Fully transitive $p$-groups with finite first Ulm subgroup

By
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Abstract

An abelian $p$-group $G$ is called (fully) transitive if for all $x, y \in G$ with $U_G(x) = U_G(y)$ ($U_G(x) \leq U_G(y)$) there exists an automorphism (endomorphism) of $G$ which maps $x$ onto $y$. It is a long-standing problem of A.L.S. Corner whether there exist non-transitive but fully transitive $p$-groups with finite first Ulm subgroup. In this paper we restrict to $p$-groups of type $A$, this is to say $p$-groups satisfying $\text{Aut}(G) \lhd p^\omega G = U(\text{End}(G) \lhd p^\omega G)$. We show that the answer to Corner’s question is no if $p^\omega G$ is finite and $G$ is of type $A$.

Introduction

The notions of transitivity and full transitivity originated in the book “Infinite Abelian Groups” by I. Kaplansky [11] and extensive classes of abelian $p$-groups which are both transitive and fully transitive were found in [8] and [10]. Moreover, $p$-groups with neither property were constructed in [13] and for larger cardinalities in [7]. In [5] Files and Goldsmith proved the surprising result that a $p$-group $G$ is fully transitive if and only if its square $G \oplus G$ is transitive. Nevertheless, for $p = 2$ the independence of both concepts was shown by Corner in [4] and already Kaplansky had shown in [11] that for $p > 2$ transitivity always implies full transitivity. Therefore it is natural to ask which fully transitive non-transitive $p$-groups appear. By a fundamental observation of Corner [4] one can reduce the decision of whether or not a $p$-group $G$ is (fully) transitive to its first Ulm subgroup $p^\omega G$. In [4] it was shown that $G$ is (fully) transitive if and only if $\text{End}(G) \lhd p^\omega G$ acts (fully) transitively on the first Ulm subgroup $p^\omega G$ of $G$, i.e., for any $x, y \in p^\omega G$ such that $U_{p^\omega G}(x) = U_{p^\omega G}(y)$ ($U_{p^\omega G}(x) \leq U_{p^\omega G}(y)$) there exists an automorphism (endomorphism) $\alpha$ of $G$ such that $\alpha(x) = y$. Corner constructed a fully transitive non-transitive $p$-group with countable first Ulm subgroup in [4] and it is a long-standing problem whether there exists a fully transitive non-transitive $p$-group with finite first Ulm subgroup. Partial results were obtained by Carroll and Goldsmith in [2], [3] and by Hennecke in [9], but a general solution hasn’t been found yet. It is the aim of this paper to show that the answer to Corner’s question is no for a large class of $p$-groups, namely the class of $p$-groups of type $A$. Here a $p$-group $G$ is

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*Supported by project No. G-0545-173.06/97 of the German-Israeli Foundation for Scientific Research & Development
†Supported by the Graduiertenkolleg Theoretische und Experimentelle Methoden der Reinen Mathematik of Essen University.
of type $A$ if $\text{Aut}(G) \mid_{p^\omega G} = U(\text{End}(G) \mid_{p^\omega G})$. Our approach is ring theoretic and uses the structure of the Jacobson radical of a ring which acts (fully) transitively on a finite $p$-group.

We follow standard notations found in Fuchs [6] and Kaplansky [11]. Multiplication by an integer $n$ on some group $H$ is denoted by $n \ast id \mid_H$ while applying two endomorphisms $\varphi$ and $\psi$ successively is denoted by $\varphi \psi$.

1 A question by A.L.S. Corner

All groups considered will be abelian $p$-groups, where $p$ is a prime. Let $G$ be such a group. For each ordinal $\alpha$ we define $p^\alpha G$ to be $p^0 G = G$ and $p^\alpha G = \bigcap_{\beta < \alpha} p(p^{\beta+1} G)$. Then the height $|x|_G$ in $G$ of an element $x \in G$ is defined to be $\infty$ if $x \in p^\alpha G$ for all ordinals $\alpha$ and otherwise $\alpha$ if $x \in p^\alpha G \setminus p^{\alpha+1} G$, where infinity exceeds any ordinal. The Ulm sequence of an element $x \in G$ is given by $U_G(x) = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ where $\alpha_i = |p^i x|_G$. The Ulm sequences are partially ordered by agreeing that $U_G(x) \leq U_G(y)$ if $|p^i x|_G \leq |p^i y|_G$ for all $i \geq 0$. By $G(U_G(x))$ we denote the set of all elements $y \in G$ such that $U_G(y) \geq U_G(x)$.

