau^*\lambda_\nu = \det(g_*\omega^\nu_{Z/Y'})^{N_\nu}.$$

Writing $B = Y \setminus U$ for the boundary divisor, let us fix an ample invertible sheaf $\mathscr{A}$ such that $\mathscr{A}(-B)$ is ample. Since

$$\kappa(\lambda_\nu) = \kappa\big(\det(g_*\omega^\nu_{Z/Y'})\big) = \dim(Y),$$

there exists some $\eta > 0$ and some effective divisor D with $\lambda^\eta_\nu = \mathscr{A}(D)$. Replacing N_ν by some multiple, we can assume

$$\det(g_*\omega^\nu_{Z/Y'})^{N_\nu} = \tau^*\mathscr{A}(D)^{\nu\cdot(\nu-1)\cdot e}.$$

Define $r_0 = \mathrm{rank}(f_*\omega^\nu_{X/Y})$ and $r = N_\nu \cdot r_0$.

PROPOSITION 4.1
Let $X^{(r)}$ denote a desingularization of the rth fibre product $X \times_Y \cdots \times_Y X$, and let $f^{(r)} : X^{(r)} \to Y$ be the induced morphism. Then, for all β sufficiently large and divisible, the sheaf

$$f^{(r)}_*(\omega^{\beta\cdot\nu}_{X^{(r)}/Y}) \otimes \mathscr{A}^{-\beta\cdot\nu\cdot(\nu-2)} \otimes \mathscr{O}_Y\big(-\beta\cdot\nu\cdot(\nu-1)\cdot D\big)$$

is globally generated over U and

$$\omega^{\beta\cdot\nu}_{X^{(r)}/Y} \otimes f^{(r)*}\big(\mathscr{A}^{-\beta\cdot\nu\cdot(\nu-2)} \otimes \mathscr{O}_Y(-\beta\cdot\nu\cdot(\nu-1)\cdot D)\big)$$

is globally generated over $V^r = f^{(r)-1}(U)$.

Proof
We again use the notation from Lemma 2.3. By Lemma 2.2(ii), mildness of a morphism is compatible with fibre products; hence

$$g'^r : Z'^r = Z' \times_{Y'} \cdots \times_{Y'} Z' \to Y'$$

is again mild.

For the normalization $X'^{(r)}$ of $X^{(r)} \times_Y Y'$, we choose a desingularization $Z^{(r)}$ with centers in the singular locus of $X'^{(r)}$, and we choose a nonsingular blow-up $X''^{(r)}$ which dominates both $Z^{(r)}$ and Z'^r. We again obtain a diagram

$$\begin{array}{ccccccccccc}
V^r & \xrightarrow{\subset} & X^{(r)} & \xleftarrow{\tau^{(r)}} & X'^{(r)} & \xleftarrow{\sigma^{(r)}} & Z^{(r)} & \xleftarrow{\rho^{(r)}} & X''^{(r)} & \xleftarrow{\delta^{(r)}} & Z'^r \\
\downarrow & & \downarrow{\scriptstyle f^{(r)}} & & \downarrow & & {\scriptstyle g^{(r)}}\downarrow & & \downarrow{\scriptstyle f''^{(r)}} & & \downarrow{\scriptstyle g'^r} \\
U & \xrightarrow{\subset} & Y & \xleftarrow{\tau} & Y' & \xleftarrow{=} & Y' & \xleftarrow{=} & Y' & \xrightarrow{=} & Y'
\end{array}$$

which satisfies the assumptions made in Lemma 2.3. One finds, for all integers $\mu \geq 0$,

$$g'^r_* \omega^{\mu}_{Z'^r/Y'} = \bigotimes^r g'_* \omega^{\mu}_{Z'/Y'}. \tag{4.1.1}$$

In fact, by flat base change and by the projection formula, both sheaves coincide over the largest subvariety of Y', where $g'_* \omega^{\mu}_{Z'/Y'}$ is locally free. By definition, the right-hand side of (4.1.1) is the reflexive hull of the tensor product on this subscheme, and by Lemma 2.2(iii) the left-hand side is reflexive; hence both are equal. Corollary 2.4 implies the following.

CLAIM 4.2

(a) *The sheaf $g^{(r)}_* \omega^{\mu}_{Z^{(r)}/Y'}$ is reflexive, and there is an isomorphism*

$$g^{(r)}_* \omega^{\mu}_{Z^{(r)}/Y'} \simeq \bigotimes^r g_* \omega^{\mu}_{Z/Y'}.$$

(b) *There is an inclusion*

$$g^{(r)}_* \omega^{\mu}_{Z^{(r)}/Y'} \longrightarrow \tau^* f^{(r)}_* \omega^{\mu}_{X^{(r)}/Y}$$

which is an isomorphism over U'.

Proof

Part (b) and the first part of (a) are nothing but Corollary 2.4(viii) and (vii). For the second part of (a), Corollary 2.4(vii) allows us to replace the left-hand side by $g'^r_* \omega^{\mu}_{Z'^r/Y'}$ and the right-hand side by $\bigotimes^r g'_* \omega^{\mu}_{Z'/Y'}$, and to apply (4.1.1). □

By construction, $g^{(r)} : Z^{(r)} \to Y'$ is smooth over $U' = \tau^{-1}(U)$, and

$$g^{(r)^{-1}}(U') = V'^r = V^r \times_U U'.$$

Now we play the usual game. For the integer $\nu \geq 3$ chosen above, and for $r_0 = \mathrm{rank}(g_*\omega^{\nu}_{Z/Y'})$, there is a natural inclusion

$$\det(g_*\omega^{\nu}_{Z/Y'}) \longrightarrow \bigotimes^{r_0}(g_*\omega^{\nu}_{Z/Y'}) \tag{4.2.1}$$

which locally splits over the open set Y'_1, where $g_*\omega^{\nu}_{Z/Y'}$ is locally free, in particular, over U'. By the choice of r, one obtains an inclusion

$$\tau^*\mathscr{A}(D)^{\nu\cdot(\nu-1)\cdot e} \longrightarrow \bigotimes^{r}(g_*\omega^{\nu}_{Z/Y'}) = g^{(r)}_*\omega^{\nu}_{Z^{(r)}/Y'}, \tag{4.2.2}$$

again locally splitting over U'. In fact, the splitting inclusions in (4.2.1) and (4.2.2) exist over Y'_1, and since the sheaves on the right-hand sides are reflexive, they extend to Y'.

For $\omega = \omega_{Z^{(r)}/Y'}$ and $\mathscr{A}' = g^{(r)*}\tau^*\mathscr{A}(D)^{\nu}$, consider $\mathscr{L} = \omega\otimes\mathscr{A}'^{-1}$. By (4.2.2), $\omega^{\nu}\otimes\mathscr{A}'^{-(\nu-1)\cdot e}$ has a section whose zero divisor Γ does not contain a whole fibre over U', and

$$\mathscr{L}^{\nu\cdot(\nu-1)\cdot e} = \omega^{\nu\cdot(\nu-1)\cdot e-\nu^2} \otimes \omega^{\nu^2} \otimes \mathscr{A}'^{-\nu\cdot(\nu-1)\cdot e} = \omega^{\nu\cdot(\nu-1)\cdot e-\nu^2} \otimes \mathscr{O}_{Z^{(r)}}(\nu\cdot\Gamma).$$

All fibres of $V^r \to U$ are of the form

$$F^r = F \times \cdots \times F,$$

and [31, Cor. 5.21] implies

$$e(\Gamma|_{F^r}) \leq e(\omega^{\nu}_{F^r}) = e(\omega^{\nu}_F) \leq e.$$

So $e(\nu\cdot\Gamma|_{F^r}) \leq \nu\cdot e$, and for $N = \nu\cdot e$, assumption (b) in Proposition 3.3 holds true. By Corollary 3.8, the sheaf $g_*\omega^{\nu\cdot(\nu\cdot(e-1)-e)}$ is weakly positive over U'. Since

$$g^{(r)*}g^{(r)}_*\omega^{\nu\cdot(\nu\cdot(e-1)-e)} \longrightarrow \omega^{\nu\cdot(\nu\cdot(e-1)-e)} = \mathscr{L}^{\nu\cdot(\nu-1)\cdot e} \otimes \mathscr{O}_{Z^{(r)}}(-\nu\Gamma)$$

is surjective over V^r, we can apply Proposition 3.3 (for $\mathscr{L}^{\nu-1}$ instead of $\mathscr{L}$) and obtain the weak positivity of

$$g^{(r)}_*(\mathscr{L}^{\nu-1} \otimes \omega_{Z^{(r)}/Y'}) = g^{(r)}_*(\omega^{\nu}_{Z^{(r)}/Y'}) \otimes \tau^*\mathscr{A}(D)^{-\nu\cdot(\nu-1)}$$

over U'. Since $g^{(r)}_*\omega^{\beta\cdot\nu}_{Z^{(r)}/Y'}$ is reflexive, one has the multiplication morphism

$$\mu_\beta : S^\beta(g^{(r)}_*\omega^{\nu}_{Z^{(r)}/Y'}) \longrightarrow g^{(r)}_*\omega^{\beta\cdot\nu}_{Z^{(r)}/Y'}.$$

By Claim 4.2(a), the left-hand side is $S^{\beta}(\bigotimes^r g_*\omega^{\nu}_{Z/Y'})$, whereas the right-hand side is $\bigotimes^r g_*\omega^{\beta\cdot\nu}_{Z/Y'}$; hence the assumption on the surjectivity of the multiplication morphism carries over, and μ_β is surjective over U'. Since

$$g^{(r)}_*(\omega^{\nu}_{Z^{(r)}/Y'}) \otimes \tau^*\mathscr{A}(D)^{-\nu\cdot(\nu-1)}$$

is weakly positive over U', for all β sufficiently large and divisible,

$$S^{\beta}\big(g^{(r)}_*(\omega^{\nu}_{Z^{(r)}/Y'}) \otimes \tau^*\mathscr{A}(D)^{-\nu\cdot(\nu-1)}\big) \otimes \tau^*\mathscr{A}^{\beta}$$

and

$$g^{(r)}_*(\omega^{\beta\cdot\nu}_{Z^{(r)}/Y'}) \otimes \tau^*\mathscr{A}(D)^{-\beta\cdot\nu\cdot(\nu-1)} \otimes \tau^*\mathscr{A}^{\beta}$$

will be globally generated over U'. By Claim 4.2(b), one has a morphism

$$\tau_*\mathscr{O}_{Y'} \otimes \mathscr{A}^{\beta\cdot(\nu-1)} \longrightarrow \tau_*\tau^*(f^{(r)}_*\omega^{\beta\cdot\nu}_{X^{(r)}/Y}) \otimes \mathscr{A}(D)^{-\beta\cdot\nu\cdot(\nu-1)} \otimes \mathscr{A}^{\beta\cdot\nu},$$

surjective over U. Although the sheaf $f^{(r)}_*\omega^{\beta\cdot\nu}_{X^{(r)}/Y}$ is not necessarily reflexive, the finiteness of τ allows us to apply the projection formula and to obtain thereby a morphism

$$\tau_*\mathscr{O}_{Y'} \otimes \mathscr{A}^{\beta\cdot(\nu-1)} \longrightarrow f^{(r)}_*(\omega^{\beta\cdot\nu}_{X^{(r)}/Y}) \otimes \mathscr{A}^{-\beta\cdot\nu\cdot(\nu-2)} \otimes \mathscr{O}_Y\big(-\beta\cdot\nu\cdot(\nu-1)\cdot D\big),$$

surjective over U. For β large enough, the sheaf on the left-hand side will be generated by global sections; hence for those β the sheaf on the right-hand side is globally generated over U. Since we assumed

$$f^{(r)*}f^{(r)}_*\omega^{\nu}_{X^{(r)}/Y} \longrightarrow \omega^{\nu}_{X^{(r)}/Y}$$

to be surjective over V, the same holds true for ν replaced by $\beta\cdot\nu$, and

$$\omega^{\beta\cdot\nu}_{X^{(r)}/Y} \otimes f^{(r)*}\big(\mathscr{A}^{-\beta\cdot\nu\cdot(\nu-2)} \otimes \mathscr{O}_{X^{(r)}}(-\beta\cdot\nu\cdot(\nu-1)\cdot D)\big)$$

is globally generated over V^r. □

From now on, we ignore the original morphism f and work only with the morphism $f^{(r)}$. To keep notation as simple as possible, we allow ourselves to change it again. Doing so, we can restate the results of the Sections 2, 3, and 4 in the following way.

