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Relation between the Hodge conjecture and singularities of normal functions

The Hodge conjecture predicts that of X is a smooth projective variety defined over the field $\mathbb{C}$ of complex numbers, then sub-Tate classes $\mathbb{Q}(0)\cdot\alpha$ of the pure Hodge structure $H^{2n}(X,\mathbb{Q}(n))$ are classes of cycles of codimension n. Upto now, it is known to be true for $n=1$, and aside of this, basically nothing very concrete is known. One direction of study, initiated by Deligne, is to say that there is a variety X_0 defined over a field of finite type $K\subset\mathbb{C}$ over $\mathbb{Q}$, such that $X=X_0\otimes_K\mathbb{C}$, that algebraic cycles are also definable over K, thus the notion of sub-Tate classes $\mathbb{Q}(0)$ should not depend on the complex emebdding $K\subset\mathbb{C}$. This yields the concept of absolute Hodge cycles. Deligne showed that of X is an abelian variety, then Hodge cycles are absolute Hodge. We will not study this in the seminar.

The goal of the seminar is to study an idea initiated by Green-Griffiths. They take a viewpoint somehow dual to the classical formulation: if the Hodge conjecture was true, then, given α, there would be a hypersurface $Y\subset X$ with $0\neq\alpha|_Y\in H^{2n}(Y,\mathbb{Q}(n))$ where $n=\dim X$. They study what singularities this hypersurface Y would have. It could be taken in an ample linear system $|\mathcal{L}|$ and would have ordinary double points. Associated to $|\mathcal{L}|$, one has a family of hypersurfaces in $\mathcal{X}\subset X\times_{\mathbb{C}}|\mathcal{L}|$, and the class α yields a normal function in $H^1(|\mathcal{L}|_{\rm sm},\mathcal{H})$ where $|\mathcal{L}|_{\rm sm}$ is the open set of smooth hypersurfaces, $\mathcal{H}$ is the admissible variation of Hodge stuctures on $|\mathcal{L}|$, defined as $R^{2n-1}f_*\mathbb{Q}(n)=\mathcal{H}, f:\mathcal{X}\to|\mathcal{L}|$. Those Y are in the boundary of f. To find such hypersurfaces is of course difficult, as this is pure geometry. The theorem we will study says that in fact, this purely geometric property has an exact cohomological translation in terms of normal functions ([2, Theorem 1.3]). Normal functions are the cohomology classes $H^1(|\mathcal{L}|_{\rm sm},\mathcal{H})$. And the statement says that α is algebraic if and only if the induced normal function is singular.

Since geometry is difficult to understand, we will rely in the seminar on [2]. It has the advantage that the language is better suited for dreaming of an ℓ-adic variant for the Tate conjecture. It has the disavantage that we will have to listen to formal lectures on perverse sheaves (which, as Deligne wrote, are neither perverse nor sheaves..), and mixed Hodge modules.

At the end of the seminar, we will report on notes sent to us by Alexander Beilinson, in which he uses work of Brylinski to give his own variant of the main theorem [2, Theorem 1.3]

The first 3 lectures are devoted to the proof of [2, Theorem 6.5] and [6, Theorem 1.1].

15.04. *Annabelle Hartmann.* **Hodge structures and geometry.** In this talk the basic definitions and properties of Hodge structures will be given. The main example is the cohomology of projective complex manifolds.

Explain [5, §1.1.3, p. 17–19] in detail. State the Hard Lefschetz Theorem [5, Theorem 1.30] and the Hodge Riemann bilinear relations [5, Theorem 1.33]. Define polarized Hodge structures [5, §2.1.2] and give [5, Example 2.10], prove semi-simplicity [5, Corollary 2.12]. Explain that the Hodge filtration can be computed via the holomorphic de Rham complex [5, Proposition 2.22] (Degeneration of the Hodge to de Rham spectral sequence). Give the definition of mixed Hodge structures [5, Definition 3.1] and discuss extensions [5, §3.5.1].

22.04. *Christian Kappen.* **Introduction to the Hodge conjecture.** In this talk we define Hodge classes and introduce the Hodge conjecture.

State the Hodge conjecture following [3, p. 1]. To this aim, use [7, §11.1.2, p.269–275] to construct the cycle class; give the definition of Hodge classes [7, §11.3] (but maybe use rational coefficients as in [3]). Prove Lefschetz' theorem on $(1,1)$ classes [7, Theorem 11.31] or [3, §2.(iii)] by using the exponential sequence and Hodge theory. Also explain the generalised Hodge conjecture [7, Conjecture 11.37]. Follow [7, §11.3.3] to construct Hodge classes in the self-product of a variety corresponding to the inverse Lefschetz operator or the Künneth projectors (these are interesting examples of Hodge classes and the Hodge conjecture); see also [3, §6] for nice corollaries of the Hodge conjecture.

29.04. *Nguyen Le Dang Thi.* **Hodge conjecture and hypersurfaces.** In this talk we "reduce" the Hodge conjecture to the case of codim $= n$ cycles in a $2n$ dimensional variety. Moreover, the Hodge conjecture is equivalent to non-vanishing of the restriction of a Hodge class to a suitable hypersurface.

Prove [6, Proposition 2.1]. State [6, Theorem 4.1] and sketch the proof, then prove [6, Theorem 4.2] in detail. Explain [2, §6.4] and show [2, Theorem 6.5] (Remark: in the proof of $(i) \Rightarrow (ii)$ choose a W such that [6, Theorem 4.1] applies and then use [6, Theorem 4.2]; for $(iii) \Rightarrow (i)$ use [4, Proposition 8.2.7]).

The next 5 lectures are devoted to perverse sheaves and mixed Hodge modules in order to understand [2, Theorem 1.3].

06.05 *Moritz Kerz*. **Applications to Abelian schemes.** This is Moritz' talk from the last semester. It is the conclusion of the previous seminar on p-divisible groups.

20.05 *Dima Doryn.* **Perverse sheaves:** [5]: Defn $j_{!*}E$ (notation [2]) $=^{\pi} \mathbb{V}_S$ (notation [5, p.309]);

Triangulated categories and t-structures p.385 bottom and Appendix A.21, Defn of ${}^p\varphi, \varphi : D^b_{cs}(X) \to D^b_{cs}(X)$; [2]: Lemma 2.3.

27.05 *Marcello Bernardara.* **Fakhruddin's proof of the improvement of the cohomological restrictions:** [2] Appendix B.

10.06 *Doan Trung Cuong.* **Variation of Hodge structures:** Chapter 10.2, thm 10.3, Chapter 10.4; Hodge modules: aim Defn 14.30, Thm 14.32.

17.06 *Abolfazl Mohajer.* **Admissible variations of Hodge structures, Mixed Hodge Modules:** Chapter 14.4, Chapter 14.4.2, Thm 14.57.

24.06 *Andre Chatzistamatiou.(I)* Proof of [2, Theorem 1.3], which is the core of the seminar.

01.07 *Andre Chatzistamatiou.(II)* Proof of [2, Theorem 1.3], which is the core of the seminar.

The next 2 lectures are devoted to Beilinson's letter.
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References

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