Forschungsseminarprogramm Wintersemester 2007/08

The general aim of this program is to become familiar with the birational classification of complex projective varieties, or, as it is often called, the minimal model program. The more concrete goal we set is to understand the relatively recent breakthrough paper 'On the existence of flips' by Christopher Hacon and James McKernan. If time permits, we will also have a quick look at the more general result 'Existence of minimal models for varieties of log general type by Birkar, Cascini, Hacon, and McKernan.

The need to classify projective varieties with respect to birational equivalence (that is, trying to classify function fields) arose from the unsurmountable difficulties one faced when trying to come up with a feasible classification of smooth varieties up to isomorphism.

Let us have a quick loot at what happens in low dimensions. For curves, birational equivalence implies isomorphism, so we obtain no simplification this way. The classification of curves is a rich and difficult subject (including various versions of moduli spaces), but one has a fairly general understanding.

On surfaces the landscape starts changing drastically. First of all, there exist nontrivial birational modifications among smooth varieties, most notably the blow-up of a point. Naturally enough, this means that notions of birational equivalence and isomorphism no longer coincide. With some effort one can describe birational maps: they are known to be compositions of blow-ups of points and their inverses. One of the most important achievements of the Italian school of geometers around the turn of the $20^{\text{th}}$ was the birational classification of smooth projective surfaces. Such results must of course rely on various geometric invariants of varieties that do not change under birational maps. Important examples of birational invariants are plurigenera (the dimensions of the spaces of global sections of $\omega_X^{\otimes m}$) or the irregularity of a variety.

Once the question of classifications has been dealt with, we face a new challenge: it would be very useful to be able to pick a 'simple' representative from a given birational equivalence class. Again, starting from a simple situation helps in the orientation: consider any smooth projective surface (the projective plane for example), and let us blow it up at a point. Then we have introduced a new rational curve — the exceptional divisor — which has self-interesction negative one; moreover we have increased the rank of the Picard group by one. In some sense the surface upstairs seems more difficult to handle. It seems like a good idea to try to reverse the process if possible. Luckily, by an elegant theorem of Castelnuovo a rational curve with self-intersection $-1$ can always be contracted to a point on a smooth surface; hence the plan works.

Following this train of thought, we pick a variety, and blow down any $(-1)$ curve we find, one after the other. At each step the Picard number will decrease by one, therefore after finitely many such operations the process will stop: we have found a smooth projective surface birational to the original one which has no $(-1)$-curves on it. Such a variety is called a minimal model of the original surface. In dimension two, minimal models exist and are unique. For future use, we will use a slightly stronger version of the concept of minimality: we will require that the canonical bundle $\omega_X$ be nef (meaning that it has nonnegative degree restricted to every curve on the surface).

This is the philosophy one tries to follow in higher dimensions as well. Until the late 1970's people were trying to extend the above-mentioned Kodaira–Enriques classification, but then it became obvious from the work of Mori and Reid that the plan has several major shortcomings, and it will definitely not work in its original form.

In particular, two major blows we suffer are that

(a) there exist smooth projective threefolds that are *not* birationally equivalent to any smooth threefold with nef canonical bundle;
(b) even if a minimal model exist, it might not be unique (more precisely, there are examples when it is *not* unique).

The morale is that if we want to have any kind of hope to succeed, one needs to work with certain singular varieties (what kinds of singularities we allow is a central question of this industry).

Another major issue we haven't touched upon yet is how to obtain a minimal model via a skillfully chosen sequence of birational maps. Not having a suitable generalization of Castelnuovo's theorem in sight, it is a priori not obvious how to construct birational maps that will take us 'closer' to a minimal model whatever that should mean. One of Mori's brilliant idea was to look at curves on which $\omega_X$ is negative, and try to contract these to a point. Of course, the actual situation is very complicated and it has not been completely settled up until today.

Mori considered the closure of the cone of effective curves, and its extremal rays. The crucial result in this respect was, that if an extremal ray was negative with respect to the canonical divisor, then one can produce a birational morphism contracting all curves in that ray. This came as the culmination of difficult results of Kawamata, Kollár, Mori, Reid, and Shokurov.