COROLLARY 4.3
Let $\tilde{U}$ be a quasi-projective manifold, and let $\tilde{f}:\tilde{V}\to\tilde{U}$ be a smooth projective surjective morphism with connected fibres, with $\mathrm{Var}(\tilde{f})=\dim(\tilde{U})$, and with $\omega_{\tilde{F}}$ semiample for all fibres $\tilde{F}$ of $\tilde{f}$.

Then there exist a proper birational morphism $U \to \tilde{U}$, a projective compactification Y of U, a projective morphism $f : X \to Y$, an invertible sheaf $\mathscr{A}$, and an effective divisor D on Y, such that for all ν sufficiently large and divisible one has the following:

(a) $f : V = f^{-1}(U) \to U$ *is smooth with connected fibres;*

(b) X *and* Y *are projective manifolds, and* $X \setminus V$ *and* $B = Y \setminus U$ *are normal crossing divisors;*

(c) $\mathscr{A}$ *is ample, and* $D \geq B$*;*

(d) $f_*(\omega^{\nu}_{X/Y}) \otimes \mathscr{A}(D)^{-\nu}$ *is globally generated over* U*;*

(e) $\omega^{\nu}_{X/Y} \otimes f^*\mathscr{A}(D)^{-\nu}$ *is globally generated over* V*.*

Proof

By Lemma 2.3, we find a smooth birational model $f : V \to U$ of $\tilde{f} : \tilde{V} \to \tilde{U}$ which fits into the diagram (2.3.1) in Lemma 2.3. We may replace $V \to U$ by $V^r \to U$ and apply Proposition 4.1. Properties (a) and (b) obviously hold true. Since we assumed $\mathscr{A}(-B)$ to be ample and $\nu \geq 3$, for the invertible $\mathscr{A}' = \mathscr{A}^{\nu-2}(-B)$ and for the divisor

$$D' = (\nu - 1) \cdot D + B,$$

one obtains property (c) and, by Proposition 4.1, (d) and (e). □

If one starts with any smooth morphism in Proposition 1.1, one knows by Remark 3.11 that the variation is maximal. Lemma 1.2 allows us to blow up the base; hence Corollary 4.3 allows us to replace the original morphism by a new one, satisfying assumptions (a)–(e). Therefore Proposition 1.1 and hence Theorem 0.1 are immediate consequences of the next proposition, which is shown at the end of Section 7.

PROPOSITION 4.4

Given U, let $f : X \to Y$ be a projective surjective morphism satisfying conditions (a)–(e) in Corollary 4.3 for some ν, $\mathscr{A}$, and D. Assume moreover that $n = \dim(F)$ is even, that $r = \dim(U) \geq 1$, and that ω^{ν}_F is very ample for all fibres F of $V \to U$. Then there exists no holomorphic map $\gamma : \mathbb{C} \to U$ with dense image.

5. Construction of cyclic coverings

Starting from a morphism $f : X \to Y$ satisfying the assumptions in Proposition 4.4 for an invertible sheaf $\mathscr{A}$, a divisor D, and a natural number ν, let us consider

$$\mathscr{L} = \omega_{X/Y} \otimes f^*\mathscr{A}(D)^{-1}.$$

Blowing up X with centers outside of V, we may assume that the global sections of $\mathscr{L}^{\nu}$ generate an invertible sheaf $\mathscr{H}$. If E denotes the divisor on X with $\mathscr{H}(E) = \mathscr{L}$, then E has support in $X \setminus V$ and hence it is a normal crossing divisor.

Let us assume there exists a holomorphic map $\gamma : \mathbb{C} \to U$ with dense image, contrary to Proposition 4.4. In this section we choose some divisor and some cyclic covering of X, depending on γ, and finally this construction helps to show that such a holomorphic map cannot exist.

By Corollary 4.3(d), we have for some ℓ a morphism $\bigoplus^{\ell+1} \mathscr{O}_Y \to f_*\mathscr{L}^\nu$, surjective over U, and by Corollary 4.3(e), the induced morphisms

$$\bigoplus^{\ell+1} \mathscr{O}_X \longrightarrow f^* f_* \mathscr{L}^\nu \longrightarrow \mathscr{L}^\nu$$

are both surjective over V. By assumption, one obtains embeddings

$$V \longrightarrow \mathbb{P} = \mathbb{P}(f_*\mathscr{L}^\nu|_V) \longrightarrow \mathbb{P}^\ell \times U.$$

The projection to $\mathbb{P}^\ell$ extends to the morphism $\pi : X \to \mathbb{P}^\ell$, defined by the sections of the sheaf $\mathscr{H}$. For all hyperplanes H in $\mathbb{P}^\ell$, one has

$$\mathscr{L} = \mathscr{O}_X\big(\pi^*(H) + E\big).$$

Let $\check{\mathbb{P}}^\ell$ denote the dual projective space. For a hyperplane $H \subset \mathbb{P}^\ell$, we write $[H] \in \check{\mathbb{P}}^\ell$ for the corresponding point. For each $u \in U$ and for $F_u = f^{-1}(u)$, the set of all $[H] \in \check{\mathbb{P}}^\ell$ with $F_u \cap H$ nonsingular and not equal to F_u is open. Let S_u denote the complement. By [6, exp. 17, Sec. XVII, Prop. 3.2], for general points u of $(\ell-1)$-dimensional components of S_u, the intersection $F_u \cap H$ will have just one ordinary double point of type A_1, that is, a singularity given locally analytic as the zero set of the equation $x_1^2 + \cdots + x_n^2$ in $\mathbb{C}^n$. Hence the locus T_u, consisting of hyperplanes H with $F_u \cap H$ having other types of singularities or with $F_u \subset H$, is of codimension at least two in $\check{\mathbb{P}}^\ell$.

As in [6, exp. 17, Sec. XVII, Sec. 6.1], those properties can also be considered in families, and the corresponding sets depend algebraically on the parameter. In particular,

$$S = \big\{([H], u);\ F_u \subset H \ \text{ or } \ F_u \cap H \ \text{ singular}\big\}$$

is a closed subset of $\check{\mathbb{P}}^\ell \times U$. Let us choose a codimension-two closed subscheme T of $\check{\mathbb{P}}^\ell \times U$, contained in S such that $S \setminus T$ is nonsingular, of pure codimension-one, and

$$S \setminus T \subset \big\{([H], u);\ F_u \not\subset H \ \text{ and } \ F_u \cap H \ \text{ has one ordinary double point of type } A_1\big\}.$$

Given $[H] \in \check{\mathbb{P}}^\ell$, let S_H and T_H be the intersections of $\{[H]\} \times U$ with S and T, respectively.

LEMMA 5.1
There exists some $[H] \in \check{\mathbb{P}}^\ell$ *such that* $T_H \cap \gamma(\mathbb{C}) = \emptyset$, *such that* S_H *meets* $\gamma(\mathbb{C})$ *transversally, and such that* $\pi^*(H)$ *is nonsingular and* $\pi^*(H) + E$ *a normal crossing divisor.*

Here “S_H meets $\gamma(\mathbb{C})$ transversally” means that for a local section σ of $\mathscr{O}_U$ with zero set $(S_H)_{\mathrm{red}}$, the holomorphic function $\gamma^*(\sigma)$ has zeros of order one.

Proof
The given map $\gamma : \mathbb{C} \to U$ induces a holomorphic map

$$\tilde{\gamma} : \check{\mathbb{P}}^\ell \times \mathbb{C} \longrightarrow \check{\mathbb{P}}^\ell \times U.$$

Since $\tilde{\gamma}$ is holomorphic, $\Delta^{(1)} = \tilde{\gamma}^{-1}(T)$ is a complex subspace of $\check{\mathbb{P}}^\ell \times \mathbb{C}$. Let $\Delta^{(2)}$ be the complex subspace of $\tilde{\gamma}^{-1}(S)$ given locally by the following condition. Let σ be a local equation of S on $\check{\mathbb{P}}^\ell \times U$. Then $\Delta^{(2)}$ is the analytic subspace of the zero set of $\tilde{\gamma}^*\sigma$, where the multiplicity of $\tilde{\gamma}^*\sigma$ is larger than or equal to two. We choose $\Delta = \Delta^{(1)} \cup \Delta^{(2)}$.

By [9, p. 172], Δ has a decomposition

$$\Delta = \bigcup_{i \in I} \Delta_i$$

in irreducible components. The index set I is countable since each point $p \in \mathbb{C}$ has a small neighborhood $U(p)$ such that $\check{\mathbb{P}}^\ell \times U(p)$ meets only finitely many of those components. As usual,

$$\dim(\Delta) = \mathrm{Max}\big\{\dim(\Delta_i);\ i \in I\big\}.$$

CLAIM 5.2
We have $\dim(\Delta) \leq \ell - 1$.

Proof
If γ is not an embedding of a small neighborhood of a point $p \in \mathbb{C}$, then

$$\check{\mathbb{P}}^\ell \times \{p\} \cap \Delta^{(2)}$$

consists of all hyperplanes H passing through p, and its dimension is $\ell - 1$. The set of those points is discrete. For all other points p and for all components Δ_i of $\Delta^{(2)}$, one has

$$\dim\big(\check{\mathbb{P}}^\ell \times \{p\} \cap \Delta_i\big) \leq \ell - 2.$$

In fact, let $U(p)$ denote a sufficiently small neighborhood of p. A general $[H] \in \check{\mathbb{P}}^\ell$ does not pass through $\gamma(p)$, and for those that do, the intersection is transversal, except for all $[H]$ in a codimension-two subset of $\check{\mathbb{P}}^\ell$.

If Δ_i is one of the components of $\Delta^{(1)}$, then for all $p \in \mathbb{C}$,

$$\dim\left(\Delta_i \cap \check{\mathbb{P}}^\ell \times \{p\}\right) = \dim\left(T \cap \check{\mathbb{P}}^\ell \times \{\gamma(p)\}\right) \le \ell - 2.$$

In both cases, if Δ_i is a component of Δ with $\Delta_i \subset \check{\mathbb{P}}^\ell \times \{p\}$, we are done. Otherwise, choose for $j = 1, 2$ two points $p_j \in \mathbb{C}$ with

$$\check{\mathbb{P}}^\ell \times \{p_j\} \cap \Delta_i \ne \emptyset.$$

Then $\check{\mathbb{P}}^\ell \times \{p_1\} \cap \Delta_i$ is not dense in Δ_i. Obviously, the dimension of $\check{\mathbb{P}}^\ell \times \{p_1\} \cap \Delta_i$ is larger than or equal to $\dim(\Delta_i) - 1$, and by Ritt's lemma (see [9, p. 102]), both must be equal. Hence

$$\dim(\Delta_i) = \dim\left(\check{\mathbb{P}}^\ell \times \{p_1\} \cap \Delta_i\right) + 1 \le \ell - 1. \qquad \square$$

CLAIM 5.3
The image $\mathrm{pr}_1(\Delta)$ *does not contain an open analytic subset* $W \subset \check{\mathbb{P}}^\ell$.

Proof
We show Claim 5.3 by induction on ℓ, using Claim 5.2 but not the definition of $\check{\mathbb{P}}^\ell$ as a dual projective space. If $\ell = 1$, the set $\mathrm{pr}_1(\Delta)$ is countable.

