Programm zum Forschungsseminar WS 2005/2006

Hitchin discovered in [3] that the cotangent bundle $T^* \operatorname{Bun}_G$ to the moduli space of principal $G-$bundles is a completely integrable Hamiltonian system. In algebraic terms, there is a map $T^* \operatorname{Bun}_G \to \mathbb{A}^{\dim \operatorname{Bun}_G}$ whose generic fibre is an open subset of an abelian variety (actually there is an open subset of the affine space, over which the fibres can be describes in terms of Jacobian varieties).

This map is called the Hitchin fibration. The study of this map has found many applications, most recently in the proof of the so called fundamental lemma for unitary groups by Laumon and Ngo. The goal of the seminar will be to try to understand this proof but, since this looks quite technical at first, we will begin with some older applications to get some idea, what the fibration might look like and then approach the fundamental lemma.

On the way we will hopefully learn a bit about moduli spaces of principal bundles, some general things about algebraic groups, one of the questions coming from the theory of automorphic forms and how geometry helps to approach this question.

I hope that most of the talks will be shorter than 90min. If one of the speakers manages to finish much earlier, it would be nice if the next speaker could already start his talk.

1. Spectral curves for GL_n

To get a first idea, how the Hitchin fibration is defined and what it looks like, I would suggest to start with the very nice and accessible article "Spectral curves and the generalized theta divisor" [1]. The references given below refer to this article.

I would suggest, that we simply assume the existence of nice moduli spaces. Actually the formulations get cleaner if one thinks of moduli stacks, which will become more important in the work of Ngo and Laumon. However the first ideas are quite elementary, therefore we should not use to much machinery here.

25.10. Talk 1: Spectral curves. (Franziska Heinloth)

Basic idea: Take an affine curve C and an endomorphism $\phi : \mathcal{O}^n \to \mathcal{O}^n$, then the eigenvalues of ϕ define a (ramified) covering of C, the total space is called spectral curve.

If C is projective, and $\phi : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}$ a morphism, the same applies and therefore ϕ defines a spectral curve $C_\phi \to C$. Give an explicit description of this as a finite covering.

3.2 genus of C_ϕ, Example 3.4, 3.5: the smooth locus of the spectral curve can also be described explicitly.

The main Propsition is 3.6: If C_ϕ is integral, then (rank 1 sheaves on C_ϕ) $\Leftrightarrow$ $((\mathcal{E}', \phi')$ on C such that the characteristic polynomial of ϕ' is the same as the one of $\phi)$.

(3.7 alternative description. 3.10 induced map on Jacobian)

4.1 Moduli-Interpretation: Hitchin space and maps. Give a reformulation of 3.6 as in [6] p.19 (there is nothing to be proved, but the two equivalent statements look different at first sight).

Proof of Theorem 1, which says, that most vector bundles on a curve can be obtained from line bundles on a suitably chosen ramified covering of C. (This uses a theorem of Laumon and Drinfeld, one should just assume this here, we will come back to this.)

References: [1] section 2.1,3,4.

8.11. Talk 2: Applications (Andre Chatzistamatiou)

As an application of the result of the first talk one can get some results on the dimensions of global sections of a natural line bundle on the moduli space of vectorbundles ([1] Theorem 2 and 3).

For this one needs to recall some facts on jacobians (section 2), to compare the universal vector bundle with the Poincaré bundle of the jacobian of a spectral curve.

References: [1] section 2,3.10,5.

2. More general groups: Hitchins article

After having seen the theory in a very explicit form in the case of vector bundles, we should be ready to see the more general formalism for principal bundles.

15.11. Talk 3: The Hitchin fibration for Bun_G. (Christian Liedtke)

Principal G-bundles, Invariant polynomials - Chevalley's theorem, connection with $H^*(G)$. General definition of the Hitchin map - Here one should recall the description of the tangent space $T\,\mathrm{Bun}_G$ by cohomology.

Example: Sp, SO.

References: [3] section 4, from 5.10 to the end of section 5 is a collection of examples, at least some of them should be explained in detail. (The Remark on page 109 has been used...)

22.11. Talk 4: Properties of the Hitchin map. (Xiao Tao Sun)

We still have not shown the theorem used in the first talk. (Proposition 3.6 or really Theorem 3.1 from [5]).