**Definition 1.1** Let $G$ be a $p$-group and $R$ a unital subring of $\text{End}(G)$. We say that

1. $R$ acts **fully transitively** on $G$ if for any $x, y \in G$ with $U_G(x) \leq U_G(y)$ there exists an element of $R$ which maps $x$ to $y$;
2. $R$ acts **transitively** on $G$ if for any $x, y \in G$ with $U_G(x) = U_G(y)$ there exists a unit of $R$ which maps $x$ to $y$;
3. $G$ is **(fully) transitive** if $\text{End}(G)$ acts (fully) transitively on $G$.

We note that any direct summand of a fully transitive $p$-group $G$ is again fully transitive. Moreover, the question of deciding when a group is (fully) transitive is often made easier by the following fundamental observation of Corner ([4]).

**Lemma 1.2** A $p$-group $G$ is (fully) transitive if and only if $\text{End}(G)$ acts (fully) transitively on the first Ulm subgroup $p^\omega G$ of $G$.

In the same paper [4] Corner showed that transitivity implies full transitivity if the first Ulm subgroup is a direct sum of cyclic groups of one and the same order $p^n$. We call such groups homogeneous. Motivated by this result it was shown in [4] that the converse does not hold. In particular Corner constructed an example of a $p$-group which is fully transitive but not transitive and $p^\omega G$ is homogeneous of cardinality $\aleph_0$. Thus the following question is natural:

**Question 1.3** Is there a fully transitive non-transitive $p$-group with finite first Ulm subgroup?

If $G$ is a $p$-group then $\text{End}(G) \mid_{p^\omega G}$ is a subring $R$ of $\text{End}(p^\omega G)$ and similarly $S = \text{Aut}(G) \mid_{p^\omega G}$ is a subgroup of $\text{Aut}(p^\omega G)$. Clearly $S \subseteq U(R)$, the units of $R$, but in general the inclusion may be strict. Hence Carroll and Goldsmith introduced the following notion in [3].

**Definition 1.4** A $p$-group $G$ is said to be of **type $A$** if $\text{Aut}(G) \mid_{p^\omega G} = U(\text{End}(G) \mid_{p^\omega G})$. 
For instance any \( p \)-group with cyclic first Ulm subgroup is of type A and all examples constructed in [4] and [13] are of type A. Nevertheless, examples of \( p \)-groups which are not of type A may appear. The following result from [3] provides a helpful tool to handle \( p \)-groups of type A.

Lemma 1.5 Let \( G \) be a \( p \)-group such that any unital subring of \( \text{End}(G) \) that acts fully transitively on \( G \) also acts transitively on \( G \). Then every fully transitively \( p \)-group \( H \) of type A having \( G \) as its first Ulm subgroup is also transitive.

The best results on Corner's Question 1.3 which are known up to now (to the authors' knowledge) are the following which can be found in [2], [3] and [9].

Theorem 1.6 Let \( G \) be a fully transitive \( p \)-group. Then \( G \) is transitive if one of the following conditions hold:

(i) \( p^\omega G = Z(p^n) \oplus Z(p^n) \) for some \( n \in \mathbb{N} \);

(ii) \( p^\omega G = \bigoplus_{i=1}^{n} Z(p^{m_i}) \) for some \( n, m_i \in \mathbb{N} \) such that \( m_i < m_j \) if \( i < j \).

2 \( p \)-Groups with finite first Ulm subgroup

In this section we will mainly consider abelian \( p \)-groups \( G \) of type A with finite first Ulm subgroup, i.e., \( p^\omega G = \bigoplus_{i=1}^{n} Z(p^{m_i}) \) for some \( m_i, n \in \mathbb{N} \). We show that for this class of groups, full transitivity implies transitivity. If \( R \) is a ring then we denote by \( J(R) \) the Jacobson radical of \( R \) and abbreviate \( J(R) \) by \( J \) if there is no danger of confusion.

