SIAM J. DISCRETE MATH.
Vol. 17, No. 2, pp. 196–209

THE DETERMINATION OF THE CHAIN GOOD WEIGHT HIERARCHIES WITH HIGH DIMENSION*

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Abstract. There are a large number of linear block codes satisfying the chain condition. Their weight hierarchies are called chain good and form an important group in classifying all possible weight hierarchies. In this paper, we present a series of new sufficient conditions to determine which kinds of sequences are chain good weight hierarchies. Our results are efficient for the determination of the chain good weight hierarchies with high dimension.

Key words. generalized Hamming weight, weight hierarchy, chain condition

AMS subject classifications. 94B05, 94B65

DOI. 10.1137/S0895480102416684

1. Introduction.

The generalized Hamming weight and weight hierarchy were first introduced by Wei in [7] and Helleseth, Kløve, and Mykkeltveit in [4]. The rth generalized Hamming weight of a q-ary $[n, k]$ linear block code C is defined as

$$d_r = \min\{|\chi(D_r)| : D_r \text{ is an } [n, r] \text{ linear subcode of } C\},$$

where $\chi(D_r)$ is the support set of D_r, i.e.,

$$\chi(D_r) = \bigcup_{c \in D_r} \{e : c_e \neq 0, \text{ where } c = (c_1, \dots, c_n)\}.$$

The weight hierarchy of C is denoted by $(d_1, \dots, d_k)$. The chain condition was first introduced by Wei and Yang in [8]. We say that the code C satisfies the chain condition if there exist k subcodes $D_r (1 \le r \le k) \subseteq C$ such that

$$\dim(D_r) = r, \quad |\chi(D_r)| = d_r, \quad \text{and} \quad D_1 \subset D_2 \subset \cdots \subset D_k = C.$$

An integer sequence $(a_1, \dots, a_k)$ is called a "chain good weight hierarchy over $GF(q)$" if it is a weight hierarchy of an $[n, k]$ $(n = a_k)$ linear block code over $GF(q)$ satisfying the chain condition. In this paper, q is a fixed prime power. A chain good weight hierarchy over $GF(q)$ is also called a "chain good weight hierarchy."

There are a large number of linear block codes satisfying the chain condition; see [1, 2, 3, 5, 6, 8]. Their chain good weight hierarchies form an important group in classifying all possible weight hierarchies and they receive much attention. In [1] and [6], some sufficient conditions were given for the determination of the chain good weight hierarchies with general dimension over $GF(q)$. However, these conditions are not efficient for the determination of the chain good weight hierarchies with high

*Received by the editors October 24, 2002; accepted for publication (in revised form) May 19, 2003; published electronically November 4, 2003. This work was supported by National Science Foundation of China grant 10271116.
http://www.siam.org/journals/sidma/17-2/41668.html

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dimension. In many cases, the lower bounds of these conditions increase exponentially with the dimension k; see the remarks of Theorems 2.1 and 2.2 in section 2.

In this paper, we present a series of new sufficient conditions to determine the chain good weight hierarchies with general dimension over $GF(q)$. The lower bounds of our new conditions increase linearly with the dimension k; see Corollaries 2.5 and 2.6 of section 2. They are more efficient than previous methods for the determination of the chain good weight hierarchies with high dimension.

Some preliminaries and our main results are introduced in section 2. In section 3, some interesting properties are shown. The proofs of our main results are presented in sections 4 and 5. For $q = 3$ and $k = 6, 7, 8$, the improvements on [1] and [6] are listed in section 6. Section 7 is the conclusion.

2. Preliminaries and main results. A positive integer sequence $(a_1, \dots, a_k)$ is called chain permissible over $GF(q)$ if $qi_{r-1} \geq i_r \geq 0$ for $1 \leq r \leq k-1$, where

$$i_r = a_{k-r} - a_{k-r-1} \quad \text{and} \quad a_0 = 0. \tag{2.1}$$

We know that the chain good weight hierarchies are chain permissible [3] and there also exist some chain permissible sequences which do not correspond to any weight hierarchies [2]. From (2.1), it is easy to see that the parameter sequence $(i_0, \dots, i_{k-1})$ can be determined from the sequence $(a_1, \dots, a_k)$ and vice versa. Let

$$\pi_r = (1-q)\sum_{j=0}^{r-1} i_j + i_r = \sum_{j=1}^{r}(i_j - qi_{j-1}) + i_0 \quad \text{for} \quad 0 \leq r \leq k-1. \tag{2.2}$$

Then

$$i_r = \sum_{j=0}^{r} \pi_j S_{r,j} \quad \text{for} \quad 0 \leq r \leq k-1, \tag{2.3}$$

where $S_{j,l} = (q-1)q^{j-l-1}$ for $j > l$, and $S_{j,j} = 1$. For a chain permissible sequence $(a_1, \dots, a_k)$, it is easy to see from (2.2) that

$$\pi_0 \geq \cdots \geq \pi_{k-1}. \quad (\pi_r \text{ may be negative for } r \geq 1.) \tag{2.4}$$

Denote

$$\iota_r = \lfloor i_r/q^r \rfloor \quad \text{and} \quad p_r = i_r - \iota_r q^r \quad \text{for} \quad 0 \leq r \leq k-1 \tag{2.5}$$

and

$$\delta_r = \begin{cases} 0 & \text{if} \quad 0 \leq p_{r+1} \leq p_r q, \\ 1 & \text{if} \quad p_r q < p_{r+1} < q^{r+1}. \end{cases} \tag{2.6}$$

Then for any chain permissible sequence, it was shown in [1] that

$$\iota_r \geq \iota_{r+1} + \delta_r. \tag{2.7}$$

The following theorems, Theorems 2.1 [1] and 2.2 [6], are two methods for the determination of the chain good weight hierarchies.

THEOREM 2.1 (see [1]). *A chain permissible sequence* $(a_1, \dots, a_k)$ *is a chain good weight hierarchy if*

$$\iota_{k-3} \geq (q-1) + \sum_{r=0}^{k-4}(\delta_r(q^{r+1}-1) + qp_r - p_{r+1}). \tag{2.8}$$

Remark. For fixed q, it is easy to see that the lower bound of condition (2.8) is greater than $q^{k-3}/2$ if $\delta_{k-4} = 1$ and $p_{k-3} - qp_{k-4}$ is small positive; it is also greater than $q^{k-4}/2$ if $\delta_{k-5} = 1$ and $p_{k-4} - qp_{k-5}$ is small positive, and so on. Therefore, in many cases, the lower bound of the condition (2.8) increases exponentially with the dimension k. By using (2.7) for a chain permissible sequence, we know that the exponential increase of ι_{k-3} with k implies the exponential increase of ι_r with k for $0 \le r \le k-4$.

