# On the Optimum Distance Profiles About Linear Block Codes

Yuan Luo, A. J. Han Vinck, *Fellow, IEEE*, and Yanling Chen

***Abstract*—In this paper, for some linear block codes, two kinds of optimum distance profiles (ODPs) are introduced to consider how to construct and then exclude (or include) the basis codewords one by one while keeping a distance profile as large as possible in a dictionary order (or in an inverse dictionary order, respectively). The aim is to improve fault-tolerant capability by selecting subcodes in communications and storage systems. One application is to serve a suitable code for the realization of the transport format combination indicators (TFCIs) of code-division multiple-access (CDMA) systems. Another application is in the field of address retrieval on optical media.**

***Index Terms*—Generalized Reed–Solomon code, Golay code, optimum distance profile (ODP) of linear block code, transport format combination indicator.**

## I. Introduction

The transport format combination indicators (TFCIs) are widely applied in code-division multiple-access (CDMA) systems; see [6] and [12]. In the realization of TFCI, some input bits are used to combine some basis codewords of a linear block code. When the number of the input bits increases or decreases, some basis codewords will be included or excluded, respectively. In this process, a general consideration is how to realize large minimum distances of the generated subcodes. In this paper, for some given linear block codes, we consider how to construct and then exclude the basis codewords one by one while keeping a corresponding distance profile as large as possible in a dictionary order. We also consider how to construct and then include the basis codewords one by one while keeping a corresponding distance profile as large as possible in an inverse dictionary order. Another application is in the field of address retrieval on optical media [1]. A summary of this paper without proof was presented at the 2008 IEEE International Symposium on Information Theory [4].

In Section II, two kinds of optimum distance profiles (ODPs) of a linear block code are introduced to describe the main objects of this paper. A linear block code has many distance profiles, and each profile is a sequence of minimum distances of subcodes nested one by one with decreasing dimensions (ranks). The optimum profile of the code is determined by a comparison of different profiles. We will introduce two kinds of comparisons and two kinds of corresponding ODPs, respectively. The generator matrices with respect to the optimum profile show how to exclude or include the basis codewords one by one under the requirement.

As to convolutional codes, the optimal distance profile is over a class of convolutional codes with given rate and given constraint length; see [7] and [8]. One convolutional code has one distance profile, which is also a sequence of minimum distances, but for the code with increasing punctured lengths from 0 to the first constraint. The application is about a criterion for selecting codes in sequential decoding.

Another concept is the weight hierarchy of a linear block code satisfying a chain condition, which was introduced in [14] and studied for high dimension in [9]. The code satisfying the chain condition can present a sequence of subcodes nested one by one with increasing dimensions (ranks) such that the support size of the subcode with dimension $i$ is equal to the $i$th generalized Hamming weight. Although the sequence of nested subcodes is also considered, the minimum distance of each subcode is not investigated.

In Sections III–V, we present the optimum profiles and the corresponding generator matrices of the (binary, ternary and extended) Golay codes, the generalized Reed–Solomon codes, and some codes from the Hadamard matrices, respectively. Final conclusions are provided in Section VI.

## II. Preliminary

### A. Distance Profiles and Subcode Chains

Let $C$ be an $[n,k]$ linear code over $\mathrm{GF}(q)$ and denote $C_0 = C$. A sequence of linear subcodes

$$C_0 \supset C_1 \supset \cdots \supset C_{k-1} \tag{1}$$

is called a subcode chain, where $\dim[C_i] = k-i$. An increasing sequence

$$\tau[C_0] \le \tau[C_1] \le \cdots \le \tau[C_{k-1}] \tag{2}$$

is called a distance profile of the linear block code $C$, where $\tau[C_i]$ is the minimum Hamming distance of the subcode $C_i$. It is easy to see that a distance profile, which is also denoted by $\tau_0, \ldots, \tau_{k-1}$, is with respect to a subcode chain.

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Y. Luo is with the Computer Science and Engineering Department, Shanghai Jiao Tong University, Shanghai 200240, China and also with the MOE-Microsoft Key Laboratory for Intelligent Computing and Intelligent Systems, Shanghai 200240, China (e-mail: yuanluo@sjtu.edu.cn).

A. J. Han Vinck and Y. Chen are with the Institute for Experimental Mathematics, Duisburg-Essen University, Essen 45326, Germany (e-mail: vinck@iem.uni-due.de; julia@iem.uni-due.de).

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For a given $[n, k]$ code $C$ over $\mathrm{GF}(q)$, the number of its subcode chains is

$$\prod_{t=2}^{k} Q[t, t-1] = \prod_{t=2}^{k} \frac{q^t - 1}{q - 1} \quad (3)$$

where $Q[t, r]$ is the $q$-ary Gaussian binomial coefficient

$$\prod_{j=0}^{r-1} \frac{q^{t-j} - 1}{q^{r-j} - 1}.$$

Among the $\prod_{t=2}^{k}[(q^t-1)/(q-1)]$ chains, we focus on the chain having the ODP defined in the following paragraphs, and try to find a generator matrix $G$ such that its first $k - i$ rows generate $C_i$ for $0 \le i \le k-1$.

### B. Optimum Distance Profiles

For any two integer sequences of length $k$, $a = a_0, \ldots, a_{k-1}$ and $b = b_0, \ldots, b_{k-1}$, we say that $a$ is an upper bound on $b$ in the dictionary order if $a$ is equal to $b$ or there is an integer $t$ such that

$$\begin{aligned} a_i &= b_i, \qquad \text{for } 0 \le i \le t-1 \\ a_t &> b_t. \end{aligned}$$

We say that $a$ is an upper bound on $b$ in the inverse dictionary order if $a$ is equal to $b$ or there is an integer $t$ such that

$$\begin{aligned} a_i &= b_i, \qquad \text{for } t+1 \le i \le k-1 \\ a_t &> b_t. \end{aligned}$$