In general, if $W \subset \mathrm{pr}_1(\Delta)$, we choose a point $p \in \mathbb{C}$ such that none of the countably many components of Δ is contained in $\check{\mathbb{P}}^\ell \times \{p\}$. Moreover, for each $i \in I$, we choose a point $q_i \in \mathrm{pr}_1(\Delta_i)$. Let $H \simeq \check{\mathbb{P}}^{\ell-1}$ be a hyperplane, passing through p but not containing any of the points q_i. Then, for each component Δ_i, the intersection $\Delta_i \cap H \times \mathbb{C}$ cannot be dense in Δ_i and

$$\dim(\Delta_i \cap H \times \mathbb{C}) < l - 1.$$

Hence

$$\dim(\Delta \cap H \times \mathbb{C}) \le \ell - 2 = \dim(H) - 1,$$

and since $W \cap H$ is an open analytic subset of H, contained in $\mathrm{pr}_1(\Delta \cap H \times \mathbb{C})$, this contradicts the induction hypotheses. $\square$

Recall that we assumed that the global sections of $\mathscr{L}$ generate the invertible subsheaf $\mathscr{H}$ of $\mathscr{L}$. In particular,

$$H^0(X, \mathscr{H}) = H^0(X, \mathscr{L}^\nu) = H^0\left(\mathbb{P}^\ell, \mathscr{O}_{\mathbb{P}^\ell}(1)\right),$$

and for $[H]$ in some Zariski-open subscheme $\check{\mathbb{P}}^\ell_0$ of $\check{\mathbb{P}}^\ell$, the preimage $\pi^*(H)$ will be nonsingular and $\pi^*(H) + E$ a normal crossing divisor. By Claim 5.3, we can find points $[H]$ in $\check{\mathbb{P}}^\ell_0 \setminus \mathrm{pr}_1(\Delta)$, and for all of them the properties asked for in Lemma 5.1 hold true. □

From now on H is fixed, and we write $T = B \cup T_H$ and S for the closure of S_H in Y. We do not use anymore the fact that T_H is of codimension-one, and in the next step we replace Y by a blow-up with the centers partly contained in T_H.

LEMMA 5.4
Assume that, contrary to Proposition 4.4, there exists $\gamma : \mathbb{C} \to U$ with a dense image. Then we may assume, in addition to Corollary 4.3(a), (b), (d), and (e), that there exists a general section of $\mathscr{L}^\nu = \omega^\nu_{X/Y} \otimes f^*\mathscr{A}(D)^{-\nu}$ with zero divisor $H + E$, and divisors S and T in Y such that
(i) *$S \cap U$ is dense in S, and $S + T$ and $f^*(S + T)$ are normal crossing divisors;*
(ii) *$X \to Y$ and $H \to Y$ are both smooth over $U_0 = Y \setminus (S \cup T)$;*
(iii) *the fibres of $H \to Y$ over $Y_0 = Y \setminus T$ are reduced with at most an ordinary double point;*
(iv) *$\gamma(\mathbb{C}) \cap T = \emptyset$;*
(v) *H is nonsingular, and $f(E)$ is contained in B;*
(vi) *$\mathscr{A}$ is semiample, ample with respect to Y_0, and $D \geq B$.*

Proof
All this can be done by blowing up Y in centers not contained in $\gamma(\mathbb{C})$ and replacing $f : X \to Y$ by a desingularization of the pullback family. □

The section of $\mathscr{L}^\nu$ with zero divisor $H + E$ gives rise to a cyclic covering $\psi' : Z' \to X$ (see, e.g., [8, Sec. 3]). Condition (ii) of Lemma 5.4 implies that

$$g : Z_0 = \psi'^{-1} f^{-1}(U_0) \longrightarrow U_0$$

is smooth; hence it gives rise to a variation of Hodge structures $\mathbb{V}_0 = R^n g_* \mathbb{C}_{Z_0}$.

LEMMA 5.5
The monodromy of $\mathbb{V}_0 = R^n g_* \mathbb{C}_{Z_0}$ around the components of S is finite.

Proof
Here we use the assumption, that the dimension n of the fibres of f is even. A general curve C meets S transversally. Replacing C by some open subset, we can assume that

for a given component S_i of S,

$$C \cap (S \cup T) = C \cap S_i = \{p\}.$$

The restriction

$$\psi_C : Z_C = Z' \times_Y C \;>>\; X_C = X \times_Y C$$

of the finite morphism $\psi' : Z' \to X$ is a cyclic covering of order ν, totally ramified along $H_C = H \times_Y C$. By the definition of S and T, the fibre $H_p = H_C \cap F_p$ has one singular point q, and we can choose locally analytic parameters t in a neighborhood of $p \in C$ and $t, x_1, \dots, x_n$ in a neighborhood of $q \in X_C$ such that H_C is the zero-set of $\sum_{i=1}^n x_i^2 + t$ near q. Then locally near $\psi_C^{-1}(q)$ the covering Z_C is given by the equation

$$\sum_{i=1}^{n} x_i^2 + t + z^\nu.$$

So $g^{-1}(p)$ has one isolated singularity, a double point of type $A_{\nu-1}$. As is well known (see, e.g., [20, p. 132]), in even dimension the local monodromy group of an $A_{\nu-1}$ singularity is finite, and as in [6] or [20, p. 41], one obtains the same for the global monodromy. □

6. Higgs bundles

Notation 6.1
In this section we assume that $f : X \to Y$ satisfies the conditions stated in Proposition 4.4, except possibly that $\mathscr{A}$ is not ample but only semiample and big. For the given holomorphic map $\gamma : \mathbb{C} \to U$, we assume, moreover, the existence of the divisors S, T, H, and E satisfying the conditions in Lemma 5.4.

We define $\Delta = f^*(T)$ and $\Sigma = f^*(S)$. Recall that the original boundary divisor B is contained in T. So the nonreduced components of Δ or the components of $\Delta + \Sigma$, mapping to codimension-two subvarieties of Y, are all supported in $f^{-1}B$.

Let $\delta : X' \to X$ be a blow-up of X with centers in $\Delta + \Sigma$ such that $H' + \Delta' + \Sigma'$ is a normal crossing divisor, where $\Delta' = \delta^*\Delta$, $\Sigma' = \delta^*\Sigma$, and H' is the proper transform of H. For $\mathscr{L} = \omega_{X/Y} \otimes f^*\mathscr{A}(D)^{-1}$, we write $\mathscr{L}' = \delta^*\mathscr{L}$. For $E' = \delta^*(H + E) - H'$, one finds $\mathscr{L}'^\nu = \mathscr{O}_{X'}(H' + E')$.

Let $g : Z_0 \to U_0$ be the fibre space (considered at the end of Sec. 5) obtained by restricting the cyclic covering $\psi' : Z' \to X$, given by the divisor $H + E$ in Lemma 5.4. We choose Z to be a desingularization of the normalization of the fibre product

$X' \times_X Z'$. Let us denote the induced morphisms by

$$\begin{array}{ccccc} Y & \xleftarrow{g} & Z & \xrightarrow{\delta'} & Z' \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ Y & \xleftarrow{f'} & X' & \xrightarrow{\delta} & X \end{array}$$

Finally, we write $\Pi = g^{-1}(S \cup T)$, and we identify Z_0 with $Z \setminus \Pi$.

In the sequel we write $T_*(-\log \bullet)$ for the dual of $\Omega^1_*(\log \bullet)$.

By [5], for all $k \geq 0$ the local constant system $R^k g_* \mathbb{C}_{Z_0}$ gives rise to a local free sheaf $\mathscr{V}_k$ on Y with the Gauss-Manin connection

$$\nabla : \mathscr{V}_k \longrightarrow \mathscr{V}_k \otimes \Omega^1_Y\big(\log(S+T)\big),$$

where we assume that $\mathscr{V}_k$ is the quasi-canonical extension of

$$(R^k g_* \mathbb{C}_{Z_0}) \otimes_{\mathbb{C}} \mathscr{O}_{Y \setminus (S \cup T)},$$

that is, that the real part of the eigenvalues of the residues around the components of $S+T$ lie in $[0, 1)$.

By [25], $\mathscr{V}_k$ carries a filtration $\mathscr{F}^p$ by coherent subsheaves. If the monodromies around the components of $S+T$ are not unipotent, the $\mathscr{F}^p$'s are not necessarily subbundles. However, this is the case outside of the singular locus of $S+T$. By abuse of notation, we drop the assumption that Y is projective in the first part of this section, leave out a codimension-two subscheme W, and assume that f, f', and g are flat and that $S+T$ is nonsingular.

So the induced graded sheaves $E^{p,k-p}$ are locally free, and they carry a Higgs structure with logarithmic poles along $S+T$. Let us denote it by

$$\big(\mathfrak{gr}_{\mathscr{F}}(\mathscr{V}_k), \mathfrak{gr}_{\mathscr{F}}(\nabla)\big) = (E, \theta) = \Big(\bigoplus_{q=0}^{k} E^{k-q,q}, \bigoplus_{q=0}^{k} \theta_{k-q,q}\Big).$$

As in [32], we consider a second system of sheaves related to Z and to the pair (X, H). We define

$$F^{p,q} = R^q f'_*\big(\delta^*(\Omega^p_{X/Y}(\log \Delta)) \otimes \mathscr{L}'^{(-1)}\big)/_{\text{torsion}}.$$

Here, for $\eta = 0, \ldots, \nu - 1$, the invertible sheaves $\mathscr{L}'^{(-\eta)}$ are defined as

$$\mathscr{L}'^{(-\eta)} = \mathscr{L}'^{-\eta} \otimes \mathscr{O}_{X'}\Big(\Big[\frac{\eta \cdot (H' + E')}{\nu}\Big]\Big) = \mathscr{L}'^{-\eta} \otimes \mathscr{O}_{X'}\Big(\Big[\frac{\eta \cdot E'}{\nu}\Big]\Big).$$

As is well known (see, e.g., [10, p. 130]), the bundles $E^{p,q}$ have a similar description:

$$E^{p,q} = R^q g_* \Omega^p_{Z/Y}(\log \Pi).$$

Let

$$\tau_{p,q} : F^{p,q} \longrightarrow F^{p-1,q+1} \otimes \Omega^1_Y(\log T)$$

and

$$\tilde{\theta}_{p,q} : E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \Omega^1_Y\big(\log(S+T)\big)$$

be the edge morphisms of the tautological exact sequences

$$\begin{aligned} 0 \to\ & f'^*\Omega^1_Y(\log T) \otimes \delta^*\big(\Omega^{p-1}_{X/Y}(\log \Delta)\big) \otimes \mathscr{L}'^{(-1)} \\ & \to \delta^*\big(\mathfrak{gr}(\Omega^p_X(\log \Delta))\big) \otimes \mathscr{L}'^{(-1)} \to \delta^*\big(\Omega^p_{X/Y}(\log \Delta)\big) \otimes \mathscr{L}'^{(-1)} \to 0 \qquad (6.1.1) \end{aligned}$$

and

$$\begin{aligned} 0 \to g^*\Omega^1_Y\big(\log(S+T)\big) \otimes \Omega^{p-1}_{Z/Y}(\log \Pi) \to \mathfrak{gr}\big(\Omega^p_Z(\log \Pi)\big) \\ \to \Omega^p_{Z/Y}(\log \Pi) \to 0, \qquad (6.1.2) \end{aligned}$$

respectively, where

$$\mathfrak{gr}\big(\Omega^p_X(\log \Delta)\big) = \Omega^p_X(\log \Delta)/f^*\Omega^2_Y(\log T) \otimes \Omega^{p-2}_{X/Y}(\log \Delta)$$

and

$$\mathfrak{gr}\big(\Omega^p_Z(\log \Pi)\big) = \Omega^p_Z(\log \Pi)/g^*\Omega^2_Y(\log S+T) \otimes \Omega^{p-2}_{Z/Y}(\log \Pi).$$

The Gauss-Manin connection is the edge morphism of

$$0 \to g^*\Omega^1_Y\big(\log(S+T)\big) \otimes \Omega^{\bullet-1}_{Z/Y}(\log \Pi) \to \mathfrak{gr}\big(\Omega^\bullet_Z(\log \Pi)\big) \to \Omega^\bullet_{Z/Y}(\log \Pi) \to 0;$$

hence $\theta_{p,q} = \tilde{\theta}_{p,q}$.