THEOREM 2.2 (see [6]). *A chain permissible sequence* $(a_1, \ldots, a_k)$ *is a chain good weight hierarchy if*

$$\iota_{k-2} \ge \sum_{r=0}^{k-3} (\delta_r(q^{r+1} - 1) + qp_r - p_{r+1}). \tag{2.9}$$

Remark. By the same arguments as in the remark for Theorem 2.1, we know that the lower bound of condition (2.9) also increases exponentially with the dimension k in many cases. The exponential increase of ι_{k-2} with k implies the exponential increase of ι_r with k for $0 \le r \le k-3$.

Therefore, Theorems 2.1 and 2.2 are not so efficient for large k. In this paper, we present a series of new sufficient conditions, the lower bounds of which increase linearly with the dimension k; see Corollaries 2.5 and 2.6. These new conditions are more efficient for the determination of the chain good weight hierarchies with high dimension. The following theorem provides an original idea about how to give a sufficient condition by using the parameters $\pi_0, \ldots, \pi_\Gamma$, where $0 \le \Gamma \le k-2$.

THEOREM 2.3. *For a chain permissible sequence* $(a_1, \ldots, a_k)$ *and an integer* Γ *such that* $0 \le \Gamma \le k-2$, *if there exist some integers* $\theta_0 \ge \theta_1 \ge \cdots \ge \theta_{k-2} \ge 0$ *satisfying*

$$i_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-2,l}, \quad \text{where} \quad 0 \le \theta_l \le \pi_l \quad \text{for} \quad 0 \le l \le \Gamma, \tag{2.10}$$

and

$$i_{j-1} \ge i_j/q + S_{j-1,0} \quad \text{for} \quad \Gamma + 2 \le j \le k-3, \tag{2.11}$$

then the chain permissible sequence is a chain good weight hierarchy.

In Theorem 2.3, condition (2.11) does not exist for $\Gamma = k-2$, $k-3$, and $k-4$. For $\Gamma = k-2$, the corresponding result of Theorem 2.3 was obtained in [6]; i.e., a chain permissible sequence $(a_1, \ldots, a_k)$ is a chain good weight hierarchy if $\pi_{k-2} \ge 0$. For $\Gamma = k-4$, the corresponding result of Theorem 2.3 includes the cases where $\Gamma = k-2$ and $k-3$.

Note that the integers $\theta_0, \ldots, \theta_{k-2}$ *satisfying* (2.10) *do not exist if* $\pi_\Gamma < 0$. In fact, if π_Γ and ι_{k-2} are large positive, we can find some suitable $\theta_0, \ldots, \theta_{k-2}$ and get the following theorem.

THEOREM 2.4. *Let* $(a_1, \ldots, a_k)$ *be a chain permissible sequence and let* Γ *be an integer such that* $0 \le \Gamma \le k-4$. *Then* $(a_1, \ldots, a_k)$ *is a chain good weight hierarchy if*

$$\pi_\Gamma \ge (k-2)q, \quad \iota_{k-2} \ge (k-2)(q-1), \tag{2.12}$$

and

$$i_{j-1} \ge i_j/q + S_{j-1,0} \quad \text{for} \quad \Gamma + 2 \le j \le k-3. \tag{2.13}$$

Furthermore, $(a_1, \ldots, a_k)$ *is a chain good weight hierarchy if*

$$\iota_\Gamma \geq (k-2)q + \sum_{r=0}^{\Gamma-1} (\delta_r(q^{r+1}-1) + qp_r - p_{r+1}), \tag{2.14}$$

$$\iota_{k-2} \geq (k-2)(q-1), \tag{2.15}$$

and

$$i_{j-1} \geq i_j/q + S_{j-1,0} \quad \textit{for} \quad \Gamma+2 \leq j \leq k-3. \tag{2.16}$$

Note that, when $\Gamma = k-4$, (2.13) and (2.16) do not exist.

Theorem 2.4 presents a series of new sufficient conditions by using a different Γ. A large number of new chain good weight hierarchies are found by using this theorem (see section 6). In particular, from the second part of Theorem 2.4, we get two important results, Corollaries 2.5 and 2.6.

COROLLARY 2.5. *Let* Γ *be a fixed nonnegative integer. A chain permissible sequence* $(a_1, \ldots, a_k)$*, where* $k \geq \Gamma + 6$*, is a chain good weight hierarchy if*

$$\iota_{k-2} \geq (k-2)(q-1) + \sum_{r=0}^{\Gamma-1} q^{r+1} \quad \textit{and} \tag{2.17}$$

$$\iota_{j-1} \geq \iota_j + 2 \quad \textit{for} \quad \Gamma+2 \leq j \leq k-3. \tag{2.18}$$

Remark. In Corollary 2.5, the lower bound of condition (2.17) increases linearly with the dimension k. The linear increase of ι_{k-2} with k only implies the linear increase of ι_r with k for $0 \leq r \leq k-3$. Therefore, in the determination of the chain good weight hierarchies with high dimension, Corollary 2.5 is more efficient than Theorems 2.1 and 2.2. By the same arguments, we have Corollary 2.6. In Corollary 2.6, the lower bound on the condition for ι_{k-2} is smaller, but a larger k is needed.

COROLLARY 2.6. *Let* Γ *be a fixed nonnegative integer. A chain permissible sequence* $(a_1, \ldots, a_k)$*, where* $k \geq \sum_{r=0}^{\Gamma-1} q^{r+1} + 6$*, is a chain good weight hierarchy if*

$$\iota_{k-2} \geq (k-2)(q-1) \quad \textit{and} \tag{2.19}$$

$$\iota_{j-1} \geq \iota_j + 2 \quad \textit{for} \quad \Gamma+2 \leq j \leq k-3. \tag{2.20}$$

3. Some basic lemmas. In this section, we give some interesting properties, which are useful in establishing our main results. In section 3.1, two types of expressions are introduced. We show that a nonnegative integer having a type I expression can also be expressed in type II. Then, in section 3.2, a symbol $R(\cdot,\cdot)$ is used to describe the relation of two expressions. In the last subsection, we introduce two new parameters, π_j^* and T_j, of a chain permissible sequence.

