Problem sheet 7

Due date: Nov. 27, 2018.

Problem 29
Let $X$ be a topological space and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X$.

(1) We define the image $\text{im}(f)$ of $f$ as the sheaf associated with the presheaf

$$U \mapsto \text{im}(\mathcal{F}(U) \to \mathcal{G}(U)), \quad U \subseteq X \text{ open}.$$  

Prove that $f$ induces a surjective morphism $\mathcal{F} \to \text{im}(f)$ of sheaves.

(2) Now assume that $f$ above is a morphism of sheaves of abelian groups. We define the kernel $\ker(f)$ of $f$ as the sheaf

$$U \mapsto \ker(\mathcal{F}(U) \to \mathcal{G}(U)), \quad U \subseteq X \text{ open}.$$  

Prove that this is in fact a sheaf and that $f$ induces an injective morphism $\ker(f) \to \mathcal{F}$.

Problem 30
Give an example of affine schemes $X$, $Y$ and a morphism $X \to Y$ of ringed spaces which is not a morphism of locally ringed spaces.

Problem 31
Let $A$ be a domain with field of fractions $K$. We view all (non-trivial) localizations of $A$ as subrings of $K$. Let $f \in A$, $f \neq 0$. Show that

$$A_f = \bigcap_{f \in A, f \notin p} A_p.$$  

Hint. Given $g \in \bigcap_{f \in A, f \notin p} A_p$, consider the ideal

$$\mathfrak{a} = \{h \in A; \ hg \in A\} \subseteq A.$$
Problem 32

Let $k$ be an algebraically closed field of characteristic $\neq 2$. Let $X = D(T + 1) \subseteq \mathbb{A}^1_k$ (where $T$ is the coordinate on $\mathbb{A}^1_k$, i.e., $\mathbb{A}^1_k = \text{Spec} \, k[T]$), and let $Y = V(U^2 - T^2(T + 1)) \subseteq \mathbb{A}^2_k$ (with coordinates $T, U$). We view $Y$ as the scheme $\text{Spec} \, k[T, U]/(U^2 - T^2(T + 1))$.

Show that there is a morphism $f: X \to Y$ of schemes which on closed points is given as $t \mapsto (t^2 - 1, t(t^2 - 1))$.

Show that $f$ is a bijection on the underlying topological spaces, but not an isomorphism of schemes.

Hint. You may make use of Hilbert’s Nullstellensatz and of the fact that $\dim X = \dim Y = 1$, i.e., all non-zero prime ideals in the affine coordinate rings of $X$ and $Y$ are maximal.