It would be nice, if some general results on the Hitchin map (it is flat and the fibres are Lagrangian in characteristic 0) could be explained. (This implies Laumon's Proposition 3.6.)

The Talk should start with a brief introductions of some notions of symplectic geometry. ([3] 3.1, [5] A.1-A.3) The cotangent bundle T^*M has a canonical symplectic form. Describe this explicitely for T^*Bun_G in terms of the Serre-pairing. (This really should be in one of the sources, somewhat strangely I couldn't find the explicit statement.)

Give the trivial example for $G = \mathbb{G}_m$ ([3] p.96).

Now there are at least two possibilities to proceed:

I would suggest to give the proof of [2] Theorem II.5 (one doesn't have to read more of this article). (I have to apologize, this proof is very short (one page) but not very easy to read. However the argument is a reduction to the case $G = \mathbb{G}_m$ - it is a bit surprising, that this is possible.)

Then explain that the analog [5] Proposition 3.1/3.5, implies the result we used in Talk 1.

If one follows Faltings' article the language of stacks is not used, this is different in Laumon's article, which is an alternative. Actually one should not be afraid of the language, because the objects considered are not so complicated and one could avoid the use of stacks (by ignoring them). So alternatively one can directly try to explain the proof of [5] Theorem 3.1 - according to the taste of the speaker that might be easier to read (you don't need algebraic groups) or more difficult (because one shouldn't be afraid of stacks).

29.11. Talk 5: This is an optional digression (Martin Möller)

Surprisingly there are other natural complex structures on Hitchin's space. These and a connection between the Hitchin space and Teichmüller space, were discovered in [4].

Later these were explained by Simpson, interpreting this space as representations of the fundamental group of the surface.

To understand this would fill most of the seminar, so this talk should only give an idea and focus on [4].

3. Unitary Groups and the fundamental lemma: The work of Laumon-Ngo

The aim of the article [6] is to prove a conjecture which comes from the theory of automorphic forms, "the fundamental lemma").

To understand, where this comes from, we would have to learn a little bit about the trace formula, which will then be sketchy and in any case come from a different part of the world. However, the statement of this conjecture is given in terms of counting lattices in the article [6]. This and the relation to the global theory should be explained.

The rough idea of the proof of the lemma hopefully is the following (Part of this comes from an idea of Goresky-Kottwitz-MacPherson): The automorphic question is concerned with the calculation of some local summation (i.e. we consider a local field $F = \mathbb{F}_q((t))$ and integrate over a conjugacy class in $G(F)$) and the fundamental Lemma claims that this should be closely related to a similar calculation for a subgroup $H \subset G$.

Now the number of points of the fibres of the Hitchin fibration gives a similar formula, however there the global field $k(C)$ appears. However, we know that the fibres are given by some compactified jacobians. On those the Jacobian of the spectral curve acts and if we quotient by this action, the result only depends on the singularities of the spectral curve, i.e. this is a product of varieties depending only on the singular points of the curve and the number of points of these varieties is what we are looking for, unfortunately we might be interested in a very singular point.

Laumon and Ngo first prove a global theorem, relating the Hitchin fibres of G to the Hitchin fibres of a subgroup H. By a deformation argument they reduce this to the case of curves which are not to singular. And finally they find that the local question appears as a contribution of a curve with a single badly singular point and some ordinary double points. They can calculate the contributions from the double points, the global theorem controls the whole object, so they can prove the result at the bad singularity.

I would like to postpone or ignore the local problems and postulate that we are interested in the Hitchin fibration and its geometry, since the main new ingredient from [6] is the analysis of this geometry. In the end we should be able to understand some words from the automorphic world (like "endoscopy") and the local question..

(There are many lecture notes of the authors available at `http://www.math.uchicago.edu/ arinkin/langlands/`. There is also a recent Bourbaki talk of J.F. Dat on the subject.) The theory concerning Hitchin's fibration has been extended to a very general setting by Ngo [7], but we will concentrate on the particular case of unitary groups. The references given below refer to [6].

6.12. Talk 6: Hitchin's fibration for unitary groups (Jochen Heinloth) Define the unitary groups over a curve, as in 2.1. Hitchin fibration in this case, 2.2 and 2.4. The Hitchin fibration has a section 2.3.