However, the battle was not yet won. If the contraction morphism obtained this way is 'small' (that is, it does not contract any divisor), then the singularities on the target space get out of control. Here the solution that Mori proposed was to remedy this situation via an operation analogous to topological surgery, which was called a flip. The two main problems with flips are that

(a) we don't know if they exist;
(b) we don't know if a sequence of flips will eventually stop.

Of these two, Mori proved that existence and termination of flips in dimension three (he was awarded a Fields Medal for this achievement in 1990). Since then termination of flips has been shown to hold in arbitrary dimension, but the existence of flips has remained an important and difficult open question.

Assuming that the canonical bundle has asymptotically enough sections, this is what Christopher Hacon and James McKernan proved first in dimension four, then in arbitrary dimension together with Bircar and Cascini. The recent work of Corti–Kawamata–Lazarsfeld as presented in [CKL] brought significant simplification as far as the proofs of the classical theorems (Non-vanishing, Base point freeness, etc.) are concerned; the same article also provides a neat streamlined proof of the work of Hacon–McKernan based on b-divisors.

A few words about the techniques one needs to use. On a very simplistic level, one can say that apart from manipulatiing divisors in a very resourceful way, the main idea is a systematic exploitation of vanishing theorems á la Kawamata–Viehweg. In practice, this will require a thorough knowledge of various classes of mild singularities, and the frequent use of multiplier ideal techniques.

As far as the literature goes, the suggested main sources are as follows:

Main Goal:

- Birkar, Cascini, Hacon, McKernan: Existence of minimal models for varieties of log general type. Preprint arXiv:math/0610203 [math.AG] (BCHM)
- Hacon, McKernan: On the existence of flips. Preprint arXiv:math/0507597 [math.AG] (HM)
- Corti, Hacking, Kollár, Lazarsfeld, Mustaţă: Lectures on Flips and Minimal Models. Preprint arXiv:math/0706.0494 [math.AG] (CHKLM)
- Corti, Kaloghiros, Lazić: Introduction to the minimal model program and the existence of flips. Preprint

Minimal Model Program:

- Corti et al.: Flips for 3-folds and 4-folds. (C)
- Debarre: Introduction to Higher-Dimensional Geometry (D)
- Kollár, Mori: Birational Geometry of Algebraic Varieties (KM)
- Matsuki: Introduction to the Mori Program (M)
- Andreatta, Mella: Morphisms of Projective Varieties from the Viewpoint of Minimal Model Theory. Preprint arXiv:math/0205224 [math.AG] (AM)

Singularities

- Kollár: Singularities of Pairs (K)
- Reid: Canonical threefolds (R)
- Reid: Yound Person's Guide to Singularities (YPG)

Multiplier Ideals

- Lazarsfeld: Positivity in Algebraic Geometry (PAG)

Other Sources

- Esnault, Viehweg: Lectures on Vanishing Theorems (EV)

- Nakayama: Invariance of the plurigenera of algebraic varieties under minimal model conjectures. Topology **25** (1986), no. 2, 237–251. (N)
- Notes from the 2007 Grenoble Summer School (GR)

All required sources have been made available.

At this point, the proposed setup of the seminar goes as follows: the role of roughly the first two-thirds of the talks would be to explain the necessary prerequisites (notably singularities of pairs, multiplier ideals, the circle of ideas around the cone theorem, and finite generation of divisorial rings), while the last four to six talks will deal with the content of the Hacon–MacKernan paper. With this in mind, the material for the second part of the term is not cut into chunks, as it is not yet obvious how it is done best. The suggestion would be that the people signing up for these talks should work together to some degree and decide among themselves, how the material is to be divided.

Here are the talks about the preparatory material.

1$^{\text{ST}}$ Talk: Introduction to the Mori program (Alex Küronya)
Outline of existence and uniqueness of minimal models in dimensions one and two, the Kodaira–Enriques classification. The cone of curves and its relation to contractions of extremal rays. Classification of extremal rays on surfaces and threefolds. The need for singularities. Divisorial contractions, flips, and fibre-type contractions. Outline of the Mori program, main problems: existence of flips, termination of flips, abundance. Relation to finite generation of the canonical ring. Results of Hacon–McKernan.

