Problem sheet 1

Due date: Oct. 16, 2018.

Problem 1

A non-empty topological space $X$ is called irreducible, if it is not equal to the union of two proper closed subsets.

1. Determine all topological spaces which are Hausdorff and irreducible. (Recall that a topological space $X$ is called Hausdorff, if for any two points $u, v \in X$, $u \neq v$, there exist disjoint open subsets $U, V \subseteq X$ with $u \in U$, $v \in V$.

2. Let $X$ be a non-empty topological space. Prove that the following properties are equivalent:

   (i) The space $X$ is irreducible.
   
   (ii) Every non-empty open subset $U \subseteq X$ is dense in $X$ (i.e., the smallest closed subset of $X$ containing $U$ is $X$).
   
   (iii) Every open subset $U \subseteq X$ is connected. (A topological space is called connected, if it cannot be written as the union of two disjoint proper closed subsets.)
   
   (iv) Any two non-empty open subsets of $X$ have non-empty intersection.

Problem 2

Let $k$ be an algebraically closed field.

(a) Let $n \geq 1$, $f \in k[T_1, \ldots, T_n]$ a polynomial such that $f(t_1, \ldots, t_n) = 0$ for all $t_1, \ldots, t_n \in k$. Prove that $f = 0$. *Hint: You can use induction on $n$.*

(b) Prove that $k^n$ (with the Zariski topology) is irreducible.
Problem 3
Let $k$ be an algebraically closed field, and let $d \geq 1$. We identify the set of all monic polynomials $f(X) = X^d + t_{d-1}X^{d-1} + \cdots + t_0$ of degree $d$ with $k^d$ by mapping $f$ to $(t_0, \ldots, t_{d-1})$.

Let $d = 2$. Prove that the subset of $k^d$ corresponding to those polynomials which have a multiple zero is of the form $V(D)$ for a polynomial $D \in k[T_0, \ldots, T_{d-1}]$.

Remark. The same result holds for $d > 2$, but is more difficult to prove. One way to do it is roughly as follows: View $f$ as a polynomial with coefficients in the field $K = k(t_0, \ldots, t_{d-1})$ of rational functions in $d$ variables over $k$. Let $L$ be the splitting field of $f$, a Galois extension of $K$. Let $\alpha_i$ be the zeros of $f$ in $L$, and let $D = \prod_{i<j}(\alpha_i - \alpha_j)^2$. Then use the main theorem on elementary polynomials. Alternatively, use Galois theory to show that $D \in K$, and use that $k[t_0, \ldots, t_{d-1}]$ is integrally closed to conclude that $D \in k[t_0, \ldots, t_{d-1}]$.

Problem 4
Let $k$ be an algebraically closed field. Let $n \geq 1$. We identify the space $M := \text{Mat}_{n \times n}(k)$ of $(n \times n)$-matrices with entries in $k$ with $k^{n^2}$ and equip it with the Zariski topology. By Problem 2 (b), it is irreducible.

(a) Prove that the subset of $M$ consisting of matrices $A$ such that $\text{charpol}_A(A) = 0$ is closed in $M$ (without using the Theorem of Cayley-Hamilton).

(b) Use Problem 3 to prove that the subset of diagonalizable matrices with $n$ different eigenvalues in $k$ is open in $M$.

(c) Prove the Theorem of Cayley-Hamilton, i.e., prove that the subset in (a) equals all of $M$. 