Lemma 2.1 Let \( n \in \mathbb{N} \) and \( H = \bigoplus_{i=1}^{n} H_i \), where each \( H_i \) is homogeneous of the form \( \bigoplus_{j=1}^{k_i} Z(p^{m_i}) \) for some fixed \( m_i \in \mathbb{N} \) such that \( m_i < m_j \) if \( i < j \) and cardinals \( k_i \). If \( x, y \in H \) and \( R \) is a unital subring of \( \text{End}(H) \) which acts (fully) transitively on \( H \), then the following hold:

(i) If \( n = 1 \), then:

(a) \( \text{ord}(x) = p^k \) if and only if \( x \in p^{m_1-k}H\backslash p^{m_1-k+1}H \);

(b) \( \text{ord}(x) \leq \text{ord}(y) \) if and only if \( U_H(y) \leq U_H(x) \);

(c) If \( \text{ord}(x) = \text{ord}(y) \), then \( x \in J^kH \) if and only if \( y \in J^kH \) for \( k \in \mathbb{N} \);

(ii) If \( U_H(x) = U_H(y) \) then \( x \in J^kH \) if and only if \( y \in J^kH \) for \( k \in \mathbb{N} \);

(iii) \( JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n \) if \( H \) is finite, i.e. all \( k_i \) are finite.
Proof. If \( n = 1 \), then \( H \) is homogeneous, hence the proof of (a) is trivial. Moreover, if \( x, y \in H \), then clearly \( U_H(y) \leq U_H(x) \) implies \( \text{ord}(x) \leq \text{ord}(y) \). Conversely, if \( \text{ord}(x) \leq \text{ord}(y) \), then \( |x| \leq |y| \) follows from (a). Thus \( U_H(y) \leq U_H(x) \) using the fact that no Ulm sequence in \( H \) has a gap before the first infinite entry. Therefore (b) holds.

To show (c) let \( \text{ord}(x) = \text{ord}(y) \). Then (b) implies that \( U_H(x) = U_H(y) \), hence it suffices to prove (ii). But (ii) is trivial since each \( J^h \) is an ideal of \( R \) and \( R \) acts (fully) transitively on \( H \). It remains to prove (iii). We have to show that \( JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n \).

Since \( p*id_H \in R \) is nilpotent we obtain \( p*id_H \in J \) and hence \( pH \subseteq JH \). Assume that there exists \( x \in JH \setminus pH \) and write \( x = (x_1, \ldots, x_n) \) with \( x_i \in H_i \). Let \( k \) be maximal such that \( x_k \neq 0 \) and \( x_k \notin pH_k \). Then any element in \( H_1 \oplus \cdots \oplus H_k \) has Ulm sequence greater or equal to \( U_H(x) \), hence \( H_1 \oplus \cdots \oplus H_k \subseteq Rx \subseteq JH \) by full transitivity. Therefore \( JH = H_1 \oplus \cdots \oplus H_{k-1} \oplus pH_k \oplus \cdots \oplus pH_n \) for some \( 1 \leq k \leq n \). Note that \( H = JH \) would imply \( H = 0 \) by Nakayama’s Lemma ([1], Corollary 15.13). If \( k < n \), then \( JH = H_1 \oplus \cdots \oplus H_k / (pH_k \oplus \cdots \oplus pH_n) \) and we choose any element \( x \) in \( H_k \setminus JH \).

It follows that \( U_H(x) = (0, 1, \ldots, m_k = 1, \ldots, \) and obviously any element \( z \in H \setminus JH \) satisfies \( U_H(z) \subseteq U_H(x) \). Hence there exists an endomorphism \( r \in R \) such that \( r(z) = x \) and thus \( Rx \subseteq Rz \). This proves that any \( R/J \) submodule \( K/JH \) of \( H/JH \) contains the non-trivial module \( (Rx + JH)/JH \). By Theorem XII.5 from [12] \( H/JH \) decomposes as an \( R/J \) module into a direct sum of simple \( R/J \) modules, thus \( H/JH = (Rx + JH)/JH \). By Nakayama’s Lemma ([1], Corollary 15.13) it follows that \( Rx = H \), a contradiction. Hence \( k = n \) and \( JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n \).