A distance profile of the linear block code $C$ is called the ODP in the dictionary order, which is denoted by

$$\mathrm{ODPB}[C]_0^{\mathrm{dic}}, \mathrm{ODPB}[C]_1^{\mathrm{dic}}, \ldots, \mathrm{ODPB}[C]_{k-1}^{\mathrm{dic}} \quad (4)$$

if it is an upper bound on any distance profile of $C$ in the dictionary order, where $B$ denotes block. A definition of the ODP in the inverse dictionary order, which is denoted by

$$\mathrm{ODPB}[C]_0^{\mathrm{inv}}, \mathrm{ODPB}[C]_1^{\mathrm{inv}}, \ldots, \mathrm{ODPB}[C]_{k-1}^{\mathrm{inv}} \quad (5)$$

follows from similar arguments of (4) but in the inverse dictionary order. It is easy to verify that $\mathrm{ODPB}[C]_0^{\mathrm{dic}} = \mathrm{ODPB}[C]_0^{\mathrm{inv}} = d_{\min}$ and $\mathrm{ODPB}[C]_{k-1}^{\mathrm{inv}} = d_{\max}$, where $d_{\min}$ is the minimum weight and $d_{\max}$ is the maximum weight.

The existence and uniqueness of the ODP ($\mathrm{ODPB}^{\mathrm{dic}}$ or $\mathrm{ODPB}^{\mathrm{inv}}$) of a linear block code are obvious since any two distance profiles can compare with each other in the corresponding order. For equivalent linear block codes, the properties about the Hamming weights, the Hamming distances and the distance profiles are the same.

### C. Generator Matrices With Respect to the Distance Profiles

For a given distance profile $\tau[C_0], \tau[C_1], \ldots, \tau[C_{k-1}]$, there is a generator matrix $G$ (maybe not unique) of the code $C$ such that its first $k - i$ rows form a generator matrix of $C_i$, which is called a generator matrix with respect to the profile. In this paper, we consider the generator matrices with respect to the $\mathrm{ODPB}^{\mathrm{dic}}$ and $\mathrm{ODPB}^{\mathrm{inv}}$.

The structures of the generator matrices with respect to $\mathrm{ODPB}^{\mathrm{dic}}$ or $\mathrm{ODPB}^{\mathrm{inv}}$ imply, respectively, how to exclude (from the last row to the first row) or include (from the first row to the last row) the basis codewords one by one, while keeping the distance profiles as large as possible. One example is about a Hamming code; see the following.

Let $C$ be the binary $[7,\ 4,\ 3]$ Hamming code. According to (3), there are $\prod_{t=2}^{4}(2^t - 1) = 315$ distinct subcode chains and 315 corresponding (not distinct) distance profiles. Two optimum profiles are of our interest. Note that each codeword has weight $0, 3, 4$, or $7$. Then, $\mathrm{ODPB}[C]_0^{\mathrm{dic}} = 3$ and

$$\mathrm{ODPB}[C]_1^{\mathrm{dic}} = \mathrm{ODPB}[C]_2^{\mathrm{dic}} = \mathrm{ODPB}[C]_3^{\mathrm{dic}} = 4$$

since the even weight codewords of $C$ generate the $[7,\ 3,\ 4]$ subcode which does not have the codeword $(1111111)$. One generator matrix with respect to the profile $\mathrm{ODPB}[C]^{\mathrm{dic}}$ is

$$G_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

If one deletes the rows of $G_1$ one by one from the last row, the minimum distances of the generated subcodes will be $3, 4, 4, 4$. This is the best way to get large minimum distances as soon as possible when deleting basis codewords.

Furthermore, it is easy to verify that $\mathrm{ODPB}[C]_3^{\mathrm{inv}} = 7$ and then $\mathrm{ODPB}[C]_i^{\mathrm{inv}} \le \lfloor 7/2 \rfloor$ for $i = 0, 1, 2$. In fact

$$\mathrm{ODPB}[C]_2^{\mathrm{inv}} = \mathrm{ODPB}[C]_1^{\mathrm{inv}} = \mathrm{ODPB}[C]_0^{\mathrm{inv}} = 3.$$

One generator matrix with respect to the profile $\mathrm{ODPB}[C]^{\mathrm{inv}}$ is

$$G_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

If one includes the rows of $G_2$ one by one from the first row to the last row, the minimum distances of the generated subcodes will be $7, 3, 3, 3$. This is the best way to get large minimum distances as soon as possible when including basis codewords. Note that the corresponding profile is denoted by $3, 3, 3, 7$.

## III. ODPs of the Golay Codes

### A. $\mathcal{G}_{24}$

Let $\mathcal{G}_{24}$ be the $[24,\ 12,\ 8]$ extended binary Golay code and $\tau_0, \tau_1, \ldots, \tau_{11}$ be a distance profile. Its weight enumerator implies that $\tau_i \in \{8,\ 12,\ 16,\ 24\}$. Then, by using the tables [3] on the maximum minimal distances of linear block codes with length 24 and dimension $12 - i$, we have

$$\tau_i = 8, \qquad \text{for } 0 \le i \le 6 \quad (6)$$

$$\tau_i \in \{8,\ 12\}, \qquad \text{for } 7 \le i \le 9 \quad (7)$$

$$\tau_i \in \{8,\ 12,\ 16\}, \qquad \text{for } i = 10 \quad (8)$$

$$\tau_i \in \{8,\ 12,\ 16,\ 24\}, \qquad \text{for } i = 11. \quad (9)$$

The $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{dic}}$ results from selecting subcode $C_i$ from $C_{i-1}$ one by one (from $i = 0$ to $i = 11$) while keeping the distance profile as large as possible in the dictionary order, where

