Coding Techniques and the Two-Access Channel

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Abstract. We consider some examples of two-user access channels that play a role in information theory. We concentrate on two non-trivial models, the 2-adder and the switching channel model, respectively. For these channel models we discuss the capacity for the feedback and feedback free situation. We also give examples of coding methods that approach or achieve the capacity.

Keywords. two-user access, adder channel, switching channel, coding, capacity, feedback

1. Introduction

Information theory considers the capacity or maximum mutual information of multi-access channels (MAC) for a fixed amount (normally 2) of users. Although the practical importance is limited, a number of interesting basic ideas and concepts can be developed. In the Introduction we give the capacity regions for the two-user access channels. The general model is given in Figure 1.

The channel capacity for a two-access channel without feedback is given in the following theorem, due to Ahlswede [1] and Liao [2].

Theorem 1. *The capacity of a memoryless two-access channel is the closure of the convex hull of all rate pairs R_1 and R_2 satisfying*
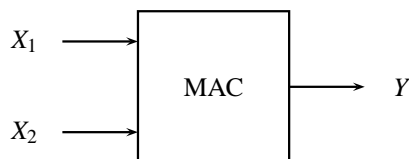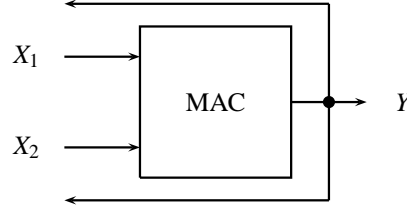

Figure 1. The two input multiple access channel model

Figure 2. The two input multiple access channel model with feedback

$$R_1 \le I(X_1;Y \mid X_2) \tag{1a}$$

$$R_2 \le I(X_2;Y \mid X_1) \tag{1b}$$

$$R_1 + R_2 \le I(X_1,X_2;Y) \tag{1c}$$

for a probability product distribution $f(x_1,x_2) = f(x_1)f(x_2)$.

Cover and Leung [3] found an achievable rate region for the discrete memoryless MAC with feedback, using superposition coding. Subsequently, Willems [4] showed that for MACs for which at least one input is a function of the output Y and the other input (class D), the region found by Cover and Leung *is* indeed the feedback capacity region.

Theorem 2. *The feedback capacity region of MACs in class D is given by*

$$R(D) := \{(R_1,R_2) :$$

$$0 \le R_1 \le I(X_1;Y \mid X_2,U) \tag{2a}$$

$$0 \le R_2 \le I(X_2;Y \mid X_1,U) \tag{2b}$$

$$R_1 + R_2 \le I(X_1,X_2;Y) = H(Y)\} \tag{2c}$$

$$\text{for } P(u,x_1,x_2,y) = P(u)P(x_1 \mid u)P(x_2 \mid u)P(y \mid x_1,x_2),$$

$$|U| \le \min(|X_1| \cdot |X_2| + 1, |Y| + 2),$$

where $P(u,x_1,x_2,y)$ is a joint probability distribution and X_i denotes the alphabet size of the random variables U, X_1 and X_2, respectively.

This capacity region is sufficient for the types of multiple access channels that we consider in the next sections. One of the information theoretical problems is the evaluation of (1) and (2) for different types of channels. An interesting problem is the design of coding methods that achieve or approach the capacity. These problems are in general very difficult and only to solve for small examples and a limited number of inputs. Another mathematical problem is to calculate the maximum obtainable rate when the decoding error probability is exactly 0.

Definition 1. *The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error.*
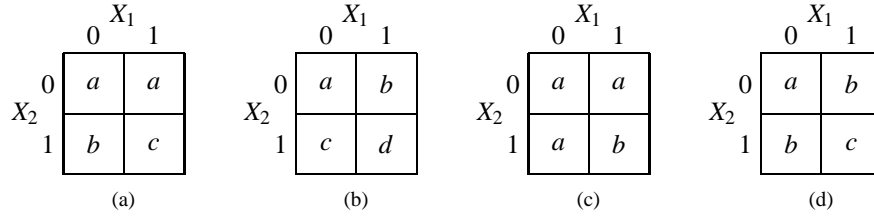
Figure 3. Four equivalence classes of binary input 2-access channels

The next section considers the different possible non-trivial channel models that can occur. Then, we discuss the capacity and coding techniques for the only two non-trivial cases: the 2-adder and the switching channel.

2. Access Models

In [5], we consider the possible equivalence classes of binary input 2-access channels. Two 2-access channels are equivalent if one channel can be converted into the other by renaming the inputs and/or outputs of the channel. It is easy to see that four non-trivial classes remain, see Figure 3.