LEMMA 6.2

Let $\bullet$ *stand either for* $\operatorname{Spec}(\mathbb{C})$ *or for* Y*. Then the group* $\mathbb{Z}/\nu$ *acts on* $\psi_*\Omega^p_{Z/\bullet}(\log(\Pi + \psi^*H'))$ *and on* $\psi_*\Omega^p_{Z/\bullet}(\log(\Pi))$*. One has a decomposition in sheaves of eigenvectors*

$$\psi_*\Omega^p_{Z/\bullet}\big(\log(\Pi + \psi^*H')\big) \cong \bigoplus_{\eta=0}^{\nu-1} \Omega^p_{X'/\bullet}\big(\log(\Delta' + \Sigma' + H')\big) \otimes \mathscr{L}'^{(-\eta)}$$

and

$$\begin{aligned} \psi_*\Omega^p_{Z/\bullet}(\log \Pi) \cong\ & \Omega^p_{X'/\bullet}\big(\log(\Delta' + \Sigma')\big) \\ & \oplus \bigoplus_{\eta=1}^{\nu-1} \Omega^p_{X'/\bullet}\big(\log(\Delta' + \Sigma' + H')\big) \otimes \mathscr{L}'^{(-\eta)}, \end{aligned}$$

compatible with the tautological sequences.

Proof
By [8, Lems. 3.21 and 3.22], there are natural inclusions

$$\psi^*\Omega^p_{X'/\bullet}\big(\log(\Delta' + \Sigma' + H')\big) \longrightarrow \Omega^p_{Z/\bullet}\big(\log(\Pi + \psi^* H')\big)$$

and $R^\beta \psi_* \Omega^p_{Z/\bullet}(\log(\Pi + \psi^* H')) = 0$ for $\beta > 0$. In fact, in [8] this is just stated for $\bullet = \mathrm{Spec}(\mathbb{C})$, but the general case follows by induction, considering the tautological sequences. Since $\mathbb{Z}/\nu$ acts on $\psi_*\mathcal{O}_Z$ with

$$\psi_*\mathcal{O}_Z = \bigoplus_{\eta=0}^{\nu-1} \mathcal{L}'^{(-\eta)}$$

as the decomposition in sheaves of eigenvectors, one obtains the first decomposition in the lemma. H' is totally ramified in Z. Hence there is an exact sequence

$$0 \to \psi_*\Omega^p_{Z/\bullet}(\log \Pi) \to \psi_*\Omega^p_{Z/\bullet}\big(\log(\Pi + \psi^* H')\big) \to \Omega^{p-1}_{H'/\bullet}\big(\log(\Delta' + \Sigma')|_{H'}\big),$$

and the two sheaves on the right-hand side differ only in the $\mathbb{Z}/\nu$ invariant part. □

LEMMA 6.3
Using the notation introduced above, let

$$\iota : \Omega^1_Y(\log T) \longrightarrow \Omega^1_Y\big(\log(S + T)\big)$$

be the natural inclusion. Then there exist morphisms $\rho_{p,q} : F^{p,q} \to E^{p,q}$ such that
(i) *the diagram*

$$\begin{array}{ccc} E^{p,q} & \xrightarrow{\theta_{p,q}} & E^{p-1,q+1} \otimes \Omega^1_Y(\log(S+T)) \\ \uparrow \rho_{p,q} & & \uparrow \rho_{p-1,q+1} \otimes \iota \\ F^{p,q} & \xrightarrow{\tau_{p,q}} & F^{p-1,q+1} \otimes \Omega^1_Y(\log T) \end{array}$$

commutes;
(ii) *there is an invertible sheaf $\mathcal{A}$, semiample and ample with respect to $Y \setminus T$, an effective divisor D', and an injection $\mathcal{A}(D') \to F^{n,0}$, which is an isomorphism over $Y \setminus T$;*
(iii) *$\tau_{n,0}$ induces a morphism*

$$\tau^\vee : T_Y(-\log T) = \big(\Omega^1_Y(\log T)\big)^\vee \longrightarrow F^{n,0\,\vee} \otimes F^{n-1,1},$$

which coincides over $Y \setminus (S \cup T)$ with the Kodaira-Spencer map

$$T_Y(-\log T) \longrightarrow R^1 f_* T_{X/Y}(-\log \Delta);$$

in particular, this morphism is injective;

(iv) *the morphisms* $\rho_{n-m,m}$ *are injective, for all* m*;*

(v) *the pair*

$$\Big(\bigoplus_{q=0}^{n} E^{n-q,q},\ \bigoplus_{q=0}^{n} \theta_{n-q,q}\Big)$$

is a Higgs bundle with logarithmic poles along $S+T$*, induced by a variation of Hodge structures with finite monodromy around the components of* S*.*

Remark 6.4

Instead of Lemma 6.3(iii) and (iv), we later use just the injectivity of $\tau^\vee$ and of $\rho_{n-m,m}$ for $m=0$ and $m=1$.

Proof

The proof is similar to [32, proof of Lem. 3.2]. It is well known that the bundle in (v) is the Higgs bundle for the variation of Hodge structures on $R^n g_* \mathbb{C}_{Z_0}$. The condition on the monodromy follows from Lemma 5.5. By Lemma 6.2, the sheaf

$$R^q f'_*\big(\Omega^p_{X'/Y}(\log(H'+\Delta'+\Sigma'))\otimes \mathscr{L}'^{(-1)}\big)$$

is a direct factor of $E^{p,q}$. The morphism $\rho_{p,q}$ is induced by the natural inclusions

$$\begin{aligned}\delta^*\Omega^p_{X/Y}(\log\Delta) &\to \delta^*\Omega^p_{X/Y}\big(\log(\Delta+\Sigma)\big)\\ &\to \Omega^p_{X'/Y}\big(\log(\Delta'+\Sigma')\big)\to \Omega^p_{X'/Y}\big(\log(H'+\Delta'+\Sigma')\big).\end{aligned} \tag{6.4.1}$$

Over $Y\setminus(S\cup T)$, the kernel of $\rho_{n-m,m}$ is a quotient of the sheaf

$$R^{m-1}(f'|_{H'})_*(\Omega^{n-m-1}_{H'/Y}\otimes \mathscr{L}'^{-1}|_{H'}).$$

Since the relative dimension of H' over Y is $n-1$ and since $\mathscr{L}'$ is fibrewise ample, the latter is zero by the Akizuki-Kodaira-Nakano vanishing theorem. So $\rho_{n-m,m}$ is injective, as claimed in (iv).

The injective morphism in (6.4.1) also exists for Y replaced by $\operatorname{Spec}(\mathbb{C})$, and the exact sequence (6.1.1) is a subsequence of

$$\begin{aligned}0\to\ & f'^*\Omega^1_Y\big(\log(S+T)\big)\otimes\Omega^{p-1}_{X'/Y}\big(\log(H'+\Delta+\Sigma)\big)\otimes\mathscr{L}'^{(-1)}\\ &\to \mathfrak{gr}\big(\Omega^p_{X'}(\log(H'+\Delta+\Sigma))\big)\otimes\mathscr{L}'^{(-1)}\\ &\to \Omega^p_{X'/Y}\big(\log(H'+\Delta+\Sigma)\big)\otimes\mathscr{L}'^{(-1)}\to 0.\end{aligned} \tag{6.4.2}$$

Finally, by Lemma 6.2 this sequence is obtained by taking the sheaves of eigenvectors in the direct image of the exact sequence (6.1.2) under $\psi: Z\to X'$. One obtains (i).

By definition, $F^{n,0} = f'_*\big(\delta^*(\Omega^n_{X/Y}(\log\Delta))\otimes\mathcal{L}'^{(-1)}\big)$. Comparing the first Chern classes for the tautological sequence for f, one finds

$$F^{n,0} = f'_*\big(\delta^*(\omega^n_{X/Y}(\Delta_{\mathrm{red}} - \Delta)) \otimes \mathcal{L}'^{(-1)}\big).$$

Recall that f is smooth over $Y \setminus B$, for the divisor B considered in Corollary 4.3(c). Hence

$$f^*B \geq -\Delta_{\mathrm{red}} + \Delta,$$

and $\Omega^n_{X/Y}(\log\Delta)$ contains $\omega_{X/Y}(-f^*B)$. Moreover, by Corollary 4.3(c), $D' = D - B$ is effective. By definition, $\mathcal{L} = \omega_{X/Y} \otimes f^*\mathcal{A}(D'+B)^{-1}$ and

$$\mathcal{L}'^{(-1)} = \mathcal{L}'^{-1} \otimes \mathcal{O}_{X'}\Big(\Big[\frac{E'}{\nu}\Big]\Big).$$

Therefore $\delta^*(\Omega^n_{X/Y}(\log\Delta))\otimes\mathcal{L}'^{(-1)}$ contains $\omega_{X/Y}(-f^*B)\otimes\mathcal{L}'^{(-1)}$ and hence the sheaf

$$f'^*\big(\mathcal{A}(D')\big) \otimes \mathcal{O}_{X'}\Big(\Big[\frac{E'}{\nu}\Big]\Big),$$

and (ii) holds true. For (iii), recall that over $Y \setminus (S \cup T)$ the sheaf $\mathcal{L}'^{(-1)}$ is nothing but

$$\mathcal{L}'^{-1} = \delta^*(\mathcal{L}^{-1}).$$

Since $R^\mu\delta_*\mathcal{O}_{X'} = 0$, by the projection formula the morphism

$$(\tau_{n,0} \otimes \mathrm{id}_{\mathcal{A}(D')^{-1}})|_{Y\setminus(S\cup T)}$$

is the restriction of the edge morphism of the short exact sequence

$$0 \to f^*\Omega^1_U \otimes \Omega^{n-1}_{V/U} \otimes \omega^{-1}_{V/U} \to \mathfrak{gr}(\Omega^n_V) \otimes {\omega_{V/U}}^{-1} \to \Omega^n_{V/U} \otimes \omega^{-1}_{V/U} \to 0.$$

Since $f|_V$ is smooth with n-dimensional fibres, the sheaf on the right-hand side is $\mathcal{O}_V$ and the one on the left-hand side is $f^*\Omega^1_U \otimes T_{V/U}$. Tensoring with

$$f^*T_U = f^*(\Omega^{r-1}_U \otimes \omega^{-1}_U)$$

and dividing by the kernel of the wedge product

$$f^*\Omega^1_U \otimes f^*(\Omega^{r-1}_U \otimes \omega^{-1}_U) \longrightarrow \mathcal{O}_V$$

on the left-hand side, one obtains an exact sequence

$$0 \longrightarrow T_{V/U} \longrightarrow \mathcal{G} \longrightarrow f^*T_U \longrightarrow 0, \tag{6.4.3}$$

where $\mathcal{G}$ is a quotient of $\mathfrak{gr}(\Omega^n_V) \otimes {\omega_V}^{-1} \otimes f^*\Omega^{r-1}_U$. By definition, the restriction to $Y \setminus (S \cup T)$ of the morphism considered in (iii) is the first edge morphism in the long exact sequence, obtained by applying $R^\bullet f_*$ to (6.4.3).