Next we will show that for the homogeneous case it is enough to consider groups of exponent \( p \). Therefore we need the following Proposition which enables us to consider \( R/J \) acting on \( H/JH \) (\( Rx/Jx \) for some \( x \in H \)) instead of \( R \) acting on \( H \).

**Proposition 2.2** Let \( n \in \mathbb{N} \) and \( H = \bigoplus_{i=1}^{n} H_i \), where each \( H_i \) is of the form \( \bigoplus_{j=1}^{k_i} \mathbb{Z}(p^{m_j}) \) for some fixed \( m_i \in \mathbb{N} \) such that \( m_i < m_j \) if \( i < j \) and cardinals \( k_i \). If \( R \) is a unital subring of \( \text{End}(H) \), then the following hold:

(i) If \( H \) is finite and \( R \) acts (fully) transitively on \( H \), then \( R/J \) acts (fully) transitively on \( H/JH \);

(ii) If \( n = 1 \) and \( R \) acts fully transitively on \( H \) and \( R/J \) acts transitively on \( H/JH \), then \( R \) acts transitively on \( H \);

(iii) If \( x, y \in H \) such that \( U_H(x) = U_H(y) \), then \( r(x) = y \) for some unit \( r \in R \) if and only if \( (s + J)(x + Jx) = y + Jx \) for some unit \( (s + J) \in R/J \).

Here the action of \( R/J \) on \( H/JH \) (\( Rx/Jx \)) is given by \( (r + J)(h + JH) = r(h) + JH \)

\( (r + J)(h + Jx) = r(h) + Jx \).

**Proof.** First we prove (i) and assume that \( H \) is as stated, hence finite. Let \( R \) be a unital subring of \( \text{End}(H) \), then it is easy to check that the action \( (r + J)(h + JH) = (r(h) + JH) \) of \( R/J \) on \( H/JH \) is well-defined. Now assume that \( R \) acts (fully) transitively on \( H \) and let \( (h_1 + JH), (h_2 + JH) \in H/JH \) be two non-zero elements of \( H/JH \). Then \( U(h_1 + JH) = U(h_2 + JH), \) since \( JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n \) by Lemma 2.1 (iii) and hence \( H/JH \) is a
direct sum of copies of \( \mathbb{Z}(p) \). Since \( h_1, h_2 \notin JH = H_1 \oplus \cdots \oplus H_{n-1} \oplus pH_n \) we easily obtain \( U(h_1) = U(h_2) \). Therefore there exists an (endomorphism) automorphism \( r \in R \) such that \( r(h_1) = h_2 \). Thus \( (r + J)(h_1 + JH) = r(h_1) + JH = h_2 + JH \) and \( r + J \) is a non trivial (endomorphism) automorphism of \( R/J \) since otherwise \( h_2 \in JH \), a contradiction. Therefore \( R/J \) acts (fully) transitively on \( H/JH \).

To show (ii) let \( n = 1 \) and assume that \( R/J \) acts transitively on \( H/JH \) with the action given as above and \( R \) acts fully transitively on \( H \). Let \( h_1, h_2 \) be two non-zero elements of \( H \) such that \( U(h_1) = U(h_2) \). We have to distinguish between two cases. Note that \( h_1 \in JH \) if and only if \( h_2 \in JH \) by Lemma 2.1 (ii):

Case 1: If \( h_1, h_2 \notin JH \), then \( (h_i + JH) \neq 0 \) for \( i = 1, 2 \), and hence \( \text{ord}(h_1) = \text{ord}(h_2) = p^m \) by Lemma 2.1 (i) and \( U(h_1 + JH) = U(h_2 + JH) \). Therefore there exists an automorphism \( (r_0 + J) \in R/J \) such that \( (r_0 + J)(h_1 + JH) = (h_2 + JH) \). Since automorphisms lift modulo the Jacobson radical, \( r_0 \) is an automorphism of \( H \) such that \( r_0(h_1) - h_2 \in JH \). Note that \( \text{ord}(h_1) = p^m \) implies that \( U(h_1) \leq U(x) \) for any \( x \in H \) (Lemma 2.1 (i)(b)) and hence \( Rh_1 = H \) since \( R \) acts fully transitively on \( H \). Thus \( JH = Jh_1 \) and there exists \( r_1 \in J \) such that \( r_0(h_1) - h_2 = r_1(h_1) \). It follows that \( (r_0 - r_1)(h_1) = h_2 \) and \( (r_0 - r_1) \in R^* \).