3.1. Two types of expressions. For nonnegative integers $z_0, \ldots, z_J$, let $[z_0, \cdots, z_J]$ be the expression $\sum_{l=0}^{J} z_l S_{J,l}$, where $J \geq 1$. We say that $[z_0, \ldots, z_J] = [y_0, \ldots, y_J]$ if $z_l = y_l$ for $0 \leq l \leq J$. We say that $[z_0, \ldots, z_J]$ and $[y_0, \ldots, y_J]$ have the same value if $\sum_{l=0}^{J} z_l S_{J,l} = \sum_{l=0}^{J} y_l S_{J,l}$. Let

$$D[z_0, \ldots, z_J] = [z_0 - \Delta, z_1 + \Delta(q-1), \ldots, z_J + \Delta(q-1)], \tag{3.1}$$

where $\Delta = \lfloor \frac{z_0 - z_1}{q} \rfloor$. Then $D[z_0, \ldots, z_J]$ and $[z_0, \ldots, z_J]$ have the same value. The expression $[z_0, \ldots, z_J]$ is called type I if

$$z_l \geq z_{l+1} \geq 0 \quad \text{for all} \;\; 0 \leq l \leq J-1. \tag{3.2}$$

It is called type II if

$$z_{l+1} + q > z_l \geq z_{l+1} \geq 0 \quad \text{for all } 0 \leq l \leq J-1. \tag{3.3}$$

Furthermore, we have the following property.

LEMMA 3.1. *Let* $[z_0, \dots, z_J]$ *be an expression of type* I; *then an expression of type* II *having the same value can be given by* $[z_0^{(J)}, \dots, z_J^{(J)}]$, *where*

$$[z_{J-l}^{(l)}, \dots, z_J^{(l)}] = D[z_{J-l}, z_{J-l+1}^{(l-1)}, \dots, z_J^{(l-1)}] \textit{ for } 1 \leq l \leq J, \quad z_J^{(0)} = z_J. \tag{3.4}$$

Proof. For $l = 1$, it is easy to see that $[z_{J-1}^{(1)}, z_J^{(1)}] = D[z_{J-1}, z_J]$ is type II. For $l = t$, suppose $[z_{J-t}^{(t)}, \dots, z_J^{(t)}]$ is type II. Then for $l = t+1$, the expression

$$[z_{J-t-1}^{(t+1)}, \dots, z_J^{(t+1)}] = D[z_{J-t-1}, z_{J-t}^{(t)}, \dots, z_J^{(t)}]$$

is also type II. Therefore, by induction, $[z_0^{(J)}, \dots, z_J^{(J)}]$ is an expression of type II. Furthermore, $[z_0^{(J)}, \dots, z_J^{(J)}]$ and $[z_0, z_1, \dots, z_J]$ have the same value since the operator D does not change the value of an expression. □

3.2. A relation *R* of two expressions. Let SUM_j and SUM_{j+1} be two expressions such that

$$SUM_j : \sum_{l=0}^{j} \alpha_{j,l} S_{j,l} + \lambda_{j,l} \quad \text{and} \quad SUM_{j+1} : \sum_{l=0}^{j+1} \alpha_{j+1,l} S_{j+1,l} + \lambda_{j+1,l}, \tag{3.5}$$

where $\alpha_{j,l}, \alpha_{j+1,l}, \lambda_{j,l} (< S_{j,l})$, and $\lambda_{j+1,l} (< S_{j+1,l})$ are nonnegative integers. We say that

$$R(SUM_j, SUM_{j+1}) \text{ is true} \tag{3.6}$$

if the coefficients of SUM_j and SUM_{j+1} satisfy

$$\alpha_{j,l} \geq \alpha_{j,l+1} + \epsilon(\lambda_{j,l+1}), \tag{3.7}$$

$$\alpha_{j+1,l} \geq \alpha_{j+1,l+1} + \epsilon(\lambda_{j+1,l+1}), \tag{3.8}$$

$$\alpha_{j,l} \geq \alpha_{j+1,l} + \epsilon(\lambda_{j+1,l}), \tag{3.9}$$

where $\epsilon(x) = 0$ for $x = 0$ and $\epsilon(x) = 1$ otherwise. By using the symbol $R(\cdot, \cdot)$, Theorem 2 of [6] can be given as follows.

LEMMA 3.2 (see [6]). *For a chain permissible sequence* $(a_1, \dots, a_k)$, *if there exist nonnegative integers* $\alpha_{j,l}$ *and* $\lambda_{j,l} (< S_{j,l})$ *such that*

$$E_j : \quad i_j = \sum_{l=0}^{j} \alpha_{j,l} S_{j,l} + \lambda_{j,l} \quad \textit{for} \quad 0 \leq j \leq k-1 \textit{ and} \tag{3.10}$$

$$R(E_j, E_{j+1}) \textit{ is true for } 0 \leq j \leq k-2, \tag{3.11}$$

then it is a chain good weight hierarchy.

3.3. New parameters: π_j^* and T_j. For a chain permissible sequence $(a_1, \dots, a_k)$, the relation between the parameter sequences $(i_0, \dots, i_{k-1})$ and $(\pi_0, \dots, \pi_{k-1})$ is obtained in (2.2) and (2.3). Now, we introduce a new parameter sequence $(\pi_0^*, \dots, \pi_{k-1}^*)$, which is useful for studying the bound of i_j $(0 \le j \le k-1)$. For $0 \le \Gamma \le k-4$, let

$$\pi_l^* = \pi_l \quad \text{for} \quad 0 \le l \le \Gamma \quad \text{and} \quad \pi_l^* = \pi_\Gamma \quad \text{for} \quad \Gamma+1 \le l \le k-1. \tag{3.12}$$

Denote

$$T_j = \sum_{l=0}^{j} \pi_l^* S_{j,l} \quad \text{for} \quad 0 \le j \le k-1. \tag{3.13}$$