The description via compactified jacobians over "$\mathbb{A}^{red}$". 2.5-2.6

References: [6] *2.1 -2.6., for 2.6 compare* [1]*3.6, to get an elementary formulation.*, 2.7.

13.12. Talk 7: Hitchin's fibration is proper over the elliptic locus (Manuel Blickle)

This is section 2.8. Since the result is very important one should take some time to explain the proof.

20.12. Talk 8: The subgroups - "Endoscopic groups" (Hô Hai Phung Endoscopic groups. This is nothing but the study of the subgroups $H = U(n_1) \times$

$U(n_2) \subset U(n_1+n_2)$. Explain 2.7 and 2.9, the latter section is the beginning of the deformation argument.

(This should be a shorter talk).

10.1. Talk 9: Application to the main geometric Theorem (Irene Bouw)

We want to apply the geometric results of the previous talks to study the cohomology of Hitchin fibres. Given, what we have seen so far, this should be a rather formal consequence of the general theorems on purity. One more geometric ingredient, the description of the image of the Hitchin fibration for the endoscopic group as a fixed point set.

Explain Corollaire 3.2.2. Tell us why the geometric results we have already seen allow to apply the general theorems on perverse cohomology. Then explain the endoscopic characters and the statement on the support of the corresponding sheaves.

Explain the deformation argument (3.9): This uses Talk 7 in order to reduce to a problem over $S_{\natural}$, the locus where the spectral curves intersect transversely.

Give the description of the image of $\mathcal{N} \to \mathcal{M}$ as a fixedpoint set of a torus action. (3.3)

Be sketchy about the local system L, this will hopefully get more natural in the next talk. *Reference: 3.1-3.6,3.9*

17.1. Talk 10: Proof of the main geometric theorem (Gebhard Böckle)

The proof of the Theorem is given in 3.7 and 3.8.

24.1. Talk 11: Application: Counting points - localization of the global theorem. (Jochen Heinloth)

Using the Lefschetz trace formula one can interpret the result in terms of numbers of points. (3.10.2 - be brief with Lemma 3.10.1, this is easy.)

More interestingly one can localize this formula, using the action of P on $\mathcal{M}$. (3.10.4) And the local terms can be described in an elementary way. Lemma 3.10.5 - this seems to be essential. Maybe a good point to start is to try to prove this lemma.

In order not to be to abstract it might be a good idea to suppress the categories where possible and talk only about the objects - that the constructions are functorial is fairly easy in most places.

Explain Theorem 3.10.6 - a first indication that one can deduce results on parts of Z.

If time permits, it would be nice to indicate that in special situations (4.6.1) one can even get to special points (see the remark following 4.6.1).

4. From local to global - the fundamental lemma

Finally we should try to understand what the local question is and how to deuce it from the global results.

31.1. Talk 12: The statement of the fundamental lemma (Alexander Schmitt)

By now it should be reasonable to explain section 1 of the article. Maybe we should start from the global situation and see what we get if we localize, in order to see where all the fields come from. We have already seen part of this in the previous talk.

We will run out of time here, so I suggest that we will just believe that one can calculate the missing terms (as indicated after 4.6.1). However the last page of the article - the identification of the fundamental lemma with a factor of theorem 3.10.6 should be noted.

References

[1] A. Beauville, M. S. Narasimhan and S. Ramanan. Spectral curves and the generalized theta divisor. *J. reine angew. Math*, 398, 1989, p. 169–179.

[2] G. Faltings. Stable $G-$bundles and projective connections. *J. Alg. Geom.*, 2, 1993, 507–568.

[3] N. Hitchin. Stable bundles and integrable systems. *Duke Math. J.*, 54 no 1, 1987, p. 91–114.

[4] N. Hitchin. The self-duality equations on a Riemann surface. *Proc. Lond Math. Soc.(3)*, 55, 1987, p. 59–126.

[5] G. Laumon. Un analogue global du cône nilpotent. *Duke Math. J.*, 57, 1988, p. 647–671.

[6] G. Laumon and B.C. Ngô. Le lemme fondamentale pour les groupes unitaires. `arXiv:math.AG/0404454v2`

[7] B.C. Ngo. Fibration de Htichin et endoscopie. `arXive:math.AG/0406599`