The output of the access channels can be 2-, 3-, or 4-ary, respectively. For channel 3 (b) the output is 4-ary, and thus the inputs are uniquely specifyed by the output. Channel 3 (c) has a binary output. By using time sharing (Time Division Multiple Access, TDMA), the maximum sum rate equal to 1 bit per transmission can be achieved. The channels from Figures 3 (a) and (d) are ternary output channels. The maximum output entropy is less than or equal to $\log_2 3$. We conclude that the channels from Figure 3 (a) and (d) are the two channels that need further investigation with respect to channel capacity. We will also investigate coding techniques that achieve the capacity of these channels for the situations: no feedback; feedback; zero error probability. The channel models of Figure 3 can also be seen as combinational circuits. This is illustrated in Figure 4, where we give several input/output combinations.

3. The 2-Adder Channel

The ADD+CARRY channel from Figure 4 is known as the 2-adder channel. Its transition diagram is given in Figure 5.

We first consider the capacity region as given in (1). Then we discuss the feedback capacity region, followed by some examples of error free access codes. Using the product input probability distribution, we can evaluate (1) as:

$$R_1 \leq I(X_1;Y \mid X_2) = H(X_1 \mid X_2) - H(X_1 \mid Y,X_2) = H(X_1) \leq 1 \qquad (3a)$$

$$R_2 \leq I(X_2;Y \mid X_1) = H(X_2 \mid X_1) - H(X_2 \mid Y,X_1) = H(X_2) \leq 1 \qquad (3b)$$

$$R_1 + R_2 \leq I(X_1,X_2;Y) = H(Y) - H(Y \mid X_1,X_2) = H(Y) \qquad (3c)$$

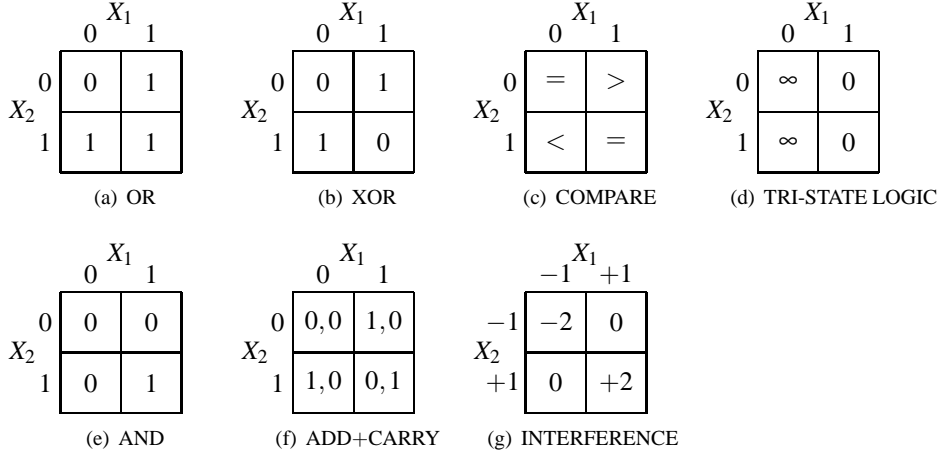The capacity region is given in Figure 6.

Figure 4 tables

(a) OR
X_2 \ X_1	0	1
0	0	1
1	1	1

(b) XOR
X_2 \ X_1	0	1
0	0	1
1	1	0

(c) COMPARE
X_2 \ X_1	0	1
0	=	>
1	<	=

(d) TRI-STATE LOGIC
X_2 \ X_1	0	1
0	∞	0
1	∞	0

(e) AND
X_2 \ X_1	0	1
0	0	0
1	0	1

(f) ADD+CARRY
X_2 \ X_1	0	1
0	0,0	1,0
1	1,0	0,1

(g) INTERFERENCE
X_2 \ X_1	-1	$+1$
-1	-2	0
$+1$	0	$+2$

Figure 4. Combinational circuit seen as two-access channel

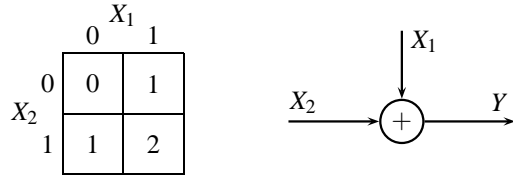| X_2 \ X_1	0	1
0	0	1
1	1	2

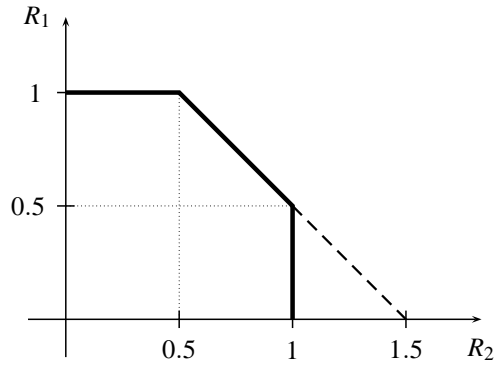**Figure 5.** Transition diagram for the 2-adder channel