LEMMA 3.3. *For a chain permissible sequence $(a_1, \dots, a_k)$, we have*

$$i_j \le T_j \quad \textit{for} \quad 0 \le j \le k-1. \tag{3.14}$$

If $i_{\Gamma+1} > i_{\Gamma+2}/q$, we have

$$i_j < T_j \quad \textit{for} \quad j \ge \Gamma+2. \tag{3.15}$$

Proof. For a chain permissible sequence $(a_1, \dots, a_k)$, it is shown in (2.4) that $\pi_0 \ge \dots \ge \pi_{k-1}$. Then $\pi_l \le \pi_l^*$ for $0 \le l \le k-1$ and

$$i_j = \sum_{l=0}^{j} \pi_l S_{j,l} \le T_j \quad \text{for} \quad 0 \le j \le k-1.$$

When $i_{\Gamma+1} > i_{\Gamma+2}/q$, if there exists an integer $j \ge \Gamma+2$ such that $i_j = T_j$, then

$$\begin{aligned} & i_j = \sum_{l=0}^{j} \pi_l S_{j,l} = \sum_{l=0}^{j} \pi_l^* S_{j,l} \\ \Rightarrow & \pi_j = \pi_\Gamma \\ \Rightarrow & \sum_{t=\Gamma+1}^{j} (i_t - q i_{t-1}) = 0 \\ \Rightarrow & i_{t-1} = i_t/q \quad \text{for} \quad \Gamma+1 \le t \le j, \end{aligned}$$

which is impossible. □

4. Proof of Theorem 2.3. In this section, the proof of Theorem 2.3 is given in two parts. The first part is presented for $\Gamma = k-4$ in Lemma 4.2, i.e., Theorem 4 of [6]. Now, we have a new description of the proof, which is useful in establishing the whole proof of Theorem 2.3. The second part is presented for $\Gamma \le k-5$. In addition, the following lemma, which is derived from Lemma 5 of [2], allows us to pay attention only to some special chain permissible sequences satisfying $i_{k-1} = q i_{k-2}$.

LEMMA 4.1 (see [2]). *For fixed integers $i_0^*, \dots, i_{k-2}^*$, let $\mathcal{A}$ be the set of chain permissible sequences with dimension k such that $i_j = i_j^*$ $(0 \le j \le k-2)$. Then all of the sequences in $\mathcal{A}$ are chain good weight hierarchies if the sequence in $\mathcal{A}$ satisfying $i_{k-1} = q i_{k-2}$ is a chain good weight hierarchy.*

LEMMA 4.2 (see [6]). *For a chain permissible sequence* $(a_1, \ldots, a_k)$*, if there exist some integers* $\theta_0 \geq \theta_1 \geq \cdots \geq \theta_{k-2} \geq 0$ *such that*

$$i_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-2,l}, \quad \textit{where} \quad \theta_l \leq \pi_l \quad \textit{for} \quad 0 \leq l \leq k-4, \tag{4.1}$$

then it is a chain good weight hierarchy.

Proof. By using Lemmas 3.1 and 4.1, we can assume that $[\theta_{k-3}, \theta_{k-2}]$ is type II and $i_{k-1} = qi_{k-2}$. Since

$$E_j: \qquad i_j = \sum_{l=0}^{j} \pi_l S_{j,l} \quad \text{for} \quad 0 \leq j \leq k-4,$$

$$E_{k-2}: \qquad i_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-2,l},$$

$$E_{k-1}: \qquad i_{k-1} = qi_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-1,l} + \theta_{k-2},$$

it follows that this lemma can be obtained by using Lemma 3.2 if there exists a suitable expression E_{k-3} for i_{k-3} such that $R(E_{k-4}, E_{k-3})$ and $R(E_{k-3}, E_{k-2})$ are both true.

In the following paragraphs, after showing two bounds of i_{k-3}, a suitable expression E_{k-3} is given in (4.4). The first bound is an upper bound obtained from Lemma 3.3:

$$i_{k-3} \leq T_{k-3} = \sum_{l=0}^{k-3} \pi_l^* S_{k-3,l}. \tag{4.2}$$

The second bound is a lower bound. Denote $\Lambda = \sum_{l=0}^{k-3} \theta_l S_{k-3,l}$; we have

$$i_{k-3} \geq \lceil i_{k-2}/q \rceil = \Lambda \tag{4.3}$$

since $i_{k-3} \geq i_{k-2}/q$ and $[\theta_{k-3}, \theta_{k-2}]$ is type II. Then a suitable expression E_{k-3} for i_{k-3} is obtained in (4.4), where the coefficients are less than or equal to those of T_{k-3} and greater than or equal to those of Λ. Denote

$$e_l = \pi_l^* - \theta_l \quad \text{for} \quad 0 \leq l \leq k-3,$$

$$L = \max\left\{\delta : i_{k-3} \geq \Lambda + \sum_{l=0}^{\delta} e_l S_{k-3,l}\right\} \quad (\text{let } L = -1 \text{ if } \delta \text{ does not exist}),$$

$$g = i_{k-3} - \Lambda - \sum_{l=0}^{L} e_l S_{k-3,l};$$

we have

$$\begin{aligned} i_{k-3} &= \Lambda + \sum_{l=0}^{L} e_l S_{k-3,l} + g \\ &= \sum_{l=0}^{L} \pi_l^* S_{k-3,l} + ((\theta_{L+1} + g_1) S_{k-3,L+1} + g_2) + \sum_{l=L+2}^{k-3} \theta_l S_{k-3,l}, \end{aligned} \tag{4.4}$$

where $g_1 = \lfloor g/S_{k-3,L+1} \rfloor < e_{k-3}$ and $g_2 = g - g_1 S_{k-3,L+1} < S_{k-3,L+1}$. For $L = k-4$, the last part of (4.4) does not exist. For $L = k-3$, the last two parts of (4.4) do not exist. ☐

Proof of Theorem 2.3 *when* $\Gamma \le k-5$. By using Lemmas 3.1 and 4.1, we can assume that $[\theta_{\Gamma+1}, \ldots, \theta_{k-2}]$ is type II and $i_{k-1} = q i_{k-2}$.