$C_0 = \mathcal{G}_{24}$ and $\dim[C_i] = 12 - i$. Consider a Turyn construction of $\mathcal{G}_{24}$ [10, p. 588]

$$\{(\alpha + x, \beta + x, \alpha + \beta + x) : \alpha, \beta \in \mathcal{D}_1, x \in \mathcal{D}_2\}$$

where $\mathcal{D}_1$ is an $[8,\ 4,\ 4]$ code and $\mathcal{D}_2$ is another $[8,\ 4,\ 4]$ code. A $[24,\ 5,\ 12]$ subcode of $\mathcal{G}_{24}$ is obtained by replacing $\mathcal{D}_1$ with an $[8,\ 1,\ 8]$ subcode and replacing $\mathcal{D}_2$ with an $[8,\ 3,\ 4]$ equidistant subcode generated by

$$\begin{pmatrix} 10100110 \\ 01001110 \\ 00111010 \end{pmatrix}. \tag{10}$$

Thus, it follows from (7) that

$$\mathrm{ODPB}[\mathcal{G}_{24}]_7^{\mathrm{dic}} = 12. \tag{11}$$

A generator matrix of the $[24,\ 5,\ 12]$ subcode is

$$\begin{pmatrix} 00000000 & 11111111 & 11111111 \\ 11111111 & 00000000 & 11111111 \\ 10100110 & 10100110 & 10100110 \\ 01001110 & 01001110 & 01001110 \\ 00111010 & 00111010 & 00111010 \end{pmatrix}. \tag{12}$$

It is well known that binary $[24,\ 5,\ 12]$ code is unique [13] and the weights of its codewords are $0, 12$, or $16$.

In the structure of the subcode chain, formula (11) and the weight distribution of the unique $[24,\ 5,\ 12]$ code imply that

$$\mathrm{ODPB}[\mathcal{G}_{24}]_{11}^{\mathrm{dic}} \neq 24. \tag{13}$$

Applying (11) and (13) to (6)–(9), we have

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &= 8, && \text{for } 0 \le i \le 6 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &= 12, && \text{for } 7 \le i \le 9 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &\le 16, && \text{for } i = 10,\ 11. \end{aligned}$$

These upper bounds are achieved in Proposition 1 by using a corresponding generator matrix.

*Proposition 1:* The $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{dic}}$ is

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &= 8, && \text{for } 0 \le i \le 6 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &= 12, && \text{for } 7 \le i \le 9 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{dic}} &= 16, && \text{for } i = 10,\ 11. \end{aligned}$$

A generator matrix $G_{12\times 24}$ with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{dic}}$ follows where its first $12 - i$ rows generate the subcode $C_i$ (from $i = 0$ to $i = 11$) in the corresponding subcode chain.

- The first five rows of $G_{12\times 24}$ are from (12).
- The last seven rows of $G_{12\times 24}$ can be constructed easily.

The $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{inv}}$ results from selecting subcode $C_i$ containing $C_{i+1}$ one by one (from $i = 11$ to $i = 0$) while keeping the corresponding distance profile as large as possible in the inverse dictionary order, where $C_0 = \mathcal{G}_{24}$ and $\dim[C_i] = 12 - i$. It is easy to see that $\mathrm{ODPB}[\mathcal{G}_{24}]_{11}^{\mathrm{inv}} = 24$, and then we know from the structure of the subcode chain that

$$\mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} \le \left\lfloor \frac{24}{2} \right\rfloor = 12, \qquad \text{for } i \le 10. \tag{14}$$

In addition, the uniqueness of binary $[24,\ 5,\ 12]$ code implies

$$\mathrm{ODPB}[\mathcal{G}_{24}]_7^{\mathrm{inv}} \neq 12 \tag{15}$$

since $\mathrm{ODPB}[\mathcal{G}_{24}]_{11}^{\mathrm{inv}} = 24$. Applying (14) and (15) to (6)–(9), we have

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{24}]_{11}^{\mathrm{inv}} &= 24 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} &\le 12, && \text{for } 8 \le i \le 10 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} &= 8, && \text{for } 0 \le i \le 7. \end{aligned}$$

Then the $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{inv}}$ follows in Proposition 2 by a construction of a generator matrix.

*Proposition 2:* The $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{inv}}$ is

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} &= 24, && \text{for } i = 11 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} &= 12, && \text{for } 8 \le i \le 10 \\ \mathrm{ODPB}[\mathcal{G}_{24}]_i^{\mathrm{inv}} &= 8, && \text{for } 0 \le i \le 7. \end{aligned}$$

A generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{24}]^{\mathrm{inv}}$ is presented such that:

- its first four rows are

$$\begin{pmatrix} 11111111 & 11111111 & 11111111 \\ 10100110 & 10100110 & 10100110 \\ 01001110 & 01001110 & 01001110 \\ 00111010 & 00111010 & 00111010 \end{pmatrix}$$

  which is composed of the generator matrix (10) of the $[8,\ 3,\ 4]$ equidistant code and an all-one vector $\mathbf{1}^{24}$;
- its last eight rows can be constructed easily.

### *B.* $\mathcal{G}_{23}$

Let $\mathcal{G}_{23}$ be the $[23,\ 12,\ 7]$ binary Golay code and $\tau_0, \tau_1, \ldots, \tau_{11}$ be a distance profile. The weight enumerator of $\mathcal{G}_{23}$ implies that $\tau_i \in \{7,\ 8,\ 11,\ 12,\ 15,\ 16,\ 23\}$. Then, by using the tables [3] on the maximum minimal distances of linear block codes with length 23 and dimension $12 - i$, we have

$$\tau_0 = 7 \tag{16}$$

$$\tau_i \in \{7,\ 8\}, \qquad \text{for } 1 \le i \le 6, \tag{17}$$

$$\tau_i \in \{7,\ 8,\ 11\}, \qquad \text{for } i = 7 \tag{18}$$

$$\tau_i \in \{7,\ 8,\ 11,\ 12\}, \qquad \text{for } i = 8,\ 9 \tag{19}$$

$$\tau_i \in \{7,\ 8,\ 11,\ 12,\ 15\}, \qquad \text{for } i = 10 \tag{20}$$

$$\tau_i \in \{7,\ 8,\ 11,\ 12,\ 15,\ 16,\ 23\}, \qquad \text{for } i = 11. \tag{21}$$

The $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{dic}}$ results from selecting subcode $C_i$ from $C_{i-1}$ one by one (from $i = 0$ to $i = 11$) while keeping the corresponding distance profile as large as possible in the dictionary order, where $C_0 = \mathcal{G}_{23}$ and $\dim[C_i] = 12 - i$. Since all even-weight codewords of $\mathcal{G}_{23}$ form a [23, 11, 8] subcode, it follows from (17) that $\mathrm{ODPB}[\mathcal{G}_{23}]_1^{\mathrm{dic}} = 8$. The weight enumerator of the unique [23, 11, 8] code [2, Th. 2] implies that its subcodes only have even weight codewords. Therefore, by using (16)–(21), we have

$$\begin{aligned}
&\mathrm{ODPB}[\mathcal{G}_{23}]_0^{\mathrm{dic}} = 7\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{dic}} = 8, \qquad \text{for } 1 \le i \le 7\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{dic}} \le 12, \qquad \text{for } i = 8,\ 9,\ 10\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{dic}} \le 16, \qquad \text{for } i = 11.
\end{aligned}$$

In fact, these upper bounds are tight; see Proposition 3. A generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{dic}}$ is also obtained.