2$^{\text{ND}}$ Talk: The canonical class and the singularities in the Mori program (Wioletta Syzdek)
The canonical divisor class on a normal variety (KM,R,YPG). $\mathbb{Q}$-factoriality and intersection theory (M,KM). Divisors as valuations of the function field of the variety (KM). Discrepancies, various important singularity classes (terminal, canonical, klt, dlt, log canonical), their behaviour under divisorial contractions and flips (KM). Illustration: the case of surfaces (M). The role of terminal and canonical singularities. The need for pairs (KM,M). Examples (KM,M,D). $\mathbb{R}$-divisors.

3$^{\text{RD}}$ Talk: Vanishing theorems and the Cone theorem (Kay Rülling)
The Hodge-to-de Rham spectral sequence as the mother of all vanishing theorems (EV,PAG,KM). Cyclic covers and the Kawamata–Viehweg vanishing theorem (KM,PAG). The rough sketch of the cohomological proof of the cone theorem: non-vanishing, basepoint-free theorem, rationality theorem with bounded denominators (D,KM). The contraction theorem (KM,D,M). The proof of the basepoint-free theorem in a simple situation (AM). First applications: global generation, invariance

of plurigenera provided one has finite generation (N).

4TH AND 5TH TALK: MULTIPLIER IDEALS I.-II. (Franziska Heinloth, ?)
Log resolutions, restriction and rounding of $\mathbb{Q}$-divisors. Definition and basic properties of multiplier ideals. Independence of log resolution. Examples. Vanishing with multiplier ideals as original motivation. Connection with multiplicities and discrepancies. Multiplier ideals of singular pairs, log canonical thresholds and log canonical centers. The support of the multiplier ideal equals the non-klt locus. Restriction theorem of Esnault–Viehweg, behaviour in families, inversion of adjunction, subadditivity and summation theorems. Asymptotic multiplier ideals. (The suggested reading for all the material related to multiplier ideals is PAG, Chapters 9-11.)

6TH TALK: BASE POINT FREE THEOREM VIA KAWAMATA SUBADJUNCTION, INVARIANCE OF PLURIGENERA I. (Tomek Szemberg)
The first main point is the new proof of the base point free theorem via multiplier ideals and subadjunction due to Kawamata and Corti–Lazarsfeld [Section 2 of CKL]. Since the availability of the Grenoble lecture notes of Corti, time should permit the inclusion of the proofs of the non-vanishing and/or the rationality theorem (if there is no time for both, one should go with the rationality theorem). As technical preparation one needs log canonical centers, tie breaking, Kawamata subadjunction (without proof) for example as in [Section 1 of CKL].

Invariance of plurigenera on smooth varieties of general type: the result and the reduction to a statement about multiplier ideals (PAG).

7TH TALK: INVARIANCE OF PLURIGENERA II.
The proof of invariance of plurigenera on smooth varieties of general type [PAG, Chapter 11]. One should treat the case of singular pairs as in the works of Hacon–McKernan [HM, CHKLM], and the lifting lemma of Hacon–Mckernan [Lecture 1 in CHKLM, CKL] with proof if time permits (in not then the proof will come in the last third of the seminar).

8TH TALK: BASICS OF FINITE GENERATION OF DIVISORIAL RINGS
The language of b-divisors [Section 1 of CKL, C], divisorial rings, restriction of divisorial rings, restricted algebras, adjoint algebras, pl flips [C,HM,Section 4 of CKL].

9TH-12TH TALK: DISCUSSION OF THE ARTICLE HACON–MCKERNAN: EXISTENCE OF FLIPS [HM, CHKLM] (Georg Hein, Stefan Kukulies, Alex Küronya, Christian Liedtke)
We will go through the rest of the excellent paper [CKL]. If time permits, which

seems likely, we plan to cover some of the material in [BCHM], probably in a more expository fashion based on Kollár's talks in [CHKLM].