The wedge product induces a morphism

$$\Omega_V^n \otimes \omega_V{}^{-1} \otimes f^*\Omega_U^{r-1} \longrightarrow \Omega_V^{n+r-1} \otimes \omega_V^{-1} = T_V.$$

Since $r = \dim(U)$, this morphism factors through $\mathscr{G}$. Hence the exact sequence (6.4.3) is isomorphic to the tautological sequence

$$0 \longrightarrow T_{V/U} \longrightarrow T_V \longrightarrow f^*T_U \longrightarrow 0. \tag{6.4.4}$$

The edge morphism $T_U \to R^1 f_* T_{V/U}$ of (6.4.4) is the Kodaira-Spencer map. Since we assumed U to be generically finite over the moduli space, this morphism is injective. □

Let us return to the case of "Y projective." We choose for $E^{p,q}$ and $F^{p,q}$ the maximal coherent extension of the sheaves defined above outside of a codimension-two subvariety of Y. Of course, the morphisms $\theta_{p,q}$, $\tau_{p,q}$, and $\rho_{p,q}$ extend, and the properties (i) – (v) in Lemma 6.3 remain true.

By [26, p. 12], $\theta \wedge \theta = 0$; hence the image of the composite

$$\theta_{n-q+1,q-1} \circ \cdots \circ \theta_{n,0} : E^{n,0} \longrightarrow E^{n-q,q} \otimes \bigotimes^{q} \Omega_Y^1\big(\log(S+T)\big)$$

factors through

$$\theta^q : E^{n,0} \longrightarrow E^{n-q,q} \otimes S^q\Omega_Y^1\big(\log(S+T)\big).$$

By Lemma 6.3(ii), $\mathscr{A}(D')$ is a subsheaf of $F^{n,0}$ and hence of $E^{n,0}$, and one obtains a morphism

$$\mathscr{A}(D') \longrightarrow \rho_{n-q,q}(F^{n-q,q}) \otimes S^q\Omega_Y^1(\log T) \xrightarrow{\subset} E^{n-q,q} \otimes S^q\Omega_Y^1(\log T) \xrightarrow{S^m(\iota)} E^{n-q,q} \otimes S^q\Omega_Y^1\big(\log(S+T)\big)$$

and thereby a morphism

$$\tau'^q : S^q\big(T_Y(-\log T)\big) \longrightarrow E^{n-q,q} \otimes \mathscr{A}(D')^{-1}.$$

The pullback of τ'^q, via $\gamma : \mathbb{C} \longrightarrow Y \setminus T \longrightarrow Y$, composed with the qth tensor power of the differential of γ,

$$d\gamma^q : T_{\mathbb{C}}^q \longrightarrow \gamma^*\big(S^q T_Y(-\log T)\big),$$

gives

$$\tilde{\tau}^q : T_{\mathbb{C}}^q \longrightarrow \gamma^*\big(E^{n-q,q} \otimes \mathscr{A}(D')^{-1}\big).$$

We choose

$$m = \mathrm{Min}\big\{q \in \mathbb{N};\ \tilde{\tau}^{q+1}(T_{\mathbb{C}}^{q+1}) = 0\big\}$$

and put $\tau = \tau'^{m}$ and $\tilde{\tau} = \tilde{\tau}^{m}$.

The morphism τ'^{1} factors as

$$T_Y(-\log T) \longrightarrow F^{n-1,1} \otimes \mathscr{A}(D')^{-1} \xrightarrow{\rho_{n-1,1}} E^{n-1,1} \otimes \mathscr{A}(D')^{-1}.$$

By Lemma 6.3(iii), the first of those morphisms is injective, and by Lemma 6.3(iv), the second one is as well. Therefore τ'^{1} is injective. Since we assumed $\gamma(\mathbb{C})$ to be dense, the pullback of an injective morphism of sheaves under γ remains injective. Hence $\tilde{\tau}^{1}$ is injective, and $m > 0$.

Altogether, starting from the morphism in Proposition 4.4 and from a holomorphic map $\gamma : \mathbb{C} \to U$ with dense image, we constructed divisors S and T with the properties stated in Lemma 5.4, and we constructed Higgs bundles that satisfy properties (a) – (d) given in the following lemma.

LEMMA 6.5

For some $m > 0$ there exist an invertible sheaf $\mathscr{A}$, an effective divisor D', and a morphism of sheaves

$$\tau : S^m T_Y(-\log T) \longrightarrow E^{n-m,m} \otimes \mathscr{A}(D')^{-1} \longrightarrow E^{n-m,m} \otimes \mathscr{A}^{-1},$$

such that the composite

$$\tilde{\tau} = \gamma^*\tau \circ d\gamma^m : T_{\mathbb{C}}^m \longrightarrow \gamma^*\big(S^m T_Y(-\log T)\big) \longrightarrow \gamma^*(E^{n-m,m} \otimes \mathscr{A}^{-1})$$

satisfies

(a) *$\tilde{\tau}$ is injective;*

(b) *$\tilde{\tau}(T_{\mathbb{C}}^m) \subset \mathscr{N} \otimes \gamma^*(\mathscr{A}^{-1})$ for a sub–line bundle $\mathscr{N}$ of*

$$\mathrm{Ker}\,\big(\gamma^*(\theta_{n-m,m}) : \gamma^*(E^{n-m,m}) \longrightarrow \gamma^*(E^{n-m-1,m+1}) \otimes \Omega^1_{\mathbb{C}}(\log \gamma^{-1}(S))\big);$$

(c) *the pair*

$$(E, \theta) = \Big(\bigoplus_{p+q=n} E^{p,q}, \theta_{p,q}\Big)$$

is the Higgs bundle, corresponding to the quasi-canonical extension $\mathscr{V}$ of $\mathbb{V}_0 \otimes_{\mathbb{C}} \mathscr{O}_{Y\setminus(S\cup T)}$ for a geometric variation of Hodge structures $\mathbb{V}_0$, with finite monodromies around the components of S;

(d) *$\gamma(\mathbb{C})$ does not meet T;*

(e) *$\mathscr{A}$ is ample.*

At the end of Section 7 we show that those properties lead to a contradiction to the Ahlfors-Schwarz lemma. Hence the holomorphic map γ cannot exist.

Proof
All properties hold true, for the Higgs bundles constructed above, with $\mathscr{A}$ semiample and big. Choose some $\eta > 0$ such that $\mathscr{A}^\eta$ contains an ample invertible sheaf $\mathscr{A}'$, and consider the Higgs bundles

$$(E', \theta') = (E^{\otimes\eta}, \theta') \qquad \text{and} \qquad (F', \tau') = (F^{\otimes\eta}, \tau').$$

Again, we first consider them on $Y - W$, where W is the singular locus of $S \cup T$, and then we take the maximal extension to Y. By [26, p. 70], the morphism

$$\theta' : E^{\otimes\eta} \longrightarrow E^{\otimes\eta} \otimes \Omega_Y\big(\log(S+T)\big)$$

is given by

$$\theta' = \theta \otimes \mathrm{id}_E \otimes \cdots \otimes \mathrm{id}_E + \mathrm{id}_E \otimes \theta \otimes \cdots \otimes \mathrm{id}_E + \cdots + \mathrm{id}_E \otimes \cdots \otimes \mathrm{id}_E \otimes \theta,$$

and similarly for F' and τ'. The decomposition as a direct sum is

$$\bigoplus_{p+q=k} E'^{p,q} \qquad \text{and} \qquad \bigoplus_{p+q=k} F'^{p,q},$$

with

$$E'^{p,q} = \bigoplus \bigotimes_{i=1}^{\eta} E^{p_i,q_i} \qquad \text{and} \qquad F'^{p,q} = \bigoplus \bigotimes_{i=1}^{\eta} F^{p_i,q_i},$$

where the direct sums are taken over all $p_1, \ldots, p_\eta, q_1, \ldots, q_\eta$ with

$$\sum_{i=1}^{\eta} p_i = p \qquad \text{and} \qquad \sum_{i=1}^{\eta} q_i = q.$$

Again, we have morphisms

$$\rho'_{p,q} = \bigoplus \bigotimes_{i=1}^{\eta} \rho_{p_i,q_i} : F'^{p,q} \longrightarrow E'^{p,q},$$

compatible with $\theta'_{p,q}$ and $\tau'_{p,q}$. In particular, $\rho'_{n\eta,0}$ is the ηth tensor product of $\rho_{n,0}$, and hence injective. The same holds true for $\rho'_{n\eta-1,1}$, which is the direct sum of morphisms of the form

$$\rho_{n,0} \otimes \cdots \otimes \rho_{n-1,1} \otimes \cdots \otimes \rho_{n,0}.$$

Properties (i) and (v) in Lemma 6.3 remain true, with E and F replaced by E' and F', for $n\eta$ instead of n in (v). In (ii) one has an injection

$$\mathscr{A}' \longrightarrow \mathscr{A}(D')^{\eta} \longrightarrow F'^{n\eta,0} = (F^{n,0})^{\otimes\eta}.$$

The morphism

$$\tau'_{n\eta,0} : F^{n,0^{\otimes\eta}} \longrightarrow \big(F^{n-1,1} \otimes F^{n,0} \otimes \cdots \otimes F^{n,0} \oplus F^{n,0} \otimes F^{n-1,1} \otimes \cdots \otimes F^{n,0} \\ \oplus \cdots \oplus F^{n,0} \otimes F^{n,0} \otimes \cdots \otimes F^{n-1,1}\big) \otimes \Omega^1_Y(\log T)$$

is a direct sum of morphisms of the form

$$\mathrm{id}_{F^{n,0}} \otimes \cdots \otimes \theta_{n,0} \otimes \cdots \otimes \mathrm{id}_{F^{n,0}};$$

hence it induces the diagonal morphism

$$\bigoplus \tau^{\vee} : T_Y(-\log T) \longrightarrow F'^{n\eta,0^{\vee}} \otimes F'^{n-1,1} = \bigoplus^{\eta} F^{n,0^{\vee}} \otimes F^{n-1,1}.$$

In particular, the injectivity of the morphisms in Lemma 6.3(iii) carries over.

As noted in Remark 6.4, the injectivity of $\bigoplus \tau^{\vee}$, $\rho'_{n\eta,0}$, and $\rho'_{n\eta-1,1}$ is sufficient to perform the constructions with E' and F' instead of E and F, and to obtain some $m > 0$ and the morphisms τ and $\tilde{\tau}$ satisfying properties (a) and (b), with $\mathscr{A}$ replaced by $\mathscr{A}^{\eta}$. The latter contains the ample sheaf $\mathscr{A}'$; hence (e) holds true.

Finally, the Higgs bundle (E', θ') comes from the locally free extension $\mathscr{V}' = \mathscr{V}^{\otimes\eta}$ of $\mathbb{V}_0^{\otimes\eta} \otimes_{\mathbb{C}} \mathscr{O}_{Y\setminus(S\cup T)}$. The eigenvalues of the residues of the induced connection lie in $\mathbb{R}_{\geq 0}$; hence $\mathscr{V}'$ is contained in the quasi-canonical extension $\mathscr{V}''$. Replacing $\mathscr{V}'$ by $\mathscr{V}''$, we enlarge the sheaves $E'^{p,q}$, which is allowed without changing properties (a) and (b). □

7. Curvature estimates and the Ahlfors-Schwarz lemma

Let T be the normal crossing divisor in Lemma 6.5, and let $T = \sum_{i=1}^{\ell} T_i$ be its decomposition in irreducible components. Let s_i be the section of $\mathscr{L}_i = \mathscr{O}_Y(T_i)$ with zero set T_i. We choose a Hermitian metric g_i on $\mathscr{L}_i$ and define

$$r_i = -\log ||s_i||^2_{g_i} \qquad \text{and} \qquad r = r_1 \cdot \ldots \cdot r_{\ell}.$$

Given any constant $c > 1$, by rescaling the sections s_i, that is, by replacing s_i by $\epsilon \cdot s_i$ for ϵ sufficiently small, one may assume that $r_i \geq c$.

On the ample invertible line bundle $\mathscr{A}$ in Lemma 6.5, we choose a metric g such that the curvature form $\Theta(\mathscr{A}, g)$ is positive definite. For a positive integer α, we define a new metric $g_{\alpha} = g \cdot r^{\alpha}$ on $\mathscr{A}|_{Y\setminus T}$.