*Proposition 3:* The ODP $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{dic}}$ is

$$\begin{aligned}
&\mathrm{ODPB}[\mathcal{G}_{23}]_0^{\mathrm{dic}} = 7\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{dic}} = 8, \qquad \text{for } 1 \le i \le 7\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{dic}} = 12, \qquad \text{for } 8 \le i \le 10\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_{11}^{\mathrm{dic}} = 16.
\end{aligned}$$

A generator matrix $G_{12\times 23}$ with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{dic}}$ follows where its first $12 - i$ rows generate the subcode $C_i$ (from $i = 0$ to $i = 11$) in the corresponding subcode chain.

- The first four rows of $G_{12\times 23}$ are from (12) but delete one row and one column

$$\begin{pmatrix} 11111111 & 0000000 & 11111111\\ 10100110 & 1010011 & 10100110\\ 01001110 & 0100111 & 01001110\\ 00111010 & 0011101 & 00111010 \end{pmatrix}.$$

- From the fifth row to the eleventh row, the weight of each codeword must be even. In the twelfth row, the weight must be seven. These eight rows can be constructed easily.

The $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{inv}}$ results from selecting subcode $C_i$ containing $C_{i+1}$ one by one (from $i = 11$ to $i = 0$) while keeping the corresponding distance profile as large as possible in the inverse dictionary order, where $C_0 = \mathcal{G}_{23}$ and $\dim[C_i] = 12 - i$. It is easy to see that $\mathrm{ODPB}[\mathcal{G}_{23}]_{11}^{\mathrm{inv}} = 23$, and then we know from the subcode chain that $\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} \le \lfloor 23/2 \rfloor = 11$ for $i \le 10$. Furthermore, by using (16)–(21)

$$\mathrm{ODPB}[\mathcal{G}_{23}]_{11}^{\mathrm{inv}} = 23 \tag{22}$$
$$\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} \in \{7, 8, 11\}, \qquad \text{for } i = 7,\ 8,\ 9,\ 10 \tag{23}$$
$$\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} \in \{7, 8\}, \qquad \text{for } 1 \le i \le 6 \tag{24}$$
$$\mathrm{ODPB}[\mathcal{G}_{23}]_0^{\mathrm{inv}} = 7. \tag{25}$$

In the following lemma, the exact value of $\mathrm{ODPB}[\mathcal{G}_{23}]_4^{\mathrm{inv}}$ is obtained.

*Lemma 1:* In the Golay code $\mathcal{G}_{23}$, there is no [23, 8, 8] self-complementary subcode.

*Proof:* Assume that $S$ is a [23, 8, 8] self-complementary subcode of $\mathcal{G}_{23}$. It follows from the weight enumerator of $\mathcal{G}_{23}$ that the weight enumerator of $S$ is

$$1 + A_8 z^8 + A_{11} z^{11} + A_{12} z^{12} + A_{15} z^{15} + z^{23}$$

where $A_i$ is the number of codewords with weight $i$, $A_8 = A_{15}$ and $A_{11} = A_{12}$. One expression of the MacWilliams identities (see [10, p. 129]) of $S$ is

$$2^8 A'_w = (P_w(0) + P_w(23)) + A_8(P_w(8) + P_w(15)) + A_{11}(P_w(11) + P_w(12)) \tag{26}$$

where $A'_w$ is the number of codewords with weight $w$ in the dual code of $S$, and $P_w(i)$ is the Krawtchouk polynomial $\sum_{j=0}^{w}(-1)^j\binom{i}{j}\binom{23-i}{w-j}$. For the cases of $w = 0$ and $w = 2$ in (26), we have

$$\begin{cases} 2^8 = 2 + 2A_8 + 2A_{11}\\ 2^8 A'_2 = 506 + 26A_8 - 22A_{11}. \end{cases} \tag{27}$$

Considering each nonnegative integer value of $A'_2$, the solutions of (27) show that

$$A_8 \in \{53,\ 69,\ 85,\ 101,\ 117\}. \tag{28}$$

For the cases of $w = 0$ and $w = 4$ in (26), we have

$$\begin{cases} 2^8 = 2 + 2A_8 + 2A_{11}\\ 2^8 A'_4 = 17710 - 210A_8 + 110A_{11}. \end{cases} \tag{29}$$

Considering each nonnegative integer value of $A'_4$, the solutions of (29) show that

$$A_8 \in \{99 - 4l : l \text{ is an nonnegative integer}\}. \tag{30}$$

Since the sets of (28) and (30) are disjoint, the code $S$ does not exist. □

Using (22) and (24), a corollary of Lemma 1 is

$$\mathrm{ODPB}[\mathcal{G}_{23}]_4^{\mathrm{inv}} = 7. \tag{31}$$

In addition, we know from the uniqueness of the [24, 5, 12] code that there is no [23, 5, 11] self-complementary code, and then (22) and (23) imply that

$$\mathrm{ODPB}[\mathcal{G}_{23}]_7^{\mathrm{inv}} \in \{7,\ 8\}. \tag{32}$$

Using (31) and (32) to simplify (22)–(25), we have

$$\begin{aligned}
&\mathrm{ODPB}[\mathcal{G}_{23}]_{11}^{\mathrm{inv}} = 23\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} \le 11, \qquad \text{for } i = 8,\ 9,\ 10\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} \le 8, \qquad \text{for } i = 5,\ 6,\ 7\\
&\mathrm{ODPB}[\mathcal{G}_{23}]_i^{\mathrm{inv}} = 7, \qquad \text{for } 0 \le i \le 4.
\end{aligned}$$

Then, the profile $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{inv}}$ is provided in Proposition 4 and these upper bounds can be achieved by using a corresponding generator matrix.