Case 2: If \( h_1, h_2 \notin JH \), then there exist \( g_1, g_2 \notin pH = JH \) such that \( h_i = p^{m-k_i} g_i \) for some \( k \in \mathbb{N} \) and \( U_H(g_1) = U_H(g_2) \) for \( i = 1, 2 \). By case 1 we obtain \( r \in R^* \) such that \( r(g_1) = g_2 \), and hence \( r(h_1) = h_2 \).

Finally, (iii) is easy to check as above since units modulo the Jacobson radical lift and the action of \( R/J \) on \( Rx/Jx \) is well-defined.

The following example shows that in Proposition 2.2 (ii) the assumption that \( R \) acts fully transitively on \( H \) can not be avoided.

Example 2.3 Let \( p > 2 \) be a prime, \( H = \mathbb{Z}(p) \oplus \mathbb{Z}(p) \) and \( R \) be the unital subring of \( \text{End}(H) \) consisting of all lower triangular \( 2 \times 2 \) matrices over \( \mathbb{Z}(p) \). Then \( R \) acts neither fully transitively nor transitively on \( H \) but \( R/J \) acts (fully) transitively on \( H/JH \).

Proof. Clearly no element of \( R \) can map the element \((0, 1)\) onto \((1, 0)\), hence \( R \) acts neither fully transitively nor transitively on \( H \) since \((1, 0) \) and \((0, 1)\) have the same Ulm sequence. Moreover, it is easy to see that \( J \) is the set of all \( 2 \times 2 \) matrices over \( \mathbb{Z}(p) \) which have a non-zero entry only in the left lower corner. Hence \( JH = \{ (0, x) : x \in \mathbb{Z}(p) \} \) and thus \( H/JH \cong \mathbb{Z}(p) \). Now, if \( (r, 0) + JH \) and \((s, 0) + JH \) are two non-zero elements in \( H/JH \), then both elements have the same Ulm sequence and \((s^{-1}_0 r_0^{-1}) + J \) is an automorphism which maps \((r, 0) + JH \) onto \((s, 0) + JH \). Therefore \( R/J \) acts (fully) transitively on \( H/JH \).

By Proposition 2.2 it suffices to look at \( Rx/Jx \) for some \( x \in H \). For this we prove a more general theorem on finite semisimple rings.

Theorem 2.4 Let \( 1 \in R \) be a finite semisimple ring and \( M \) a finitely generated \( R \)-module. If \( u, v \in M \) and \( r, s \in R \) such that \( ru = v \) and \( sv = u \), then there exists a unit \( t \in R \) such that \( tu = v \).
Proof. Since $R$ is semisimple it follows by the Artin-Wedderburn Theorem ([1], Theorem 13.6) that $R$ is the ring direct product $R \cong \prod_{i=1}^{k} \text{Mat}_{n_i}(F_i)$ for some $k, n_i \in \mathbb{N}$ and some finite fields $F_i$ $(1 \leq i \leq k)$. Since $M$ is finitely generated over $R$ it is projective ([12], Theorem XII.5). Hence [12][Theorem XII.4] implies that $M \cong \bigoplus_{i=1}^{l} \text{Re}_i$, where the $e_i \in R$ are minimal idempotents. Let $i \in \{1, \ldots, k\}$ and $E_i \in \text{Mat}_{n_i}(F_i)$ be the matrix with 1 in the $(j,j)$ entry and zeros elsewhere ($j = 1, \ldots, n_i$). Then $E_i$ is a minimal idempotent in $\text{Mat}_{n_i}(F_i)$ and it is easy to see that $\{E_i : j = 1, \ldots, n_i\}$ is a complete set of minimal idempotents of $\text{Mat}_{n_i}(F_i)$. Thus each $\text{Re}_i$ in the decomposition of $M$ is a copy of $F_i^{E_i}$ if $e_i \in Mat_{n_i}(F_i)$. We let $M_i$ be the collection of all summands $\text{Re}_j$ belonging to $\text{Mat}_{n_i}(F_i)$, i.e., $M_i = \bigoplus_{e_j \in \text{Mat}_{n_i}(F_i)} \text{Re}_j$. Then any $M_i$ is a direct sum of copies of $F_i^{n_i}$. Let $u, v \in M$ and $r, s \in R$ such that $ru = v$ and $sv = u$. We write $u = (u_1, \ldots, u_k)$ and $v = (v_1, \ldots, v_k)$ corresponding to the decomposition $M = \bigoplus_{i=1}^{k} M_i$. Moreover, we write $r = (r_1, \ldots, r_k)$ and $s = (s_1, \ldots, s_k)$ corresponding to the decomposition of $R = \prod_{i=1}^{k} \text{Mat}_{n_i}(F_i)$. It follows that $ru = (r_1u_1, \ldots, r_ku_k) = v$ and $sv = (s_1v_1, \ldots, s_kv_k) = u$. Hence we obtain $r_iu_i = v_i$ and $s_iv_i = u_i$ for all $1 \leq i \leq k$. If we can find units $t_i \in \text{Mat}_{n_i}(F_i)$ such that $t_iu_i = v_i$ for all $i$, then $t = (t_1, \ldots, t_k)$ would be a unit in $R$ mapping $u$ onto $v$ and we are done. Thus we may assume w.l.o.g. that $r, s \in R = \text{Mat}_{n}(F)$ for some finite field $F$ and $u, v \in M \cong \bigoplus_{i=1}^{k} F^{n_i}$.