Figure 6. Capacity region for the binary 2-adder channel

The point $(R_1, R_2) = (1, 0.5)$ can be achieved as follows. Suppose that transmitter 1 transmits information with $P(0) = P(1) = 0.5$. Hence, $I(X_1; Y \mid X_2) = H(X_1) - H(X_1 \mid Y, X_2) = 1$ bit per transmission. For the other user the channel looks like an erasure channel, see Figure 7. The capacity for this channel is 0.5 bit per transmission. According to classical information theory, we can achieve this single channel capacity with error probability going to zero. Hence, the points $(1, 0.5)$ and $(0.5, 1)$ or $R_1 + R_2 = 1.5$
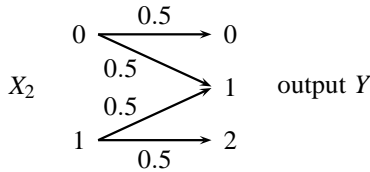
Figure 7. Binary erasure channel for user 2

bits/transmission can be achieved. The points on the straight line can also be achieved by using time sharing or Time Division Multiple Access (TDMA).

For $P(x_1 = 0) = P(x_2 = 0) = \frac{1}{2}$, the output probabilities $P(y = 0) = P(y = 2) = \frac{1}{4}$, respectively. Hence, the entropy $H(Y) = 1.5$ bit/channel use. In general, one can show [6], that for any product input distribution for binary X_1 and binary X_2, the maximum output entropy $H(Y) = 1.5$ bit/channel use. The proof is based on taking partial derivatives. This shows that Figure 6 indeed represents the capacity region of the 2-adder channel. The capacity region specifies the total amount of information that can be transmitted with a vanishing small probability of error. In literature, many researchers also investigate the development of *error free* codes for particular multi-access situations. We now give an example of these efforts.

Example 1. *For the 2-adder channel, Figure 5, the following code achieves a total efficiency of* $0.5 \log_2 6 = 1.29$ *bits/transmission. The code is uniquely decodable.*

Table 1. Code book for the 2-adder channel

		X_1		
		11	10	01
X_2	00	11	10	01
	11	22	21	12

It is very difficult to improve on this short length codebook. The problem is to design codebooks for both users such that the sum of two codewords gives a unique output Y. Coebergh and van Tilborg [7], developed a strategy with sum rate 1.30565. The so called zero-capacity region is still an open problem. The obtained code efficiency can be used as a lower bound to the zero-error capacity region. Since the rate points $(0.5, 1.29)$ and $(1.29, 0.5)$ can be achieved, the connecting straight line can also be achieved, by using the time sharing argument.

Gaarder and Wolf [8] showed that feedback may increases the capacity region of the multi-access channel. They used the 2-adder channel as a demonstrating example and developed a simple two stage coding strategy.

- During the first stage the channel accepts N independent input digits from both users. On the average $\frac{N}{2}$ ambiguous receptions are known to both users, due to the presence of the feedback links. Note: we can also assume transmission until exactly $\frac{N}{2}$ ambiguous transmissions occur (value $y = 1$).
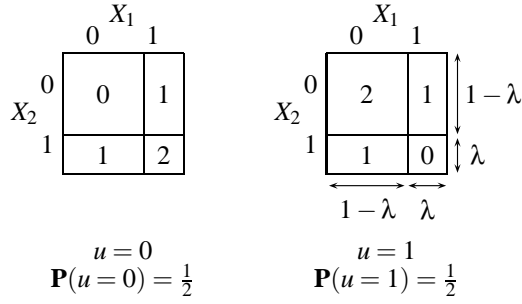
Figure 8. The input/output relation for the 2-adder channel dependent on U

- After a block of N transmissions, both encoders transmit at the cooperative channel capacity of $\log_2 |Y| = \log_2 3$ bits per transmission in order to resolve the receiver's uncertainty. In total cooperation, they can choose the output to be 0, 1, or 2.

