Problem sheet 2

Due date: Oct. 23, 2018.

Problem 5
Let $\varphi : A \to B$ be a ring homomorphism, and let $f : \text{Spec } B \to \text{Spec } A$ denote the map attached to $\varphi$.

(a) Let $b \subseteq B$ be an ideal. Prove that

$$ f(V(b)) = V(\varphi^{-1}(b)) $$

(where $\bar{\cdot}$ denotes closure).

(b) Assume that $\varphi$ is surjective. Prove that $f$ induces a homeomorphism from Spec $B$ onto $V(\ker(\varphi))$.

(c) Prove that the image of $f$ is dense in Spec $A$ if and only if every element of $\ker(\varphi)$ is nilpotent.

Problem 6
Let $A$ be a ring, $f \in A$ and $X := D(f) \subseteq \text{Spec } A$ with the subspace topology. Prove that the topological space $X$ is quasi-compact (i.e., for every cover $X = \bigcup_{i \in I} U_i$ by open subsets $U_i$, there exists a finite subset $I' \subseteq I$ with $X = \bigcup_{i \in I'} U_i$).

Problem 7
Let $(I, \leq)$ be a partially ordered set. An inductive system of sets (with index set $I$) is a family $X_i, i \in I$, of sets together with maps $\varphi_{ji} : X_i \to X_j$ for all pairs $i, j \in I$ with $i \leq j$, such that $\varphi_{ii} = \text{id}_{X_i}$, $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ for all $i \leq j \leq k \in I$.

A set $C$ together with maps $\psi_i : X_i \to C$ such that $\psi_j \circ \varphi_{ji} = \psi_i$ for all $i \leq j$ is called a colimit (or direct limit or inductive limit) if it satisfies the following “universal property”: For every set $T$ together with maps $\xi_i : X_i \to T$ such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all $i \leq j$, there exists a unique map $\chi : C \to T$ such that $\chi \circ \psi_i = \xi_i$ for all $i$. The colimit is also denoted by $\text{colim}_{i \in I} X_i$ or by $\lim_{i \in I} X_i$. (The maps $\varphi_{ij}$ and $\psi_i$ are usually omitted from the notation.)
(a) Suppose that \( I \) is directed, i.e., for all \( i, j \in I \) there exists \( k \in I \) with \( i \leq k \) and \( j \leq k \). Let \((X_i, \varphi_{ji})\) be an inductive system of sets, let \( U \) be the disjoint union

\[
U = \bigsqcup_{i \in I} X_i,
\]

and consider the following relation on \( U \): for \( x, y \in U \), say \( x \in X_i \), \( y \in X_j \), we set \( x \sim y \) if and only if there exists \( k \geq i, j \) with \( \varphi_{ki}(x) = \varphi_{kj}(y) \). Prove that \( \sim \) is an equivalence relation and that the set \( U/ \sim \) of equivalence classes together with the natural maps \( X_i \to U/ \sim \) is a colimit of the system \((X_i, \varphi_{ji})\).

(b) Assume that \( I \) is directed, that all \( X_i \) are subsets of a set \( X \), that \( i \leq j \) if and only if \( X_i \subseteq X_j \), and that the maps \( \varphi_{ji} \) are the inclusion maps. Prove that the union \( C := \bigcup_i X_i \) with the inclusion maps \( X_i \to C \) is a colimit of the \( X_i \).

**Problem 8**

Let \( I \) be a partially ordered set.

(a) Define the notion of *colimit* (with index set \( I \)) in an arbitrary category.

(b) Suppose that \( I \) is directed. Prove that all colimits with index set \( I \) in the category of abelian groups exist.

(c) Suppose that \( I \) is directed. Prove that the functor \( \text{colim}_i \) on the category of abelian groups is exact, i.e.: Let \((A_i, \varphi_{ji}), (B_i, \psi_{ji}), (C_i, \xi_{ji})\) be inductive systems of abelian groups indexed by \( I \). Suppose that for each \( i \), there are short exact sequences

\[
0 \to A_i \to B_i \to C_i \to 0
\]

which are compatible with the maps \( \varphi_{ji}, \psi_{ji}, \xi_{ji} \) in the obvious sense. Prove that these sequences induce a sequence

\[
0 \to \text{colim}_i A_i \to \text{colim}_i B_i \to \text{colim}_i C_i \to 0
\]

which is again exact.