Problem sheet 12

Due date: Jan. 15, 2018.

Problem 49

(1) Show that the properties “open immersion” and “closed immersion” are local on the target: Given a morphism \( f: X \to Y \) and a cover \( Y = \bigcup_i V_i \) by open subschemes, \( f \) is an open (closed) immersion if and only if for every \( i \), the induced morphism \( f^{-1}(V_i) \to V_i \) is an open (closed) immersion.

(2) Show that the properties “open immersion” and “closed immersion” are stable under composition of morphisms.

Problem 50

Show that open and closed immersions are monomorphisms in the category of schemes: If \( f: X \to Y \) is an open (closed) immersion, then for every scheme \( S \) the induced map \( \text{Hom}(S, X) \to \text{Hom}(S, Y) \) is injective.

Problem 51

(1) Let \( X \) be a scheme. Prove that there exists a unique reduced closed subscheme \( X_{\text{red}} \) of \( X \) which has the same underlying topological space as \( X \).

(2) Let \( f: X \to Y \) be a morphism of schemes. Show that \( f \) induces a unique morphism \( f_{\text{red}}: X_{\text{red}} \to Y_{\text{red}} \) such that the diagram

\[
\begin{array}{ccc}
X_{\text{red}} & \xrightarrow{f_{\text{red}}} & X \\
\downarrow f_{\text{red}} & & \downarrow f \\
Y_{\text{red}} & \xleftarrow{f_{\text{red}}} & Y
\end{array}
\]

Problem 52

Let \( k \) be a field, and let \( A = k[X, Y]/(XY, X^2) \). Define two morphisms \( f, g: \text{Spec } A \to \text{Spec } k[T]/(T^2) \) such that \( f \neq g \), but such that there exists a non-empty open subset \( U \subset \text{Spec } A \) such that \( f|_U = g|_U \).