Problem 9

Let $A$ be a ring. We call an element $e \in A$ idempotent if $e^2 = e$. Show that the following conditions are equivalent:

(i) $\text{Spec } A$ is not connected.

(ii) There exists an idempotent element $e \in A$ different from 0 and 1.

(iii) There exists a ring isomorphism $A \cong A_1 \times A_2$ with non-zero rings $A_1, A_2$.

Solution. In the situation of (iii), let $e = (1, 0)$. Then $e$ is idempotent and $\neq 0, 1$, and (ii) follows. Under condition (ii), with $e$ a non-trivial idempotent element, the Chinese Remainder Theorem easily implies that $A \cong A/(e) \times A/(1 - e)$ (and hence (iii)). Furthermore, it is not hard to check that for a prime ideal $\mathfrak{p} \subset A$, we have $e \in \mathfrak{p}$ if and only if $1 - e \notin \mathfrak{p}$ which shows that $V(e) = D(1 - e)$ (and symmetrically, $V(1 - e) = D(e)$). We thus obtain a decomposition

$$\text{Spec } A = V(e) \sqcup V(1 - e)$$

as a disjoint union of closed (and open) subsets, so $\text{Spec } A$ is not connected.

Remark. In fact, it is easy to check that the natural map $A \to A_e$ to the localization factors through an isomorphism $A/(1 - e) \to A_e$. This again shows that $V(1 - e) = D(e)$, but it is a stronger statement. We also see that the map $A \to A_e$ is surjective (which is rarely true for maps to a localization). It may be useful to keep this remark in mind when approaching the proof of (i) $\Rightarrow$ (iii).

Example. Before we start the proof of (i) $\Rightarrow$ (iii), let us look at the example $A = k \times k[X]/(X^2)$, where $k$ is a field. Then (iii) is visibly satisfied, but since $A$ has nilpotent elements, where are several choices for ideals $\mathfrak{a}_i$ giving us a disjoint union $\text{Spec } A = V(\mathfrak{a}_1) \sqcup V(\mathfrak{a}_2)$. For instance, consider the principal ideals

$$\mathfrak{a}_1 = (1, X)A, \quad \mathfrak{a}_2 = (0, 1)A.$$

Then we do have $\text{Spec } A = V(\mathfrak{a}_1) \cup V(\mathfrak{a}_2)$, but we do not have $A \cong A/\mathfrak{a}_1 \times A/\mathfrak{a}_2$. (It is true that the latter isomorphism holds “up to nilpotent elements”, i.e., $A/\text{nil}(A) \cong A/\text{rad}(\mathfrak{a}_1) \times \text{rad}(\mathfrak{a}_2)$.) For $\mathfrak{a}_1 = (1, X)$, $\mathfrak{a}_2 = (0, 1 - X)$ we then have $\mathfrak{a}_1 + \mathfrak{a}_2 = 1$. However, $\mathfrak{a}_1$ is not idempotent (and neither is $\mathfrak{a}_2$)! This shows that one has to use the information of (iii) in a clever way. Loosely speaking, it is better to think of
being non-connected as being a disjoint union of open subsets (rather than of closed ones) and correspondingly, to try to express A as a product of localizations, rather than a product of quotients. (We will understand this better when we will have defined the “structure sheaf” on Spec A, and even better when we discuss “closed subschemes”.)

Now let us carry out the proof of \((i) \Rightarrow (iii)\) (we will do \((i) \Rightarrow (ii)\) on the way, but I think that keeping \((iii)\) in mind will make the situation clearer). Although we would prefer to express Spec A as a disjoint union of open subsets, we start with closed subsets because we have better control on them. The assumption tells us that there are ideals \(a_1, a_2 \subseteq A\) such that

\[\text{Spec } A = V(a_1) \cup V(a_2) \quad \text{(disjoint union)},\]

with \(V(a_i) \neq \emptyset\), i.e., \(a_i \neq A\). Since \(V(a_1 + a_2) = V(a_1) \cap V(a_2) = \emptyset\), we obtain \(a_1 + a_2 = A\), say \(a_1 + a_2 = 1\) with \(a_i \in a_i, a_i \notin A^\times\). (Compare the example above.)

Since \(a_1 + a_2 = 1\) and \(V(a_1) \cup V(a_2) \supseteq V(a_1) \cup V(a_2) = \text{Spec } A\), for each \(p \in \text{Spec } A\) we have \(a_1 \in p\) if and only if \(a_2 \notin p\), and hence

\[\text{Spec } A = D(a_2) \cup D(a_1) \quad \text{(disjoint union)}.\]

Now that we can write Spec A as a disjoint union of nice open subsets (namely, principal open ones), we can obtain a statement on the level of rings:

**Claim.** The natural map \(A \rightarrow A_{a_2} \times A_{a_1}\) into the product of the localizations with respect to the \(a_i\) is an isomorphism (and hence \((iii)\) is true).

To prove the claim, we check injectivity and surjectivity of the map:

**Injective.** If \(a\) maps to 0 in \(A_{a_2}\) and \(A_{a_1}\), then there exists \(i\) such that \(a^2_a = a^i_a = 0\). Since \((a_1, a_2) = A\), we also have \((a^i_1, a^i_2) = A\) (indeed, \(a_1, a_2 \in \text{rad}(a^i_1, a^i_2)\), so some power of 1 is in \((a^i_1, a^i_2)\), and it follows that \(a = 0\).