From Lemma 3.2, we know that this theorem can be obtained if there exist the following expressions:

$$(4.5) \qquad E_j : \ i_j = \sum_{l=0}^{j} \pi_l^* S_{j,l} \quad \text{for} \quad 0 \le j \le \Gamma,$$

$$(4.6) \qquad E_j : \ i_j = \sum_{l=0}^{u_j - 1} \pi_l^* S_{j,l} + \sum_{l=u_j}^{j} \alpha_{j,l} S_{j,l} + \lambda_{j,\eta_j} \text{ for } \Gamma + 1 \le j \le k-3,$$

$$E_{k-2} : \ i_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-2,l},$$

$$E_{k-1} : \ i_{k-1} = \sum_{l=0}^{k-2} \theta_l S_{k-1,l} + \theta_{k-2},$$

where $\alpha_{j,l}$, u_j, $\eta_j (\ge u_j)$, and $\lambda_{j,\eta_j} (< S_{j,\eta_j})$ are nonnegative integers to be determined under the true condition $R(E_j, E_{j+1})$. Note that expression (4.5) is fixed.

In the following paragraphs, the construction for (4.6) is given in three steps. In Step 1, an expression E_j is obtained from E_{j+1} by induction in (4.8). Then, in Step 2, we show that $R(E_j, E_{j+1})$ is true. However, in some cases, E_j should be changed. The changes are given in the last step.

Step 1. Now, we show how to get the expression (4.6) by induction. By the same arguments as in the proof of Lemma 4.2, we get an expression E_{k-3} from E_{k-2} such that

$$R(E_{k-3}, E_{k-2}) \text{ is true and } u_{k-3} = \eta_{k-3}.$$

For any integer $j : \Gamma + 1 \le j \le k-4$, assume that E_{j+1} has been obtained from E_{j+2} satisfying

$$R(E_{j+1}, E_{j+2}) \text{ is true and } u_{j+1} = \eta_{j+1}.$$

Then, by the same arguments as in the proof of Lemma 4.2, we get an expression E_j in (4.8) from E_{j+1} if

$$(4.7) \qquad [\alpha_{j+1,u_{j+1}^*}, \ldots, \alpha_{j+1,j+1}] \text{ is type II},$$

where $u_{j+1}^* = \max\{u_{j+1}, \Gamma + 1\}$. The corresponding arguments are

$$\Lambda = \sum_{l=0}^{u_{j+1}-1} \pi_l^* S_{j,l} + \sum_{l=u_{j+1}}^{j} \alpha_{j+1,l} S_{j,l} \le \lceil i_{j+1}/q \rceil \le i_j \quad \text{(by using (4.7))},$$

$$e_l = \pi_l^* - \alpha_{j+1,l} \quad \text{for} \quad u_{j+1} \le l \le j,$$

$$L = \max \left\{ \delta : i_j \ge \Lambda + \sum_{l=u_{j+1}}^{\delta} e_l S_{j,l} \right\} \ \text{(if } \delta \text{ doesn't exist, let } L = u_{j+1} - 1),$$

$$g = i_j - \Lambda - \sum_{l=u_{j+1}}^{L} e_l S_{j,l}.$$

Denote $g_1 = \lfloor g/S_{j,L+1} \rfloor$ and $g_2 = g - g_1 S_{j,L+1}$; we have

$$\begin{aligned} i_j &= \Lambda + \sum_{l=u_{j+1}}^{L} e_l S_{j,l} + g \\ &= \sum_{l=0}^{L} \pi_l^* S_{j,l} + ((a_{j+1,L+1} + g_1) S_{j,L+1} + g_2) + \sum_{l=L+2}^{j} a_{j+1,l} S_{j,l} \\ &= \sum_{l=0}^{L} \pi_l^* S_{j,l} + \sum_{l=L+1}^{j} \alpha_{j,l} S_{j,l} + \lambda_{j,L+1}, \end{aligned} \tag{4.8}$$

where $\alpha_{j,L+1} = \alpha_{j+1,L+1} + g_1$, $\alpha_{j,l} = \alpha_{j+1,l}$ for $L+2 \leq l \leq j$ and $\lambda_{j,L+1} = g_2 < S_{j,L+1}$. Note that in (4.8) the coefficients are greater than or equal to those of Λ and less than or equal to those of T_j. In addition,

$$u_j = \eta_j = L + 1 \geq u_{j+1} = \eta_{j+1}. \tag{4.9}$$

Step 2. By analyzing two cases of (4.9), we know that $R(E_j, E_{j+1})$ is true.

- If $L + 1 > u_{j+1}$, then it is easy to verify that $R(E_j, E_{j+1})$ is true.
- Assume that $L+1 = u_{j+1}$. By using (2.11), we have $i_j - i_{j+1}/q \geq S_{j,0}$. Then $g = i_j - \Lambda \geq i_j - \lceil i_{j+1}/q \rceil \geq S_{j,0}$ and

$$g_1 \geq \lfloor S_{j,0}/S_{j,L+1} \rfloor \geq 1, \tag{4.10}$$

which implies that $R(E_j, E_{j+1})$ is true.

Step 3. In Step 1, we construct E_j from E_{j+1} by induction when E_{j+1} satisfies (4.7). For E_{k-2}, condition (4.7) is obvious since $[\theta_{\Gamma+1}, \ldots, \theta_{k-2}]$ is type II. Now we should make E_j have the same property, where $\Gamma + 2 \leq j \leq k - 3$.

Suppose E_{j+1} has property (4.7), and E_j is obtained in Step 1. In the following two cases, we present a method to make $[\alpha_{j,u_j^*}, \ldots, \alpha_{j,j}]$ a type II expression. Note that u_j^* denotes $\max\{\mu_j, \Gamma + 1\}$.

- *Case* 1. $u_j^* < j$.
  - If $u_j < \Gamma+1$, then, from (4.8) and (4.9), we know that $u_j^* = \Gamma+1 = u_{j+1}^*$, and $[\alpha_{j,u_j^*}, \ldots, \alpha_{j,j}] = [\alpha_{j+1,u_{j+1}^*}, \ldots, \alpha_{j+1,j}]$ is type II.
  - If $u_j \geq \Gamma + 1$, then $u_j^* = u_j$. Let

$$[\alpha'_{j,u_j}, \ldots, \alpha'_{j,j}] = D[\alpha_{j,u_j}, \ldots, \alpha_{j,j}].$$

Then it is easy to verify that $[\alpha'_{j,u_j}, \ldots, \alpha'_{j,j}]$ is type II since $[\alpha_{j,u_j+1}, \ldots, \alpha_{j,j}] = [\alpha_{j+1,u_j+1}, \ldots, \alpha_{j+1,j}]$ is type II. Now, we get a new expression for i_j:

$$E'_j: \quad i_j = \sum_{l=0}^{u_j-1} \pi_l^* S_{j,l} + \sum_{l=u_j}^{j} \alpha'_{j,l} S_{j,l} + \lambda_{j,u_j}. \tag{4.11}$$

E_j can be replaced with E'_j since $R(E'_j, E_{j+1})$ is true and $[\alpha'_{j,u_j}, \ldots, \alpha'_{j,j}]$ is type II.