*Proposition 4:* The $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{inv}}$ is

$$\begin{aligned}\mathrm{ODPB}[\mathcal{G}_{23}]_{11}^{\mathrm{inv}} &= 23\\ \mathrm{ODPB}[\mathcal{G}_{23}]_{i}^{\mathrm{inv}} &= 11, \quad \text{for } 8 \le i \le 10\\ \mathrm{ODPB}[\mathcal{G}_{23}]_{i}^{\mathrm{inv}} &= 8, \quad \text{for } 5 \le i \le 7\\ \mathrm{ODPB}[\mathcal{G}_{23}]_{i}^{\mathrm{inv}} &= 7, \quad \text{for } 0 \le i \le 4.\end{aligned}$$

A generator matrix $G_{12\times 23}$ with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{23}]^{\mathrm{inv}}$ follows where its first $12 - i$ rows generate the subcode $C_i$ (from $i = 11$ to $i = 0$) in the corresponding subcode chain.

- The first seven rows of the matrix $G_{12\times 23}$ are

$$\begin{pmatrix} 11111111 & 11111111 & 1111111\\ 10001101 & 10001101 & 1000110\\ 01000111 & 01000111 & 0100011\\ 00101110 & 00101110 & 0010111\\ 00111010 & 01001110 & 0111010\\ 01001110 & 10011100 & 1101001\\ 10011100 & 01110100 & 1110100 \end{pmatrix}.$$

These rows are from a Turyn construction of $\mathcal{G}_{24}$ [10, p. 588]

$$\{(\alpha + x, \beta + x, \alpha + \beta + x) : \alpha, \beta \in \mathcal{D}_1, x \in \mathcal{D}_2\}$$

but delete the last column, where $\mathcal{D}_1$ is an $[8,\ 4,\ 4]$ code generated by $G_1$ and $\mathcal{D}_2$ is another $[8,\ 4,\ 4]$ code generated by $G_2$ such that

$$G_1 = \begin{pmatrix} 00111010\\ 01001110\\ 10011100\\ 11111111 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} 11111111\\ 10001101\\ 01000111\\ 00101110 \end{pmatrix}.$$

- The last five rows of the matrix $G_{12\times 23}$ can be constructed easily where the eighth row must have weight seven.

### C. $\mathcal{G}_{12}$

Let $\mathcal{G}_{12}$ be the $[12,\ 6,\ 6]$ extended ternary Golay code, and $\tau_0, \tau_1, \ldots, \tau_5$ be a distance profile. Its weight enumerator implies that $\tau_i$ is $6, 9$, or $12$. Then, by using the tables on the maximum minimal distances of linear codes [3], we have

$$\tau_i = 6, \quad \text{for } 0 \le i \le 3 \tag{33}$$

$$\tau_4 \in \{6,\ 9\} \tag{34}$$

$$\tau_5 \in \{6,\ 9,\ 12\}. \tag{35}$$

Considering a generator ([10, p. 489], not full rank) of $\mathcal{G}_{12}$

$$\begin{pmatrix} & 0\\ \bar{G} & \vdots\\ & 0\\ 11111111111 & 1 \end{pmatrix}_{12\times 12} \tag{36}$$

where

$$\bar{G} = \begin{pmatrix} 10100011101\\ 11010001110\\ 01101000111\\ 10110100011\\ 11011010001\\ 11101101000\\ 01110110100\\ 00111011010\\ 00011101101\\ 10001110110\\ 01000111011 \end{pmatrix}_{11\times 11}$$

it is easy to verify that $\mathcal{G}_{12}$ has a $[12,\ 2,\ 9]$ subcode with generator matrix

$$\begin{pmatrix} 111002012112\\ 110120222220 \end{pmatrix}. \tag{37}$$

The weight distribution of any $[12,\ 2,\ 9]$ ternary linear code is presented in Corollary 1, which is helpful for investigating the ODPs in (38) and (39).

*Lemma 2 [5, p. 139]:* Let $C$ be an $[n, k, d]$ code over $\mathrm{GF}(3)$ with $k \ge 2$. Then:

1) $A_i = 0$ for $i > 3(n - d)$;
2) $A_i = 0$ or 2 for $i > (3n - 2d)/2$;

where $A_i$ is the number of codewords with weight $i$.

A special case of Lemma 2 for a $[12,\ 2,\ 9]$ ternary code follows.

*Corollary 1:* The weight distribution of any $[12,\ 2,\ 9]$ ternary linear code is $A_9 = 8$ and $A_{10} = A_{11} = A_{12} = 0$.

Corollary 1 and (34) and (35) imply that, if $\mathrm{ODPB}[\mathcal{G}_{12}]_4^{\mathrm{dic}} = 9$, we have

$$\mathrm{ODPB}[\mathcal{G}_{12}]_5^{\mathrm{dic}} = 9 \tag{38}$$

since $\mathrm{ODPB}[\mathcal{G}_{12}]_5^{\mathrm{dic}} \ne 12$. Further, if $\mathrm{ODPB}[\mathcal{G}_{12}]_5^{\mathrm{inv}} = 12$

$$\mathrm{ODPB}[\mathcal{G}_{12}]_4^{\mathrm{inv}} = 6 \tag{39}$$

since $\mathrm{ODPB}[\mathcal{G}_{12}]_4^{\mathrm{inv}} \ne 9$. Then, the profiles $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{dic}}$ and $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{inv}}$ are obtained from (33)–(35), (38), and (39).

*Proposition 5:* The $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{dic}}$ is

$$\begin{aligned}\mathrm{ODPB}[\mathcal{G}_{12}]_{i}^{\mathrm{dic}} &= 6, \quad \text{for } 0 \le i \le 3\\ \mathrm{ODPB}[\mathcal{G}_{12}]_{i}^{\mathrm{dic}} &= 9, \quad \text{for } 4 \le i \le 5.\end{aligned}$$

The $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{inv}}$ is

$$\begin{aligned}\mathrm{ODPB}[\mathcal{G}_{12}]_{5}^{\mathrm{inv}} &= 12\\ \mathrm{ODPB}[\mathcal{G}_{12}]_{i}^{\mathrm{inv}} &= 6, \quad \text{for } 0 \le i \le 4.\end{aligned}$$

By using the structure of (36), a generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{dic}}$ follows where:

- its first two rows are the rows of (37);
- its last four rows can be generated easily.

A generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{12}]^{\mathrm{inv}}$ satisfies the following:

- its first two rows are

$$\begin{pmatrix} 111111111111 \\ 110120222220 \end{pmatrix};$$

- its last four rows can be generated easily.

*D.* $\mathcal{G}_{11}$

Let $\mathcal{G}_{11}$ be the $[11,\ 6,\ 5]$ ternary Golay code, and $\tau_0, \tau_1, \ldots, \tau_5$ be a distance profile. Its weight enumerator implies that $\tau_i \in \{5,\ 6,\ 8,\ 9,\ 11\}$. Then, by using the tables on the maximum minimal distances of linear codes [3], we have

$$\tau_0 = 5, \tag{40}$$
$$\tau_i \in \{5,\ 6\}, \qquad \text{for } 1 \le i \le 3 \tag{41}$$
$$\tau_4 \in \{5,\ 6,\ 8\} \tag{42}$$
$$\tau_5 \in \{5,\ 6,\ 8,\ 9,\ 11\}. \tag{43}$$

Considering the following generator of $\mathcal{G}_{11}$ ([10, p. 489], not full rank)

$$\begin{pmatrix} \bar{G} \\ 11111111111 \end{pmatrix} \tag{44}$$

where $\bar{G}$ is of (36), it is easy to verify that $\mathcal{G}_{11}$ has a $[11,\ 5,\ 6]$ subcode generated by $\bar{G}$, then $\mathrm{ODPB}[\mathcal{G}_{11}]_1^{\mathrm{dic}} = 6$. In fact, in any $[11,\ 5,\ 6]$ ternary code, there is no codeword of weight 8 and weight 11; see the following proof.

Let $C$ be an $[n,k,d]$ code of defect $s = n-k+1-d$ and dimension $k$ over $\mathrm{GF}(q)$. If there exists an integer $m$ such that $k \ge m+1 \ge 2$ and

$$d = \frac{q^m(q-1)}{q^m-1}(s+m)$$

then $C$ is called a maximum–minimum distance (MMD) code [11].

*Lemma 3 [11]:* Assume that $C$ is an $[n,k,d]_q$ MMD code with $s+m > (q^m-1)/(q-1)$. Then, $m = s^{\perp}$ and the weight distribution of $C$ satisfies the following;

1)
$$A_d = \frac{\binom{n}{k-s^{\perp}}}{\binom{k+s-1}{k-s^{\perp}}}(q^{s^{\perp}}-1);$$

2) $A_{d+1} = A_{d+2} = \cdots = A_{n-k+s^{\perp}+1} = 0$;

3)
$$A_{n-k+v} = \binom{n}{k-v}\sum_{i=0}^{v-s^{\perp}}(-1)^i\binom{n-k+v}{i}(q^{v-i}-1) - \frac{(-1)^{v-s^{\perp}}(s+s^{\perp}-1)(q^{s^{\perp}}-1)}{(v+s-1)} \times \binom{n}{k-v}\binom{n-k+v}{v-s^{\perp}}.$$

Therefore, there is no codeword of weight 8 and weight 11 in any $[11,\ 5,\ 6]$ ternary linear code since it is MMD code with parameters $s=1$ and $m=1$. Furthermore, by using the fact $\mathrm{ODPB}[\mathcal{G}_{11}]_1^{\mathrm{dic}} = 6$ and (40)–(43), we have

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{11}]_0^{\mathrm{dic}} &= 5 \\ \mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{dic}} &= 6, \qquad \text{for } 1 \le i \le 4 \\ \mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{dic}} &\le 9. \end{aligned}$$

The $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{dic}}$ achieves these bounds; see Proposition 6. A generator matrix with respect to the profile is also provided.

*Proposition 6:* The ODP $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{dic}}$ is

$$\begin{aligned} \mathrm{ODPB}[\mathcal{G}_{11}]_0^{\mathrm{dic}} &= 5 \\ \mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{dic}} &= 6, \qquad \text{for } 1 \le i \le 4 \\ \mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{dic}} &= 9. \end{aligned}$$

Using the structure (44), a generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{dic}}$ is

$$\begin{pmatrix} 11012022222 \\ 01110110100 \\ 00111011010 \\ 00011101101 \\ 01101000111 \\ 11111111111 \end{pmatrix}.$$

The $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{inv}}$ results from Lemmas 4 and 5, which are used to investigate $\mathrm{ODPB}[\mathcal{G}_{11}]_4^{\mathrm{inv}}$ in (45) and $\mathrm{ODPB}[\mathcal{G}_{11}]_2^{\mathrm{inv}}$ in (46), respectively.

*Lemma 4:* For any $[11,\ 2,\ 8]$ ternary linear code, $A_8 = 6$, $A_9 = 2$, and $A_{10} = A_{11} = 0$ where $A_i$ is the number of codewords with weight $i$.

*Proof:* Let $C$ be an $[11,\ 2,\ 8]$ ternary linear code with Hamming weight enumerator $W_C(x,y) = x^{11} + A_8x^3y^8 + A_9x^2y^9 + A_{10}xy^{10} + A_{11}y^{11}$. The MacWilliams theorem shows that $W_{C^{\perp}}(x,y) = 1/|C|W_C(x+2y, x-y)$ where the integer coefficients of $x^{11}$ and $x^{10}y$ are $(1+A_8+A_9+A_{10}+A_{11})/9 = 1$ and $(22-2A_8-5A_9-8A_{10}-11A_{11})/9 = (6-3A_9-6A_{10}-9A_{11})/9$, respectively. Therefore, $A_{10} = A_{11} = 0$, $A_9 = 2$, and $A_8 = 6$ since $2|A_i$ when $i \ne 0$ □.

