Problem sheet 3

Due date: Oct. 30, 2018.

Problem 9
Let $A$ be a ring. We call an element $e \in A$ idempotent if $e^2 = e$. Show that the following conditions are equivalent:

(i) $\text{Spec } A$ is not connected.

(ii) There exists an idempotent element $e \in A$ different from 0 and 1.

(iii) There exists a ring isomorphism $A \cong A_1 \times A_2$ with non-zero rings $A_1, A_2$.

Problem 10
Let $X$ be a topological space, and suppose that $X = \bigcup_{i=1}^{n} Z_i$, where the $Z_i, i = 1, \ldots, n,$ are closed irreducible subsets of $X$ such that $Z_i \subseteq Z_j$ for $i \neq j$. Prove that the $Z_i$ are precisely the irreducible components of $X$.

Problem 11
Let $A$ be a ring, $X = \text{Spec } A$. Show that every irreducible subset $Y \subseteq X$ contains at most one generic point. Give an example of a ring $A$ and an irreducible subset of $\text{Spec } A$ which does not contain a generic point.

Problem 12
Let $A$ be a ring, $f \in A$, $M$ an $A$-module. Consider the inductive system of $A$-modules (with index set $\mathbb{Z}_{\geq 0}$)

$$M \rightarrow M \rightarrow M \rightarrow \ldots$$

where all transition maps are given by multiplication by $f$. Show that there exists a natural isomorphism between the colimit $\colim_i M$ and the localization $M_f$ of the $A$-module $M$ with respect to the element $f$. 