- *Case* 2. $u_j^* = j = u_j > \Gamma + 1$. Now E_j has the form $i_j = \sum_{l=0}^{j-1} \pi_l^* S_{j,l} + \alpha_{j,j}$ and $\lambda_{j,\eta_j} = 0$. In order to have the type II property as before, E_j should be replaced with a new expression:

$$\widetilde{E}_j: \quad i_j = \sum_{t=0}^{j-2} \pi_t^* S_{j,t} + \widetilde{\alpha}_{j,j-1} S_{j,j-1} + \widetilde{\alpha}_{j,j}, \tag{4.12}$$

where $[\widetilde{\alpha}_{j,j-1}, \widetilde{\alpha}_{j,j}] = D[\pi_{j-1}^*, \alpha_{j,j}]$, i.e.,

$$\widetilde{\alpha}_{j,j-1} = \pi_{j-1}^* - \Delta_j, \;\; \widetilde{\alpha}_{j,j} = \alpha_{j,j} + (q-1)\Delta_j, \;\; \Delta_j = \lfloor (\pi_{j-1}^* - \alpha_{j,j})/q \rfloor.$$

It is easy to see that $[\widetilde{\alpha}_{j,j-1}, \widetilde{\alpha}_{j,j}]$ is type II. However, we do not know if $R(\widetilde{E}_j, E_{j+1})$ is true. In order to make $R(\widetilde{E}_j, E_{j+1})$ true, all of the expressions $E_l (j \le l \le \omega)$ should be changed, where ω is the integer such that

$$j = u_j = u_{j+1} = \cdots = u_\omega > u_{\omega+1}.$$

The new expressions for i_l are given by

$$\widetilde{E}_l: \quad i_l = \sum_{t=0}^{j-2} \pi_t^* S_{l,t} + \sum_{t=j-1}^{l} \widetilde{\alpha}_{l,t} S_{l,t} + \lambda_{l,\eta_l} \quad \text{for} \quad j \le l \le \omega, \tag{4.13}$$

where $\widetilde{\alpha}_{l,j-1} = \pi_{j-1}^* - \Delta_j$ and $\widetilde{\alpha}_{l,t} = \alpha_{l,t} + (q-1)\Delta_j$ for $j \le t \le l$. λ_{l,η_l} is the same as the corresponding term in E_l. Let $\widetilde{u}_l = j-1$. It is easy to verify that $R(\widetilde{E}_l, \widetilde{E}_{l+1})$ is true for $j \le l \le \omega - 1$ and that $R(\widetilde{E}_\omega, E_{\omega+1})$ is also true.

Now, the induction given by Steps 1, 2, and 3 ends the proof. □

Note that, when we construct E_j from E_{j+1} by induction, if Case 2 of Step 3 occurs, then Case 1 of Step 3 will not appear in the next cycle. This is because, in the next cycle, the expression for i_{j-1} obtained by using Step 1 has the form $E_{j-1} : i_{j-1} = \sum_{l=0}^{j-2} \pi_l^* S_{j-1,l} + \alpha_{j-1,j-1}$.

5. Proofs of Theorem 2.4 and two corollaries. The proof of Theorem 2.4 is based on Theorem 2.3 and the following three lemmas: Lemmas 5.1, 5.2, and 5.3. Lemma 5.1 leads to the first part of Theorem 2.4. It tells us how to make use of Theorem 2.3.

LEMMA 5.1. *For a chain permissible sequence* $(a_1, \ldots, a_k)$ *and a fixed integer* $\Gamma : 0 \le \Gamma \le k-4$, *if*

$$\pi_\Gamma \ge (k-2)q \quad \textit{and} \quad \iota_{k-2} \ge (k-2)(q-1),$$

then there exist integers $\theta_0 \ge \theta_1 \ge \cdots \ge \theta_{k-2} \ge 0$ *such that*

$$i_{k-2} = \sum_{l=0}^{k-2} \theta_l S_{k-2,l}, \quad \textit{where} \quad \theta_l \le \pi_l \quad \textit{for} \quad 0 \le l \le \Gamma. \tag{5.1}$$

Proof. The proof of Lemma 5.1 is given in two steps. In the first step, an initial expression for i_{k-2} is presented in (5.3). In the second step, the parameters $\theta_0, \ldots, \theta_{k-2}$ satisfying (5.1) are obtained in (5.5) and (5.8), respectively. Denote

$$z = \max\left\{ \rho : \; i_{k-2} \ge \sum_{r=0}^{\rho} \pi_r^* S_{k-2,r} \right\} \quad (\text{if } i_{k-2} < \pi_0^* S_{k-2,0}, \text{ let } z = -1), \tag{5.2}$$

where π_r^* is defined in (3.12). If $z = k-2$ or $k-3$, by using Lemma 3.3, the proof is trivial since we can select $\theta_r = \pi_r^*$ for $0 \le r \le k-3$ and $\theta_{k-2} \le \pi_{k-2}^*$. In the following paragraphs, the proof is presented for $z \le k-4$.

First, by using (5.2), an initial expression for i_{k-2} is obtained:

$$i_{k-2} = \sum_{r=0}^{z} \pi_r^* S_{k-2,r} + \sum_{r=z+1}^{k-2} \sigma_r S_{k-2,r}, \tag{5.3}$$

where $\sigma_{z+1}, \ldots, \sigma_{k-2}$ are nonnegative integers such that

$$\begin{array}{lll} \sigma_{z+1} & < & \pi_{z+1}^*, \\ \sigma_r & < & S_{k-2,r-1}/S_{k-2,r} = q \quad \text{for} \quad z+2 \le r \le k-3, \\ \sigma_{k-2} & < & S_{k-2,k-3}/S_{k-2,k-2} = q-1. \end{array}$$

In particular, for $z = -1$, we have $i_{k-2} = \sum_{r=0}^{k-2} \sigma_r S_{k-2,r}$, where σ_0 is selected as $\lfloor i_{k-2}/S_{k-2,0} \rfloor$. From the condition $i_{k-2} \ge (k-2)qS_{k-2,0}$, we know that

$$\sigma_0 \ge (k-2)q \quad \text{when} \quad z = -1. \tag{5.4}$$

Second, by adjusting (5.3) in the following two cases, (5.1) is obtained in (5.5) and (5.8), respectively.