Again we write $u = (u_1, \ldots, u_l)$ and $v = (v_1, \ldots, v_l)$ with $u_i, v_i \in F^n$ for all $1 \leq i \leq l$. Since $r$ and $s$ are matrices they preserve linear dependence of vectors. Let $\{u_i : i \in I\}$ be a maximal linearly independent set among the $u_i$’s. Then $ru_i = v_i$ and $sv_i = u_i$ implies that $\{v_i : i \in I\}$ is also maximal linearly independent among the $v_i$’s. Obviously there is a non-singular matrix $t \in \text{Mat}_{n}(F)$ mapping $u_i$ onto $v_i$ for all $i \in I$ since both sets $\{u_i : i \in I\}$ and $\{v_i : i \in I\}$ can be extended to bases of $F^n$. It follows immediately that $tu_i = v_i$ for all $1 \leq i \leq l$ and this completes the proof. \hfill \Box

The following corollary contrasts Corner’s example of a fully transitive non-transitive $p$-group with countable first Ulm subgroup from [4].

Corollary 2.5 Let $n \in \mathbb{N}$ and $H = \bigoplus_{i=1}^{n} \mathbb{Z}(p)$. If $R$ is a unital subring of $\text{End}(H)$, then $R$ acts fully transitively on $H$ if and only if $R$ acts transitively on $H$.

Proof. By Lemma 2.1 (iii) we get that $JH = pH = 0$ if $R$ acts (fully) transitively on $H$. Hence $J = 0$ and $R$ is semisimple. Thus the result follows by Theorem 2.4 since $H$ is a finitely generated $R$-module. Note that transitivity implies full transitivity trivially in our situation since all non-zero elements have the same Ulm sequence. \hfill \Box

Finally we obtain
Theorem 2.6. Let $H$ be a finite $p$-group. If $R$ is a unital subring of $\text{End}(H)$ which acts fully transitively on $H$, then $R$ acts transitively on $H$. In particular every abelian $p$-group $G$ of type $A$ with finite first Ulm subgroup that is fully transitive must also be transitive.

Proof. Let $x, y \in H$ such that $U_H(x) = U_H(y)$. Then there exist $r, s \in R$ such that $r(x) = y$ and $s(y) = x$. If $r \in J$, then $x = s(r(x)) \in Jx$, a contradiction, hence $r \not\in J$ and similarly $s \not\in J$. Thus $(r + J)(x + Jx) = (y + Jx)$ and $(s + J)(y + Jx) = (x + Jx)$ and by Theorem 2.4 it follows that $(t + J)(x + Jx) = (y + Jx)$ for some unit $(t + J) \in R/J$ since $R/J$ is finite semi-simple. By Proposition 2.2 (iii) we get that $u(x) = y$ for some unit $u \in R$. The second claim follows by Lemma 1.5. 

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