Lemma 4 implies that

$$\mathrm{ODPB}[\mathcal{G}_{11}]_4^{\mathrm{inv}} \ne 8$$

since $\mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{inv}} = 11$. Then, we know from (42) that

$$\mathrm{ODPB}[\mathcal{G}_{11}]_4^{\mathrm{inv}} \in \{5,\ 6\}. \tag{45}$$

*Lemma 5:* In the ternary Golay code $\mathcal{G}_{11}$, there is no $[11,\ 4,\ 6]$ subcode satisfying $A_{11} \ne 0$, where $A_i$ is the number of codewords with weight $i$.

*Proof:* Let $C$ be an $[11,\ 4,\ 6]$ subcode of $\mathcal{G}_{11}$ with Hamming weight enumerator $W_C(x,y)$ such that $A_{11} \ne 0$. The weight distribution of $\mathcal{G}_{11}$ and Lemma 2 show that $W_C(x,y) = x^{11} + A_6x^5y^6 + A_8x^3y^8 + A_9x^2y^9 + 2y^{11}$ since $A_{11} \ne 0$.

Using the MacWilliams theorem $W_{C^\perp}(x,y) = 1/|C|W_C(x+2y, x-y)$, the weight distribution of the dual code $C^\perp$ satisfies

$$A'_0 = \frac{(3 + A_6 + A_8 + A_9)}{81} = 1$$
$$A'_1 = \frac{(4A_6 - 2A_8 - 5A_9)}{81} = 0$$
$$A'_{11} = \frac{(2046 + 32A_6 + 8A_8 - 4A_9)}{81}.$$

Then, $A'_{11} = 122/3$, which is impossible. Therefore, there is no $[11,\ 4,\ 6]$ subcode of $\mathcal{G}_{11}$ such that $A_{11} \neq 0$. □

Lemma 5 implies that

$$\mathrm{ODPB}[\mathcal{G}_{11}]_2^{\mathrm{inv}} \neq 6$$

since $\mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{inv}} = 11$. Then, we know from (41) that

$$\mathrm{ODPB}[\mathcal{G}_{11}]_2^{\mathrm{inv}} = 5. \tag{46}$$

Using (45) and (46) to simplify (40)–(43), we have

$$\begin{aligned} &\mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{inv}} = 11 \\ &\mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{inv}} \leq 6, \qquad \text{for } 3 \leq i \leq 4 \\ &\mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{inv}} = 5, \qquad \text{for } 0 \leq i \leq 2. \end{aligned}$$

Proposition 7 shows that these upper bounds are sharp by providing a generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{inv}}$.

*Proposition 7:* The $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{inv}}$ is

$$\begin{aligned} &\mathrm{ODPB}[\mathcal{G}_{11}]_5^{\mathrm{inv}} = 11 \\ &\mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{inv}} = 6, \qquad \text{for } 3 \leq i \leq 4 \\ &\mathrm{ODPB}[\mathcal{G}_{11}]_i^{\mathrm{inv}} = 5, \qquad \text{for } 0 \leq i \leq 2. \end{aligned}$$

Using the structure (44), a generator matrix with respect to the profile $\mathrm{ODPB}[\mathcal{G}_{11}]^{\mathrm{inv}}$ is

$$\begin{pmatrix} 12212111222 \\ 01002102120 \\ 01000111011 \\ 11012022222 \\ 00111011010 \\ 00011101101 \end{pmatrix}.$$

## IV. ODPs of the Generalized Reed-Solomon Codes

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a vector over $\mathrm{GF}(q)$ where $\alpha_i \neq \alpha_j$ for $i \neq j$, and $v = (v_1, v_2, \ldots, v_n)$ be another vector such that $v_i \neq 0$ for $1 \leq i \leq n$. A generalized Reed–Solomon code $\mathrm{GRS}_{n,k}(\alpha, v)$ over $\mathrm{GF}(q)$ with length $n \leq q$ and dimension $k \leq n$ is a set of vectors in the form of

$$\{(v_1 f(\alpha_1), v_2 f(\alpha_2), \ldots, v_n f(\alpha_n)) : f(x) \in F_q[x],\ \deg f \leq k-1\} \tag{47}$$

see [10]. If $n = q-1$, $v_i = 1$, and $\alpha_i = \gamma^i$ for $1 \leq i \leq n$, then the definition (47) is of a Reed–Solomon code, where $\gamma$ is a primitive element in $\mathrm{GF}(q)$.

It is known that $\mathrm{GRS}_{n,k}(\alpha, v)$ is an $[n, k, n-k+1]$ maximum distance separable (MDS) code, and $\mathrm{GRS}_{n,k-1}(\alpha, v)$ is a subcode of $\mathrm{GRS}_{n,k}(\alpha, v)$. Then, by using the Singleton bound, its $\mathrm{ODPB}[\mathrm{GRS}]^{\mathrm{dic}}$ and $\mathrm{ODPB}[\mathrm{GRS}]^{\mathrm{inv}}$ are obtained.

*Proposition 8:* For an $[n, k]$ generalized Reed–Solomon code, its $\mathrm{ODPB}[\mathrm{GRS}]^{\mathrm{dic}}$ and $\mathrm{ODPB}[\mathrm{GRS}]^{\mathrm{inv}}$ are both equal to

$$n-k+1, n-k+2, \ldots, n.$$

Furthermore, by the properties of Vandermonde matrix and the arguments of (47), one generator matrix with respect to the optimum profile is

$$\begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ v_1\alpha_1 & v_2\alpha_2 & \cdots & v_n\alpha_n \\ v_1\alpha_1^2 & v_2\alpha_2^2 & \cdots & v_n\alpha_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ v_1\alpha_1^{k-1} & v_2\alpha_2^{k-1} & \cdots & v_n\alpha_n^{k-1} \end{pmatrix}.$$

## V. ODPs of Some Codes From the Hadamard Matrices

The Hadamard matrix $H_n$ considered in this section is a normalized Sylvester-type Hadamard matrix where $n = 2^k$. By replacing $+1$ and $-1$ with 0 and 1, respectively, $H_n$ is changed into a binary Hadamard matrix $A_n$. The codes studied here are from the matrix $A_n$ as shown in the following [10].