**Surjective.** First note that the element \(a_1/1\) lies in every prime ideal of \(A_{a_2}\) (since \(D(a_2) \subseteq V(a_1)\)), so this is a nilpotent element. Likewise, \(a_2/1\) is nilpotent in \(A_{a_1}\). So the product \(a_1a_2\) maps to a nilpotent element in the product and the injectivity then implies that \(a_1a_2\) is nilpotent. (A different way to see this would be to say that \(V(a_1a_2) = V(a_1) \cup V(a_2) = \text{Spec } A\), whence \(a_1a_2 \in \text{rad}((a_1a_2)) = \text{nil}(A)\). Fix \(i\) such that \(a^i_1a^i_2 = 0\).

Since \((a_1, a_2) = A\), we also have \((a^i_1, a^i_2) = A\) (an argument we use for the second time, here), so we find \(b_1, b_2 \in A\) with \(b_1a^i_1 + b_2a^i_2 = 1\). Let \(e := b_1a^i_1\) so that \(1 - e = b_2a^i_2\). It follows that \(e(1 - e) = 0\) (so \(e\) is idempotent).

We can now also conclude the proof of surjectivity: Since \(a^i_2\) maps to 0 in \(A_{a_1}\), we see that \(b_1a^i_1\) maps to 1, i.e., \(b_1\) maps to \(a^{-i}_1\). This shows that the map \(A \mapsto A_{a_1}\) is surjective, and likewise, the map \(A \mapsto A_{a_2}\) is surjective. This shows that every

\(^1\text{The lemma that is hidden here, comparing with Eisenbud’s proposed argument at this point (cf. the Comment) is that the radical of an ideal is actually an ideal itself.}\)
element in $A_{a_2} \times A_{a_1}$ can be written in the form $(x_1/1, x_2/1)$ with $x_i \in A$. But our map maps $x_1b_1a_1^i + x_2b_2a_2^i$ to $(x_1/1, x_2/1)$, so the map is indeed surjective. (One could also identify $A/(e) = A_{1-e} = A_{a_2}$, $A/(1-e) = A_e = A_{a_1}$ and use the Chinese Remainder Theorem again.)

Comments. In [Eisenbud, Commutative Algebra] Exercise 2.25, there is a sketch of a slightly shorter proof. However, I hope that the proof given here will in the end be more enlightening, and will show the power of the construction of the “structure sheaf” on Spec $A$, enabling us to view Spec $A$ as a ringed space. Cf. [Görtz, Wedhorn, Algebraic Geometry I], Exercise 2.17 (c).

Problem 10
Let $X$ be a topological space, and suppose that $X = \bigcup_{i=1}^n Z_i$, where the $Z_i$, $i = 1, \ldots, n$, are closed irreducible subsets of $X$ such that $Z_i \not\subseteq Z_j$ for $i \neq j$. Prove that the $Z_i$ are precisely the irreducible components of $X$.

Solution. It is enough to show that for every irreducible subset $Y \subseteq X$, there exists $i$ with $Y \subseteq Z_i$ (because this clearly implies that the maximal $Z_i$ are the maximal irreducible subsets, i.e., the irreducible components; since no $Z_i$ is contained in a $Z_j$, $i \neq j$, they are all maximal). Now given $Y \subseteq X$ irreducible, write

$$Y = \bigcup_{i} (Y \cap Z_i),$$

a decomposition into closed subsets. Using induction (and that the union is finite), the irreducibility of $Y$ gives us $Y = Y \cap Z_i$ for some $i$, i.e., $Y \subseteq Z_i$, as desired.

Question. What would be an example of a topological space $X = \bigcup_{i \in I} Z_i$ where the $Z_i$ are closed irreducible subsets of $X$ such that $Z_i \not\subseteq Z_j$, but where the $Z_i$ are not irreducible components of $X$ (and of course $I$ has to be infinite)?

Problem 11
Let $A$ be a ring, $X = \text{Spec} A$. Show that every irreducible subset $Y \subseteq X$ contains at most one generic point. Give an example of a ring $A$ and an irreducible subset of $\text{Spec} A$ which does not contain a generic point.

Solution. Given $Y \subseteq \text{Spec} A$ irreducible, the closure $Z$ of $Y$ is also irreducible and hence of the form $V(p)$, $p \in \text{Spec} A$. A generic point of $V(p)$ is a point corresponding to a prime ideal which contains $p$ and is contained in all $q \in V(p)$, so $p$ is the unique generic point of $V(p)$. Since $Z$ is the closure of $Y$, $Y$ is dense in $Z$, so every generic point of $Y$ is also a generic point of $Z$. It follows that $Y$ can have at most one generic point.

If $A = \mathbb{Z}$ and $Y = \text{Spec} \mathbb{Z} \setminus \{(0)\}$, then $Y$ has no generic point, but is irreducible (since its closure $\text{Spec} \mathbb{Z}$ is irreducible).
Problem 12

Let $A$ be a ring, $f \in A$, $M$ an $A$-module. Consider the inductive system of $A$-modules (with index set $\mathbb{Z}_{\geq 0}$)

$$M \to M \to M \to \ldots$$

where all transition maps are given by multiplication by $f$. Show that there exists a natural isomorphism between the colimit $\text{colim}_i M$ and the localization $M_f$ of the $A$-module $M$ with respect to the element $f$. 