Recall that a Hermitian form ω_{α} on $T_Y(-\log T)$ is continuous and positive definite if each point in Y has a neighborhood U with local coordinates $z_1, \ldots, z_n$, such that $T \cap U$ is the zero set of $z_1 \cdot \ldots \cdot z_k$ and such that, writing $\iota_1 = \cdots = \iota_k = 1$ and $\iota_{k+1} = \cdots = \iota_n = 0$,

$$\omega_\alpha = \sqrt{-1} \sum_{1 \le i,j \le n} a_{i,j} \frac{dz_i}{z_i^{\iota_i}} \wedge \frac{d\bar{z}_j}{\bar{z_j}^{\iota_j}}$$

for a continuous and positive definite Hermitian matrix $(a_{ij})_{1 \le i,j \le n}$.

LEMMA 7.1
Rescaling the s_i, if necessary, there exists a continuous and positive definite Hermitian form ω_α on $T_Y(-\log T)$ with

$$r^2 \Theta(\mathscr{A}|_{Y \setminus T}, g_\alpha) \ge \omega_\alpha.$$

Proof
We recall the formula for the curvature calculation of a line bundle with a metric $(\mathscr{L}, g)$ (see, e.g., [7, Def. 7.1]). Let

$$\mathscr{L}|_U \simeq U \times \mathbb{C}$$

be a local trivialization of $\mathscr{L}$, and let s_U be a holomorphic section of $\mathscr{L}|_U$ which does not vanish in any point of U. Then s_U corresponds to a holomorphic function h_U on U, and the metric g is given by

$$||s_u||_g^2 = |h_U|^2 e^{-\phi}.$$

The curvature $\Theta(\mathscr{L}, g)$ is given by

$$\Theta(\mathscr{L}, g) = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi.$$

Applying this formula (see also [21, proof of Prop. 3.1]), one finds

$$\begin{aligned}
\Theta(\mathscr{A}, g_\alpha) = \Theta(\mathscr{A}, gr^\alpha) = \Theta(\mathscr{A}, ge^{-(-\alpha \log r)}) = \Theta(A, g) - \frac{\sqrt{-1}\alpha}{2\pi} \partial\bar{\partial} \log r \\
= \Theta(\mathscr{A}, g) - \sum_{i=1}^{\ell} \frac{\sqrt{-1}\alpha}{2\pi} \partial\bar{\partial} \log r_i = \Theta(\mathscr{A}, g) - \sum_{i=1}^{\ell} \frac{\sqrt{-1}\alpha}{2\pi} \partial \frac{\bar{\partial} r_i}{r_i} \\
= \Theta(\mathscr{A}, g) - \sum_{i=1}^{\ell} \frac{\alpha\Theta(L_i, g_i)}{r_i} + \frac{\sqrt{-1}\alpha}{2\pi} \frac{\partial r_i \wedge \bar{\partial} r_i}{r_i^2}.
\end{aligned}$$

Rescaling the sections s_i, one can assume that the r_i are larger than a large constant $c > 1$ and hence that

$$\Theta(\mathscr{A}, g) - \sum_{i=1}^{\ell} \omega'_\alpha := \frac{\alpha\Theta(L_i, g_i)}{r_i}$$

is a continuous and positive definite (1,1)-form on Y. Moreover,

$$\Theta(\mathscr{A}, g_\alpha) = \omega'_\alpha + \sum_{i=1}^{\ell} \frac{\sqrt{-1}\alpha}{2\pi} \frac{\partial r_i \wedge \bar\partial r_i}{r_i^2} \geq \omega'_\alpha + \frac{\sqrt{-1}\alpha}{2\pi} \sum_{i=1}^{\ell} \frac{\partial r_i \wedge \bar\partial r_i}{r^2}.$$

The (1,1)-form

$$\frac{\sqrt{-1}\alpha}{2\pi} \sum_{i=1}^{\ell} \partial r_i \wedge \bar\partial r_i$$

is clearly positive semidefinite on $Y \setminus T$.

CLAIM 7.2

Assume again that $T \cap U$ is the zero set of $z_1 \cdot \ldots \cdot z_k$ for local coordinates $z_1, \ldots, z_n$ on U. Then in a small neighborhood of $T \cap U$ the form

$$\frac{\sqrt{-1}\alpha}{2\pi} \sum_{i=1}^{\ell} \partial r_i \wedge \bar\partial r_i$$

is positive definite on the subspace of $T_Y(-\log T)|_U$ spanned by

$$\{z_1 \partial_{z_1}, \ldots, z_k \partial_{z_k}\}.$$

Proof

Near T_i the section s_i can be expressed as

$$s_i = z_i t_i, \qquad ||s_i||^2_{g_i} = z_i \bar z_i ||t_i||^2_{g_i} = z_i \bar z_i f_i,$$

where t_i is a local basis of L_i and where f_i is a positive function. So

$$r_i = -\log ||s_i||^2_{g_i} = -\log z_i - \log \bar z_i - \log f_i,$$

$$\partial r_i = -\frac{dz_i}{z_i} - \frac{1}{f_i} \sum_{j=1}^{n} \frac{\partial f_i}{\partial z_j} dz_j,$$

and

$$\bar\partial r_i = -\frac{d\bar z_i}{\bar z_i} - \frac{1}{f_i} \sum_{j=1}^{n} \frac{\partial f_i}{\partial \bar z_j} d\bar z_j.$$

So the leading term in

$$\frac{\sqrt{-1}\alpha}{2\pi} \sum_{i=1}^{\ell} \partial r_i \wedge \bar\partial r_i$$

near $T \cap U$ is

$$\frac{\sqrt{-1}\alpha}{2\pi} \sum_{i=1}^{k} \frac{dz_i}{z_i} \wedge \frac{d\bar z_i}{\bar z_i}.$$

Obviously, this form is positive definite on the subspace spanned by

$$\{z_1\partial_{z_1}, \ldots, z_k\partial_{z_k}\}. \qquad \square$$

Since we assumed that $r \geq 1$,

$$r^2\Theta(A|_{Y\setminus T}, g_\alpha) \geq r^2\omega'_\alpha + \frac{\sqrt{-1}\alpha}{2\pi}\sum_{i=1}^{\ell}\partial r_i \wedge \bar{\partial} r_i \geq \omega'_\alpha + \frac{\sqrt{-1}\alpha}{2\pi}\sum_{i=1}^{\ell}\partial r_i \wedge \bar{\partial} r_i.$$

By Claim 7.2, the (1,1)-form

$$\omega_\alpha = \omega'_\alpha + \frac{\sqrt{-1}\alpha}{2\pi}\sum_{i=1}^{\ell}\partial r_i \wedge \bar{\partial} r_i$$

is continuous and positive definite on $T_Y(-\log T)$. $\square$

Let $\gamma : \mathbb{C} \to Y \setminus T$ be the holomorphic map with Zariski-dense image considered in Lemma 6.5, and let t be the global coordinate on $\mathbb{C}$. We take the ample bundle $\mathscr{A}$ on Y with the metric g_α on $Y \setminus T$ and the Hermitian metric ω_α on $T_Y(-\log T)$ from Lemma 7.1. Writing again

$$d\gamma : T_{\mathbb{C}} \to \gamma^* T_Y(-\log T)$$

for the differential, one finds

$$\gamma^*\omega_\alpha = \sqrt{-1}||d\gamma(\partial_t)||^2_{\gamma^*\omega_\alpha}\, dt \wedge d\bar{t},$$

and Lemma 7.1 implies the following corollary.

COROLLARY 7.3
We have $\gamma^* r^2\Theta(\mathscr{A}|_{Y\setminus T}, g_\alpha) \geq \sqrt{-1}||d\gamma(\partial_t)||^2_{\gamma^*\omega_\alpha}\, dt \wedge d\bar{t}$.

Let us return to the morphism of sheaves in Lemma 6.5:

$$\tau : S^m T_Y(-\log T) \longrightarrow E^{n-m,m} \otimes \big(\mathscr{A}(D')\big)^{-1} \hookrightarrow E^{n-m,m} \otimes \mathscr{A}^{-1},$$
$$\tilde{\tau} := \gamma^*\tau \circ (d\gamma)^m : T^m_{\mathbb{C}} \longrightarrow \gamma^* S^m T_Y(-\log T) \hookrightarrow \gamma^*(E^{n-m,m} \otimes \mathscr{A}^{-1}).$$

By Lemma 6.5(c), $E^{n-m,m}$ is a subquotient of the quasi-canonical extension of a geometric variation of Hodge structures $\mathbb{V}_0$ on $Y \setminus S \cup T$. By Kawamata's construction (see [31, Lem. 2.5]), one finds a finite morphism $\pi : Y' \to Y$ with Y' nonsingular and $S' + T' = \pi^*(S + T)$ a normal crossing divisor such that the local monodromies of the pullback $\pi^*\mathbb{V}_0$ around $S' + T'$ are unipotent. For the discriminant $\Delta(Y'/Y)$ of $\pi : Y' \to Y$, both

$$\Delta(Y'/Y) + S + T \qquad \text{and} \qquad \pi^*(\Delta(Y'/Y) + S + T)$$

are normal crossing divisors. Moreover, for a component T_i of $S + T$, there exists some μ_i with

$$\pi^* T_i = \mu_i \cdot (\pi^* T_i)_{\mathrm{red}}.$$

Since we assumed the local monodromy of $\mathbb{V}_0$ around the components of S to be of finite order, the local monodromy of $\pi^*\mathbb{V}_0$ around the components of $S' = \pi^*(S)$ is trivial; hence $\pi^*\mathbb{V}_0$ extends to a variation of Hodge structures $\mathbb{V}'_0$ across S'. Let h and h' denote the Hodge metrics on $\mathbb{V}_0$ and $\mathbb{V}'_0$, respectively. We use the same notation for the induced metric on the Higgs bundles $\bigoplus E^{p,q}$ and $\bigoplus E'^{p,q}$, where the latter is given by subquotients of the canonical extension of $\mathbb{V}'_0$ across $T' = \pi^* T$. We have an inclusion of sheaves

$$\iota : (\pi^* E^{n-m,m}, \pi^* h) \hookrightarrow (E'^{n-m,m}, h')$$

such that $\pi^*(h) = \iota^*(h')$ on $Y' \setminus S' \cup T'$.

Consider the diagram of morphisms of analytic spaces

$$\begin{array}{ccc} \mathbb{C}' & \xrightarrow{\gamma'} & Y' \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{C} & \xrightarrow{\gamma} & Y \end{array} \tag{7.3.1}$$

where $\mathbb{C}'$ is obtained as a normalization of the fibre product. Hence if $U \subset \mathbb{C}$ is a sufficiently small neighborhood of $t_0 \in \gamma^{-1}(S)$, then for each $t'_0 \in \pi'^{-1}(t_0)$ there exists a connected component $U' \subset \pi'^{-1}(U)$ and a coordinate function t' on U' for which the map $\pi' : U' \to U$ is given by

$$t - t_0 = \pi'(t) = (t' - t'_0)^{\mu_0} \quad \text{for some } \mu_0 \in \mathbb{N} - \{0\}. \tag{7.3.2}$$

By Lemma 6.5(b), $\tilde{\tau}(T^m_{\mathbb{C}})$ is contained in an invertible line bundle $\mathcal{N} \otimes \gamma^*(\mathcal{A}^{-1})$, where $\mathcal{N}$ is a sub–line bundle of the kernel of $\gamma^*(\theta_{n-m,m})$. If

$$\theta'_{n-m,m} : E'^{n-m,m} \longrightarrow E'^{n-m-1,m+1} \otimes \Omega^1_{Y'}\big(\log \pi^*(T)\big)$$

denotes the Higgs structure on Y', we have a commutative diagram

$$\begin{array}{ccc} \gamma'^* E'^{n-m,m} & \xrightarrow{\gamma'^*(\theta'_{n-m,m})} & \gamma'^* E'^{n-m-1,m+1} \otimes \Omega^1_{\mathbb{C}'}(\log S') \\ {\scriptstyle \gamma'^*(\iota)}\uparrow\subset & & {\scriptstyle \gamma'^*(\iota)\otimes\pi'^*}\uparrow\subset \\ \gamma'^*\pi^* E^{n-m,m} & \xrightarrow{\gamma'^*\pi^*(\theta_{n-m,m})} & \gamma'^*\pi^* E^{n-m-1,m+1} \otimes \pi^*\Omega^1_{\mathbb{C}}(\log S) \end{array}$$

So ι induces an inclusion

$$\pi'^{*}\ker\big(\gamma^{*}(\theta_{n-m,m})\big) \xrightarrow{\subset} \ker\big(\gamma'^{*}(\theta'_{n-m,m})\big);$$

hence there exists a sub–line bundle

$$\mathscr{N}' \subset \ker\big(\gamma'^{*}(\theta'_{n-m,m})\big)$$

with

$$\pi'^{*}\tilde{\tau}(T_{\mathbb{C}}^{m}) \subset \pi'^{*}(\mathscr{N}) \otimes \gamma'^{*}(\pi^{*}\mathscr{A}^{-1}) \subset \mathscr{N}' \otimes \gamma'^{*}(\pi^{*}\mathscr{A}^{-1}). \tag{7.3.3}$$

As in [32, Lem. 1.1], using $\theta'_{n-m,m}(\mathscr{N}') = 0$ and P. Griffiths's estimates for the curvature of the Hodge metric (see [10, Chap. II]), one obtains the following lemma.