- Assume that $\sigma_{z+1} \ge \sigma_{z+2} + (k-z-4)q + 1$; then we have

$$\begin{cases} \theta_r & = & \pi_r^* \quad \text{for} \quad 0 \le r \le z, \\ \theta_{z+1} & = & \sigma_{z+1} - (k-z-4), \\ \theta_r & = & \sigma_r + (k-r-2)q - (k-r-3) \;\; \text{for} \;\; z+2 \le r \le k-3, \\ \theta_{k-2} & = & \sigma_{k-2}. \end{cases} \tag{5.5}$$

  This assumption implies that $\theta_{z+1} \ge \theta_{z+2}$. In addition, the condition $(k-2)q \le \pi_\Gamma$ implies that $\theta_{z+2} \le \pi_\Gamma$ since $\theta_{z+2} \le (k-2)q$.
- Assume that $\sigma_{z+1} < \sigma_{z+2} + (k-z-4)q + 1$; we have

$$z \ge 0. \tag{5.6}$$

  If $z = -1$, then the assumption denotes that $\sigma_0 < \sigma_1 + (k-3)q + 1 \le (k-2)q$, which is opposite to (5.4). Let $\mu = \lfloor \sigma_{z+1}/q \rfloor$; then

$$k - z - \mu - 3 \ge 0. \tag{5.7}$$

  If $k-z-\mu-3 < 0$, then the assumption denotes that $\sigma_{z+1} < (k-z-3)q \le \mu q$, which is also impossible. By using (5.6), (5.7), and the condition $(k-2)q \le \pi_\Gamma$, we have

$$\begin{cases} \theta_r & = & \pi_r^* \quad \text{for} \quad 0 \le r \le z-1, \\ \theta_z & = & \pi_z^* - (k-z-\mu-3), \\ \theta_{z+1} & = & \sigma_{z+1} + (k-z-\mu-3)q - (k-z-4), \\ \theta_r & = & \sigma_r + (k-r-2)q - (k-r-3) \;\; \text{for} \;\; z+2 \le r \le k-3, \\ \theta_{k-2} & = & \sigma_{k-2}. \end{cases} \tag{5.8}$$

  Note that, in (5.8), since $\theta_{z+1} = (\sigma_{z+1} - \mu q) + (k-z-3)q - (k-z-4)$, we have

$$(k-z-2)q - (k-z-3) \ge \theta_{z+1} \ge (k-z-3)q - (k-z-4),$$

  which implies that $\theta_z \ge \theta_{z+1} \ge \theta_{z+2}$ and $\pi_\Gamma \ge \theta_{z+1}$. ∎

In the following lemma, i.e., Lemma 5 of [1], a relation between the parameters ι_l and π_l is introduced. Using this lemma, the second part of Theorem 2.4 can be obtained from the first part of Theorem 2.4.

LEMMA 5.2 (see [1]). *For a chain permissible sequence with dimension k, if there exists a positive integer l such that $\iota_r = \iota_{r+1} + \delta_r$ for $0 \le r \le l-1$, then*

$$\iota_l = \pi_l + \sum_{r=0}^{l-1} (\delta_r(q^{r+1} - 1) + qp_r - p_{r+1}). \tag{5.9}$$

Lemma 5.3, a special case of Lemma 8 of [6], allows us to pay attention to some special chain permissible sequences.

LEMMA 5.3. *For fixed nonnegative integers $l(\le k-1)$, s, and F, if each chain permissible sequence such that*

$$\iota_j = \iota_{j+1} + \delta_j \quad for \quad 0 \le j \le l-1, \tag{5.10}$$

$$\iota_l \ge s + \sum_{r=0}^{l-1} (\delta_r(q^{r+1} - 1) + qp_r - p_{r+1}), \tag{5.11}$$

$$i_{j-1} \ge i_j/q + S_{j-1,0} \quad for \quad l+2 \le j \le k-3, \tag{5.12}$$

$$i_{k-2} \ge F \tag{5.13}$$

is a chain good weight hierarchy, then the chain permissible sequences which satisfy only (5.11), (5.12), *and* (5.13) *are chain good weight hierarchies.*

Proof of Theorem 2.4. For $0 \le \Gamma \le k-4$, by Lemma 5.1 and Theorem 2.3, we know that the chain permissible sequences such that

$$\pi_\Gamma \ge (k-2)q, \quad \iota_{k-2} \ge (k-2)(q-1),$$

and

$$i_{j-1} \ge i_j/q + S_{j-1,0} \quad \text{for} \quad \Gamma + 2 \le j \le k-3 \tag{5.14}$$

are chain good weight hierarchies. This is the first part of Theorem 2.4. Then by Lemma 5.2, the chain permissible sequences such that

$$\begin{aligned} &\iota_u = \iota_{u+1} + \delta_u \quad \text{for} \quad 0 \le u \le \Gamma - 1, \\ &\iota_\Gamma \ge (k-2)q + \sum_{r=0}^{\Gamma-1} (\delta_r(q^{r+1} - 1) + qp_r - p_{r+1}), \\ &i_{k-2} \ge (k-2)qS_{k-2,0}, \\ &i_{j-1} \ge i_j/q + S_{j-1,0} \quad \text{for} \quad \Gamma + 2 \le j \le k-3 \end{aligned} \tag{5.15}$$

are chain good weight hierarchies. Finally, by using Lemma 5.3 with parameters $l = \Gamma$, $s = (k-2)q$, and $F = (k-2)qS_{k-2,0}$, the second part of this theorem is obtained. Note that, for $\Gamma = k-4$, conditions (5.14) and (5.15) do not exist. ◻

Proof of Corollary 2.5. Corollary 2.5 follows from the second part of Theorem 2.4. Condition (2.15) is satisfied by (2.17). Condition (2.16) can be obtained by using (2.18) and the inequality $i_{j-1}/q^{j-1} \ge \iota_{j-1} \ge \iota_j + 2 \ge i_j/q^j + 1$, where $\Gamma + 2 \le j \le k-3$. We will show that the condition (2.14) is also satisfied.