- Type 1: The rows of $A_n$ with the first column deleted form a $[2^k-1, k, 2^{k-1}]$ simplex code, i.e., the dual of the binary Hamming code, which is equidistant.
- Type 2: The codewords of Type 1 and their complements form a $[2^k-1, k+1, 2^{k-1}-1]$ punctured Reed–Muller code with weight enumerator $1+(2^k-1)z^{2^{k-1}-1}+(2^k-1)z^{2^{k-1}}+z^{2^k-1}$.
- Type 3: The rows of $A_n$ and their complements form a $[2^k, k+1, 2^{k-1}]$ first-order Reed–Muller code $RM(1,k)$ with weight enumerator $1+(2^{k+1}-2)z^{2^{k-1}}+z^{2^k}$.

For the code of Type 1 (T1), its $\mathrm{ODPB}[T1]^{\mathrm{dic}}$ and $\mathrm{ODPB}[T1]^{\mathrm{inv}}$ are both equal to a sequence with length $k$: $2^{k-1}, \ldots, 2^{k-1}$ because the code is equidistant. Any generator matrix is with respect to the profile.

For the code of Type 2 (T2), its $\mathrm{ODPB}[T2]^{\mathrm{dic}}$ and $\mathrm{ODPB}[T2]^{\mathrm{inv}}$ are $2^{k-1}-1, 2^{k-1}, \ldots, 2^{k-1}$ and $2^{k-1}-1, \ldots, 2^{k-1}-1, 2^k-1$, respectively. A generator matrix with respect to $\mathrm{ODPB}[T2]^{\mathrm{dic}}$ (or $\mathrm{ODPB}[T2]^{\mathrm{inv}}$) can be obtained by appending a row of weight $2^k-1$ to the end (or to the start, respectively) of any generator matrix of Type 1.

For the code of Type 3 (T3), its ODPs $\mathrm{ODPB}[T3]^{\mathrm{dic}}$ and $\mathrm{ODPB}[T3]^{\mathrm{inv}}$ are both equal to $2^{k-1}, \ldots, 2^{k-1}, 2^k$. Any generator matrix such that the first row is of weight $2^k$ is with respect to the profile.

## VI. Conclusion

For a given $[n,k]$ linear block code over $\mathrm{GF}(q)$, there are $\prod_{t=2}^{k}[(q^t-1)/(q-1)]$ subcode chains. In this paper, we focus on the subcode chains having two kinds of ODPs, which show us how to get large minimum distances when deleting or selecting its basis codewords one by one. The specific situations are about the (binary, ternary, and extended) Golay codes, the generalized Reed–Solomon codes, and some codes from the Hadamard matrices. The aim is to improve fault-tolerant capability by selecting subcodes in communications and storage systems.

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**Yuan Luo** received the B.S., M.S., and Ph.D. degrees in applied mathematics from Nankai University, Tianjin, China, in 1993, 1996, and 1999, respectively.

From July 1999 to April 2001, he held a postdoctoral position at the Institute of Systems Science, Chinese Academy of Sciences, Beijing, China. From May 2001 to April 2003, he held a postdoctoral position at the Institute for Experimental Mathematics, University of Duisburg-Essen, Essen, Germany. Since June 2003, he has been with the Computer Science and Engineering Department, Shanghai Jiao Tong University, Shanghai, China. He held his present position as a Full Professor since 2006. His current research interests include coding theory and information theory.

**A. J. Han Vinck** (M'77–SM'91–F'06) received the Ph.D. degree in electrical engineering from the University of Eindhoven, Eindhoven, The Netherlands, in 1980.

He has been a Full Professor in Digital Communications at the University of Duisburg-Essen, Essen, Germany, since 1990. In 1991–1993, 1998–2000, and 2006–2008, he was the Director of the Institute for Experimental Mathematics in Essen. He was also the Director (1997–1999) of the Postgraduate School on Networking, "CINEMA." From 2000 to 2004, he was the Chairman of the Communication Division of the Institute for Critical Infrastructures, CRIS. In 2003, he was an Adjoint Professor at the Sun Yat-Sen University, Kaohsiung, Taiwan. His interest is in information and communication theory, coding, and network aspects in digital communications.

Dr. Vinck was elected an IEEE Fellow in 2006 for his "contributions to coding techniques." He has served on the Board of Governors of the IEEE Information Theory Society since 1997 (until 2006). In 2003, he was elected President of the IEEE Information Theory Society. He served as Member-at-Large (2001–2002) in the Meetings and Services Committee for the IEEE. In 1999, he was the Program Chairman for the IEEE Information Theory Workshop, Kruger Park, South Africa, and in 1997, he acted as Co-Chairman for the 1997 IEEE Information Theory Symposium, Ulm, Germany. He was Founding Chairman (1995–1998) of the IEEE German Information Theory chapter. In 1990, he organized the IEEE Information Theory Workshop, Veldhoven, The Netherlands. He is the initiator of the Japanese-Benelux Workshops on Information Theory (now Asia-Europe) and the International Winter Meeting on Coding, Cryptography and Information Theory. He started and still supports the organization of the series of conferences on Power Line Communications and its Applications. In 2006, he received the IEEE ISPLC2006 Achievement Award in Orlando, FL, for his contributions to Powerline Communications and for facilitating the transition of ISPLC to a fully financially and technically sponsored IEEE Communications Society conference. The SA-IEE annual award was presented to him for the best paper published in the SA-IEE Africa Research Journal in 2008. He is Co-Founder and President of the Shannon and the Gauss Foundations. These foundations stimulate research and help young scientists in the field of information theory and digital communications.

**Yanling Chen** received the B.S. and M.S. degrees in applied mathematics from Nankai University, Tianjin, China, in 2001 and 2004, respectively, and the Ph.D. degree in engineering from the University of Duisburg-Essen, Essen, Germany, in 2007.

Currently, she is an ERCIM Postdoctoral Fellow and conducts research in the Centre for Quantifiable Quality of Service in Communication Systems, Centre of Excellence at the Norwegian University of Science and Technology, Norway. Her research interests include information theory, coding theory, and cryptography.