LEMMA 7.4
The curvature $\Theta(\mathscr{N}', h'|_{\mathscr{N}'})$ of the restricted Hodge metric on $\mathscr{N}'$ is negative semidefinite on $Y' \setminus T'$.

The Hodge metric h defines a metric $h \otimes g_{\alpha}^{-1}$ on $E^{n-m,m} \otimes \mathscr{A}^{-1}|_{Y \setminus S \cup T}$. By Lemma 6.5(a), $\tilde{\tau} \neq 0$, and since $\gamma(\mathbb{C})$ is Zariski dense in Y, we may define a nonzero positive semidefinite Kähler form $\sqrt{-1}c(t)\, dt \wedge d\bar{t}$ on $\mathbb{C} \setminus \gamma^{-1}(S)$ by choosing

$$c(t) = \big|\big|\tilde{\tau}((\partial_t)^m)\big|\big|^{2/m}_{\gamma^{*}(h\otimes g_{\alpha}^{-1})}.$$

LEMMA 7.5
Let μ denote the lowest common multiple of all the ramification orders of components of $\pi^{*}(S)$ over Y. Then there exists an effective divisor Π on $\mathbb{C}$ (i.e., a locally finite sum $\sum \beta_i P_i$ with $\beta_i \geq 0$) and a line bundle $\mathscr{N}^{(\mu)}$ on $\mathbb{C}$ with

$$\tilde{\tau}(T_{\mathbb{C}}^{m})^{\mu} \otimes \mathscr{O}_{\mathbb{C}}(\Pi) = \mathscr{N}^{(\mu)} \otimes \gamma^{*}\mathscr{A}^{-\mu} \qquad \text{and} \qquad \pi'^{*}\mathscr{N}^{(\mu)} = \mathscr{N}'^{\mu}.$$

Proof
By (7.3.3), $\pi'^{*}\tilde{\tau}(T_{\mathbb{C}}^{m})$ is a subsheaf of $\mathscr{N}' \otimes \gamma'^{*}(\pi^{*}\mathscr{A}^{-1})$. Using the description of π' in (7.3.2), we choose for a given point $t'_0 \in \pi'^{-1}\gamma^{-1}(S)$ a small neighborhood U' and some $\rho \in \mathbb{N}$ with

$$\pi'^{*}\tilde{\tau}(T_{\mathbb{C}}^{m})|_{U'} \otimes \mathscr{O}_{U'}(\rho \cdot t'_0) = \mathscr{N}' \otimes \gamma'^{*}(\pi^{*}\mathscr{A}^{-1})|_{U'}.$$

The number ρ/μ_0 is determined by the monodromy of $\mathbb{V}_0$ around the component of S containing $\gamma(t_0)$; hence it is independent of the point $t'_0 \in \pi'^{-1}(t_0)$. Since the ramification order μ_0 in (7.3.2) divides μ, we may choose Π to be the effective divisor with $\Pi|_U = (\rho \cdot \mu/\mu_0) \cdot t_0$ and

$$\mathscr{N}^{(\mu)} = \tilde{\tau}(T_{\mathbb{C}}^{m})^{\mu} \otimes \mathscr{O}_{\mathbb{C}}(\Pi) \otimes \gamma^{*}\mathscr{A}^{\mu}.$$

□

Outside of $\pi'^*\Pi$, the metrics $\gamma'^* h'^\mu$ and $\pi'^*\gamma^* h^\mu$ on $\mathcal{N}'^\mu$ coincide; hence $\gamma^* h^\mu$ extends to a metric $h^{(\mu)}$ on $\mathcal{N}^{(\mu)}$ and

$$c(t) = \left|\left|\tilde{\tau}((\partial_t)^m)^\mu\right|\right|^{2/(m\cdot\mu)}_{h^{(\mu)}\otimes\gamma^* g_\alpha^{-\mu}}.$$

In particular, $\sqrt{-1}c(t)\,dt \wedge d\bar{t}$ defines a semidefinite Kähler form on $\mathbb{C}$. The induced metric F is a singular metric in the sense described in [7, Def. 7.1] or [22, Sec. 2]. The curvature current of $T_\mathbb{C}$ is then defined to be the closed (1,1)-current

$$\Theta(T_\mathbb{C}, F) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log c(t).$$

LEMMA 7.6

There exists some $\epsilon' > 0$ *with*

$$-\Theta(T_\mathbb{C}, F) \geq \epsilon'\gamma^*\Theta(\mathcal{A}|_{Y\setminus T}, g_\alpha)$$

in the sense of currents.

Proof

Let $[\Pi]$ denote the current of integration over the divisor Π. As in [7, proof of Prop. 7.2], one defines a singular metric $|s|^2$ on sections of $\mathcal{O}_\mathbb{C}(\Pi)$ by taking the square of the modulus of s viewed as a complex-valued function. By the Lelong-Poincaré equation, $[\Pi]$ is the curvature current of this metric. One finds

$$\Theta(T_\mathbb{C}^{m\cdot\mu}, F^{m\cdot\mu}) + \gamma^*\Theta(\mathcal{A}^\mu|_{Y\setminus T}, g_{\alpha\cdot\mu}) + [\Pi] = \Theta(\mathcal{N}^{(\mu)}, h^{(\mu)}).$$

By [22, Sec. 2], the curvature current of a singular metric on a holomorphic line bundle on a complex manifold is compatible with pullback under holomorphic maps. Hence

$$\pi'^*\Theta(\mathcal{N}^{(\mu)}, h^{(\mu)}) = \Theta(\mathcal{N}'^\mu, h'^\mu) = \mu\cdot\Theta(\mathcal{N}', h').$$

By Lemma 7.4, the latter is negative semidefinite; hence $\Theta(\mathcal{N}^{(\mu)}, h^{(\mu)}) \leq 0$. Moreover, $[\Pi] \geq 0$ in the sense of currents; hence

$$-\Theta(T_\mathbb{C}, F) = -\frac{1}{m\cdot\mu}\Theta(T_\mathbb{C}^{m\cdot\mu}, F^{m\cdot\mu}) \geq \frac{1}{m}\gamma^*\Theta(\mathcal{A}|_{Y\setminus T}, g_\alpha). \qquad \square$$

LEMMA 7.7

For $\alpha \gg 1$ *there exists some* $\epsilon > 0$ *with*

$$\gamma^*\Theta(\mathcal{A}|_{Y\setminus T}, g_\alpha) \geq \epsilon\sqrt{-1}c(t)\,dt \wedge d\bar{t}.$$

Proof

We use the notation from Lemma 7.5, in particular, the metric $h^{(\mu)}$ on $\mathcal{N}^{(\mu)}$. Recall that

$$c(t) = \left|\left|\tilde{\tau}((\partial_t)^m)^\mu\right|\right|^{2/(m\cdot\mu)}_{h^{(\mu)}\otimes\gamma^* g_\alpha^{-\mu}}$$

and that by 7.3, for all $\alpha > 0$,

$$\gamma^*\Theta(\mathscr{A}|_{Y\setminus T}, g_\alpha) \geq \sqrt{-1}\gamma^* r^{-2}||d\gamma(\partial_t)||^2_{\gamma^*\omega_\alpha} dt \wedge d\bar{t}.$$

Hence, in order to show Lemma 7.7, it remains to verify that for $\alpha \gg 1$ there exists some $\epsilon > 0$ with

$$\begin{aligned}\gamma^* r^{-2}||d\gamma(\partial_t)||^2_{\gamma^*\omega_\alpha} &\geq \epsilon\gamma^* r^{-\alpha/m}\left|\left|\tilde{\tau}((\partial_t)^m)^\mu\right|\right|^{2/(m\cdot\mu)}_{h^{(\mu)}\otimes\gamma^* g^{-\mu}} \\ &= \epsilon\left|\left|\tilde{\tau}((\partial_t)^m)^\mu\right|\right|^{2/(m\cdot\mu)}_{h^{(\mu)}\otimes\gamma^* g_\alpha^{-\mu}}. \qquad (7.7.1)\end{aligned}$$

Given a point $p \in Y$, choose a small polydisk U with coordinates $z_1, \ldots, z_n$ in such a way that the divisors $T \cap U$ and $S \cap U$ are defined by the equations

$$z_1 \cdot \ldots \cdot z_k = 0 \qquad \text{and} \qquad z_{k+1} \cdot \ldots \cdot z_{k+k'} = 0.$$

Let $\pi : Y' \to Y$ be the cover ramified along $S + T$ which we considered in (7.3.1). Choosing U small enough, we may assume that the connected component $U' \subset \pi^{-1}(U)$ are polydisks with coordinates $\{w_1, \ldots, w_n\}$ and that π is defined by

$$\pi(w_1, \ldots, w_n) = (z_1^{\mu_1}, \ldots, z_n^{\mu_n}).$$

Hence for $S' = \pi^*(S)_{\text{red}}$ and $T' = \pi^*(T)_{\text{red}}$, the restrictions to U' are the zero sets of

$$w_1 \cdot \ldots \cdot w_k \qquad \text{and} \qquad w_{k+1} \cdot \ldots \cdot w_{k+k'},$$

respectively.

Consider, as above, the Higgs bundle $\bigoplus E'^{p,q}$ obtained from the canonical extension of $\mathbb{V}'_0$ along T', and let $\{e'_1, e'_2, \ldots\}$ be a basis for $E'^{n-m,m}|_{U'}$.

CLAIM 7.8

For U and U' sufficiently small, there exist some $\beta' \gg 1$ and a real number $c > 0$ with

$$h'\big(e'_i(w), e'_j(w)\big) \leq c\big((-\log|w_1|)\cdot(-\log|w_2|)\cdot \ldots \cdot(-\log|w_k|)\big)^{\beta'}$$

for all $w = (w_1, \ldots, w_n) \in U' \setminus T'$.