For a chain permissible sequence $(a_1, \ldots, a_k)$, it follows from (2.7) and (2.18) that

$$\iota_\Gamma \ge \iota_{\Gamma+1} \ge \iota_{k-3} + 2(k-4-\Gamma) \ge \iota_{k-2} + 2(k-4-\Gamma). \tag{5.16}$$

Then by using (5.16) and (2.17), we have

$$\iota_\Gamma \geq (k-2)(q-1) + \sum_{r=0}^{\Gamma-1} q^{r+1} + 2(k-4-\Gamma) \geq (k-2)q + \sum_{r=0}^{\Gamma-1}(q^{r+1}-2)$$

since $k \geq \Gamma + 6$. Therefore, (2.14) is satisfied since

$$q^{r+1} - 2 \geq \delta_r(q^{r+1}-1) + qp_r - p_{r+1}. \qquad \square$$

The proof of Corollary 2.6 uses the same arguments as that of Corollary 2.5.

6. Improvements on [1] and [6]. Theorem 2.4 presents a series of sufficient conditions for determining the chain good weight hierarchies by using different Γ's. In this section, using Theorem 2.4, we find many new chain good weight hierarchies, which cannot be investigated using Theorems 2.1 and 2.2. For $q = 3$ and $k = 6, 7, 8$, three examples of the improvements are given by using Corollaries 6.1, 6.2, and 6.3, respectively.

Let $(a_1, \ldots, a_k)$ be a chain permissible sequence and let Γ be an integer such that $0 \leq \Gamma \leq k-4$. From the second part of Theorem 2.4, we know that $(a_1, \ldots, a_k)$ is chain good if (2.14), (2.15), and (2.16) are satisfied. Since $\delta_r(q^{r+1}-1)+qp_r-p_{r+1} \leq q^{r+1}-2$, it is easy to see that a chain permissible $(a_1, \ldots, a_k)$ is chain good if

$$\iota_\Gamma \geq (k-2)q + \sum_{r=0}^{\Gamma-1}(q^{r+1}-2),$$
$$\iota_{k-2} \geq (k-2)(q-1),$$

and

$$i_{j-1} \geq i_j/q + S_{j-1,0} \quad \text{for} \quad \Gamma + 2 \leq j \leq k-3.$$

Then, Corollaries 6.1, 6.2, and 6.3 are obtained for $q = 3$ and $\Gamma = k - 5 = 1$, $\Gamma = k - 6 = 1$, and $\Gamma = k - 7 = 1$, respectively.

COROLLARY 6.1. *For $q = 3$ and $k = 6$, a chain permissible sequence is a chain good weight hierarchy if*

(6.1) $$\iota_1 \geq 13, \quad \iota_4 \geq 8, \quad \textit{and} \quad i_2 \geq i_3/3 + 6.$$

Example. From Corollary 6.1, we find that, for each pair of parameters (i_3, i_4) such that $648 \leq i_4 \leq 1997$ and $i_4/3 \leq i_3 \leq 695$, there exist many new chain good weight hierarchies which cannot be investigated using Theorems 2.1 and 2.2. For instance, if $i_4 = 648$ and $i_3 = 216$, all the corresponding chain permissible sequences with dimension 6 such that $i_2 \in \{115, 116\} \bigcup \{l : 120 + 9t \leq l \leq 125 + 9t, 0 \leq t \leq 13\}$ are new chain good weight hierarchies.

COROLLARY 6.2. *For $q = 3$ and $k = 7$, a chain permissible sequence is a chain good weight hierarchy if*

(6.2) $$\iota_1 \geq 16, \quad \iota_5 \geq 10, \quad \textit{and} \quad i_{j-1} \geq i_j/3 + 2 \cdot 3^{j-2} \textit{ for } j = 3, 4.$$

Example. From Corollary 6.2, we find that, for each pair of parameters (i_4, i_5) satisfying $2430 \leq i_5 \leq 19013$ and $i_5/3 \leq i_4 \leq 6419$, many new chain good weight hierarchies cannot be checked with Theorems 2.1 and 2.2. For instance, if $i_5 = 2430$

and $i_4 = 810$, all the corresponding chain permissible sequences with dimension 7 such that $i_2 \geq i_3/3 + 6$ and $i_3 \in \{l : 409 + 27t \leq l \leq 431 + 27t, 0 \leq t \leq 65\}$ are new chain good weight hierarchies.

COROLLARY 6.3. *For $q = 3$ and $k = 8$, a chain permissible sequence is a chain good weight hierarchy if*

$$\iota_1 \geq 19, \quad \iota_6 \geq 12, \quad \textit{and} \quad i_{j-1} \geq i_j/3 + 2 \cdot 3^{j-2} \textit{ for } j = 3, 4, 5. \tag{6.3}$$

Example. From Corollary 6.3, we find that, for each pair of parameters (i_5, i_6) such that $8748 \leq i_6 \leq 174695$ and $i_6/3 \leq i_5 \leq 58475$, there are also many new chain good weight hierarchies which cannot be investigated using Theorems 2.1 and 2.2. For instance, if $i_6 = 8748$ and $i_5 = 2916$, all the corresponding chain permissible sequences with dimension 8 such that $i_2 \geq i_3/3 + 6$, $i_3 \geq i_4/3 + 18$, and $i_4 \in \{l : 1405 \leq l \leq 1457\} \bigcup \{l : 1463 + 81t \leq l \leq 1538 + 81t, 0 \leq t \leq 224\}$ are new chain good weight hierarchies.

7. Conclusion. The determination of chain good weight hierarchies was studied several years ago. For the binary codes with dimension up to 5 and the ternary codes with dimension up to 4, the problem was solved in [3] and [2], respectively. As for linear codes with general dimension over $GF(q)$, some research was done in [1] and [6]. However, these results are not efficient for the determination of the chain good weight hierarchies with high dimension since in many cases the lower bounds on the conditions for $\iota_0, \ldots, \iota_{k-3}$(or ι_{k-2}) increase exponentially with the dimension k. In this paper, we present a method to deal with the high dimension cases; see Corollaries 2.5 and 2.6. Our lower bounds on the conditions for $\iota_0, \ldots, \iota_{k-2}$ only increase linearly with the dimension k.

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