Proof
By [4, Th. 5.21], $U'_0 = U' \setminus T'$ can be decomposed into

$$U'_0 = \bigcup U'^{I}_{0,K},$$

where the open subset $U^I_{0,K}$ depends on the index of the filtration of the mixed Hodge structure (see [4, Sec. 5.7]), so that

$$h'\big(e'_i(w), e'_i(w)\big) \sim (-\log|w_1|)^{l_1/2} \cdot (-\log|w_2|)^{(l_2-l_1)/2} \cdot \ldots \cdot (-\log|w_k|)^{(l_k-l_{k-1})/2},$$

for all $w \in U'^{I}_{0,K}$, where $(l_1,\ l_2, \ldots, l_k)$ is the multi-index of the weight filtration of the mixed Hodge structure. Since this index set is finite, there exist some $\beta' \gg 1$ and some $c > 0$ such that

$$h'\big(e'_i(w), e'_i(w)\big) \le c\Big((-\log|w_1|) \cdot (-\log|w_2|) \cdot \ldots \cdot (-\log|w_k|)\Big)^{\beta'}$$

for all $w \in U'^{I}_{0,K}$ and for all I. Hence

$$h'(e'_i(w), e'_i(w)) \le c\Big((-\log|w_1|) \cdot (-\log|w_2|) \cdot \ldots \cdot (-\log|w_k|)\Big)^{\beta'}$$

for all $w \in U' \setminus T'$. By the Cauchy-Schwarz inequality, we obtain

$$h'\big(e'_i(w), e'_j(w)\big) \le c\Big((-\log|w_1|) \cdot (-\log|w_2|) \cdot \ldots \cdot (-\log|w_k|)\Big)^{\beta'}$$

for all $w \in U' \setminus T'$. □

Y is compact; hence there is a finite covering $\{U\}$ of Y such that, for all U and each of the finitely many connected components U' of $\pi^{-1}(U)$, Claim 7.8 holds. We may even assume that Claim 7.8 remains true, for the same β', for all points in a small neighborhood of the closure $\bar{U}'$ not lying on T'.

We choose some $\alpha \gg 1$ such that for all the open sets U' and for the constant β' given by Claim 7.8, one has

$$\alpha \ge \beta' + 2m.$$

In order to prove (7.7.1), it is sufficient to show that on each U' there is some $\epsilon > 0$ with

$$\Big|\Big|\pi'^{*} d\gamma(\partial_t)|_{\gamma'^{-1}(U')}\Big|\Big|^2_{\pi'^{*}\gamma^{*}\omega_\alpha} \ge \epsilon\pi'^{*}\gamma^{*}(r^{-\alpha/m+2})\Big|\Big|\tilde{\tau}((\partial_t)^m)|_{\gamma'^{-1}(U')}\Big|\Big|^{2/m}_{\pi'^{*}\gamma^{*}(h\otimes g^{-1})}. \tag{7.7.2}$$

Let us return to diagram (7.3.1). As in the beginning of this section, for each component T_i of T we consider $\mathscr{L}_i = \mathscr{O}_Y(T_i)$ with the Hermitian metric g_i, and $\pi^* L_i$ with the pullback metric $\pi^* g_i$. Let s_i be a section of $\mathscr{L}_i$ with zero locus T_i, where we assume that s_i has been rescaled as needed in Corollary 7.3 for the constant α, chosen above.

For the section $s_i' = \pi^* s_i$, define

$$r_i' = -\log ||s_i'||_{\pi * g_i}$$

and $r' = r_1' \cdot \ldots \cdot r_\ell'$. Obviously, one has $r_i' = \pi^* r_i$ and $r' = \pi^* r$.

Proof of inequality (7.7.2)

Let $\{\phi_1, \phi_2, \ldots\}$ be an orthonormal basis for $T_Y(-\log T)|_U$ with respect to ω_α. Then

$$\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_m};\ i_1 \leq \cdots \leq i_m\}$$

is an orthonormal basis for $S^m T_Y(-\log T)|_U$ with respect to ω_α and

$$\{\gamma^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m});\ i_1 \leq \cdots \leq i_m\}$$

is an orthonormal basis for $\gamma^* S^m T_Y(-\log T)|_{\gamma^{-1}U}$ with respect to $\gamma^*\omega_\alpha$. Then, using the morphisms in (7.3.1),

$$\{\gamma'^* \pi^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m});\ i_1 \leq \cdots \leq i_m\}$$

is an orthonormal basis for $\gamma'^* \pi^* S^m T_Y(-\log T)|_{\gamma'^{-1}U'}$ with respect to $\gamma'^* \pi^* \omega_\alpha$.

For the map

$$d\gamma^m : T_{\mathbb{C}}^m|_{\gamma^{-1}(U)} \to \gamma^*\big(S^m T_Y(-\log T)|_U\big), \tag{7.7.3}$$

write

$$d\gamma^m\big((\partial_t)^m|_{\gamma^{-1}(U)}\big) = \sum c_{i_1,\ldots,i_m} \gamma^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m}).$$

Then

$$\big|\big|d\gamma(\partial_t)|_{\gamma^{-1}(U)}\big|\big|^2_{\gamma^*\omega_\alpha} = \Big(\sum |c_{i_1,\ldots,i_m}|^2\Big)^{1/m}.$$

Let

$$\pi'^* d\gamma^m : \pi'^* T_{\mathbb{C}}^m|_{\gamma^{-1}(U)} \to \pi'^* \gamma^*\big(S^m T_Y(-\log T)|_U\big)$$

be the pullback of the morphism (7.7.3). By the commutativity of (7.3.1), one obtains

$$\pi'^* d\gamma^m\big((\partial_t)^m\big)\big|_{\gamma'^{-1}\pi^{-1}(U)} = \sum \pi'^*(c_{i_1,\ldots,i_m}) \gamma'^* \pi^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m})$$

and

$$\big|\big|\pi'^* d\gamma(\partial_t)|_{\gamma'^{-1}\pi^{-1}(U)}\big|\big|^2_{\pi'^*\gamma^*\omega_\alpha} = \Big(\sum \pi'^* |c_{i_1,\ldots,i_m}|^2\Big)^{1/m}.$$

Next, we consider the second map

$$\gamma^*\tau : \gamma^*\big(S^m T_Y(-\log T)|_U\big) \to \gamma^*(E^{n-m,m} \otimes \mathscr{A}^{-1}|_U)$$

and its pullback

$$\pi'^*\gamma^*\tau : \gamma'^*\pi^*\big(S^m T_Y(-\log T)|_U\big) \to \gamma'^*\pi^*(E^{n-m,m} \otimes \mathscr{A}^{-1}|_U)$$
$$\hookrightarrow \gamma'^*(E'^{n-m,m} \otimes \pi^*\mathscr{A}^{-1}|_{U'}).$$

For the connected component U' of $\pi^{-1}(U)$, let a'^{-1} be a local generator of $\pi^*\mathscr{A}^{-1}|_{U'}$. Then $\{e'_1 \otimes a'^{-1}, e'_2 \otimes a'^{-1}, \ldots\}$ is a basis of $E'^{n-m,m} \otimes \pi^*\mathscr{A}^{-1}|_{\gamma'^{-1}(U)}$ and the morphism

$$\pi^*\tau : \pi^* S^m T_Y(-\log T)|_{U'} \to E'^{n-m,m} \otimes \pi^*\mathscr{A}^{-1}|_{U'}$$

is given by

$$\pi^*\tau\big(\pi^*(\phi_{i_1} \otimes \cdots \otimes \phi_{i_m})\big) = \sum b^j_{i_1,\ldots,i_m} e'_j \otimes a'^{-1}$$

and one finds

$$\pi'^*\gamma^*\tau d\gamma^m\big((\partial_t)^m|_{\gamma'^{-1}(U')}\big) = \sum \pi'^*(c_{i_1,\ldots,i_m})\gamma'^*(b^j_{i_1,\ldots,i_m})\gamma'^*(e'_j \otimes a'^{-1}).$$

Since the metric π^*g^{-1} on $\pi^*\mathscr{A}^{-1}$ is regular on U', Claim 7.8 implies

$$\big|\gamma'^*(h' \otimes \pi^*g^{-1})\big(\gamma'^*(e'_i \otimes a'^{-1}), \gamma'^*(e'_j \otimes a'^{-1})\big)\big|$$
$$\leq c\gamma'^*\big((-\log|w_1|)\cdot(-\log|w_2|)\cdot\ldots\cdot(-\log|w_k|)\big)^{\beta'}.$$

Here and later we allow ourselves to replace the constant c by some larger constant whenever necessary.

For the ramification order μ_i of π over T_i, and for some positive function d_i on U', one has

$$|w_i| = d_i \big|\big|s'^{1/\mu_i}_i|_{U'}\big|\big|_{\pi^* g_i}.$$

This description extends to the compactification $\bar{U}'$ of U'. Since $\bar{U}'$ is compact, d_i is bounded away from zero, and one finds

$$\big|\gamma'^*(h' \otimes \pi^*g^{-1})\big(\gamma'^*(e'_i \otimes a'^{-1}), \gamma'^*(e'_j \otimes a'^{-1})\big)\big| \leq c\gamma'^* r'^{\beta'} = c\pi'^*\gamma^* r^{\beta'}.$$

On the compact set $\bar{U}'$, all $b^j_{i_1,\ldots,i_m}$ are bounded above. Hence all $\gamma'^*(b^j_{i_1,\ldots,i_m})$ are also bounded above, and the Cauchy-Schwarz inequality implies

$$\big|\big|\pi'^*\tilde{\tau}((\partial_t)^m)|_{\gamma'^{-1}(U')}\big|\big|^2_{\pi'^*\gamma^*(h\otimes g^{-1})} \leq c\pi'^*\gamma^* r^{\beta'} \sum \pi'^*|c_{i_1,\ldots,i_m}|^2$$
$$= c\pi'^*\gamma^* r^{\beta'} \big|\big|\pi'^* d\gamma(\partial_t)|_{\gamma'^{-1}(U')}\big|\big|^{2m}_{\pi'^*\gamma^*\omega_\alpha}. \qquad (7.7.4)$$

Since we assumed $r \geq 1$ and $\alpha - 2m \geq \beta'$, the right-hand side in (7.7.4) is smaller than

$$c\pi'^* \gamma^* r^{\alpha-2m} \big|\big|\pi'^* d\gamma(\partial_t)|_{\gamma'^{-1}(U')}\big|\big|^{2m}_{\pi'^*\gamma^*\omega_\alpha};$$

hence we obtain the inequality

$$\big|\big|\pi'^* d\gamma(\partial_t)|_{\gamma'^{-1}(U')}\big|\big|^{2}_{\pi'^*\gamma^*\omega_\alpha} \geq \frac{1}{c}\pi'^*\gamma^*(r^{-\alpha/m+2})\big|\big|\pi'^*\tilde{\tau}((\partial_t)^m)|_{\gamma'^{-1}(U')}\big|\big|^{2/m}_{\pi'^*\gamma^*(h\otimes g^{-1})},$$

as stated in (7.7.2). □

Therefore Lemma 7.7 is proved. □

Proof of Proposition 4.4

It remains to contradict the existence of the ample sheaf $\mathscr{A}$ and of the Higgs bundles having the properties stated in Lemma 6.5. Those led to the estimates in this section.

Recall the Ahlfors-Schwarz lemma, as stated in in [27, Lem. 1.1.1] (see also [7, Lem. 3.2]).

LEMMA 7.9

Let c be a real-valued nonnegative function on $\mathbb{C}$ which locally is of the form $\varphi|f|^2$, where φ is a local smooth positive function and f is a local holomorphic function. Then there cannot exist any positive number ρ such that

$$\partial_t \partial_{\bar{t}} \log c(t) \geq \rho \cdot c(t)$$

on $\mathbb{C}$ in the sense of currents.

Using the inequalities obtained in Lemmas 7.6 and 7.7, one has for suitable constants ϵ and ϵ',

$$\frac{\sqrt{-1}}{2\pi}\partial_t\partial_{\bar{t}} \log c(t)\, dt \wedge d\bar{t} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log c(t) = -\Theta(T_{\mathbb{C}}, F) \geq \epsilon\gamma^*\Theta(A_{Y\setminus T}, g_\alpha) \geq \epsilon\cdot\epsilon'\sqrt{-1}c(t)\, dt\wedge d\bar{t}$$

in the sense of currents. Hence

$$\partial_t\partial_{\bar{t}}\log c(t) \geq 2\pi\cdot\epsilon\cdot\epsilon'\cdot c(t),$$

contradicting Lemma 7.9. □

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