The sum rate is thus expected to be on the average

$$R = \frac{2N}{N + \frac{\frac{N}{2}}{\log_2 3}} = 1.52 \, \text{bit/transmission.}$$

For the feedback situation we can use the capacity region as defined in (2). Using the properties of the 2-adder channel, we obtain

$$0 \leq R_1 \leq I(X_1; Y \mid X_2, U) = H(X_1 \mid U) \tag{4a}$$

$$0 \leq R_2 \leq I(X_2; Y \mid X_1, U) = H(X_2 \mid U) \tag{4b}$$

$$R_1 + R_2 \leq I(X_1, X_2; Y) = H(Y) \tag{4c}$$

$$\text{for } P(u, x_1, x_2, y) = P(u)P(x_1 \mid u)P(x_2 \mid u)P(y \mid x_1, x_2), \quad |U| \leq 5.$$

The problem here is that we do not know the distribution nor the cardinality of the random variable U. We will give an intuitive interpretation of U and show its practical importance.

From [6], we know that, without feedback, for any product input distribution for binary X_1 and binary X_2, the maximum output entropy $H(Y) = 1.5$ bit/channel use. However, the maximum value for H(Y) is $H(Y) = \log_2 3$ bit for equally likely outputs Y. We show, that the variable U plays a crucial role in the attempt to increase the output entropy.

In Figure 7, we illustrate the input distribution for X_1, X_2 and U, using the unit squares, that show that $P(Y = 0) = P(Y = 1) = P(Y = 2) = \frac{1}{3}$ is indeed possible for $\lambda = 0.2113$. The lengths of the intervals along the unit squares correspond to the respective probabilities. For $\lambda = 0.2113, H(Y) = log_2 3 = 1.584$ and $H(X_1) = H(X_2) = h(\lambda) = 0.744$. Hence, the sum rate $H(X_1) + H(X_2) = 1.488$. Willems [9] optimized the sum rate and found that for $\lambda = 0.24, H(Y) = 2, h(\lambda) = 1.5822$ bits/transmission.

Suppose that both users have *common knowledge*. Then, the two users can cooperate and use the U to create an artificial channel to the receiver and transmit $I(U; Y)$ bits of
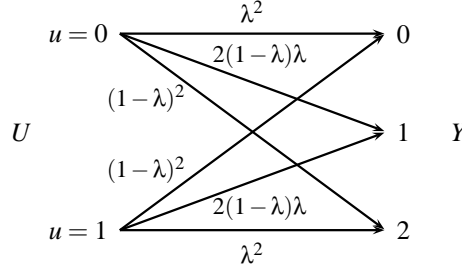
Figure 9. Artificial channel from U to Y

information per channel use about the common knowledge, see Figure 9. The capacity $C_{U \to Y}$ of the channel is

$$C_{U \to Y} = (1 - 2\lambda(1 - \lambda))\left(1 - h\left(\frac{\lambda^2}{\lambda^2 + (1-\lambda)^2}\right)\right).$$

The question is: how do the two users get their common knowledge? Suppose that user 1 and user 2 use the unit square for $u = 0$ for N subsequent transmission. After these N transmissions, the receiver has $2\lambda(1 - \lambda)N$ bits of uncertainty whether the input pair $(X_1, X_2) = (0, 1)$ or $(X1, X2) = (1, 0)$ occurred. Since we assume feedback, the transmitters know this uncertainty. In the following block of N transmissions, they use the artificial channel to transmit this common knowledge to the receiver

This leads to the following strategy of transmission:

- repeat the transmission, i.e., the same input symbol, as long as erasures $(y = 1)$ occur. This reduces the efficiency to $(1 - 2\lambda(1 - \lambda))$;
- use a constructive feedback scheme for the binary symmetric channel with transition probability $f = 1 - \left(\lambda^2/(\lambda^2 + (1 - \lambda)^2)\right)$. This strategy will then deliver U.

We are able to transmit the uncertainty of block $I - 1$ in the next block I, if $C_{U \to Y} > 2\lambda(1 - \lambda)$. For this, we use the constructive feedback scheme that Schalkwijk [10] developed. The performance, R_s curve for this scheme, where each erroneously received digit is repeated 4 times, is given by the straight line

$$R_s = \frac{0.25 - f}{0.25 - 0.0804} \cdot 0.5966 \text{ bit/transmission.}$$

For $\lambda = 0.7625, 2\lambda(1 - \lambda) < (1 - 2\lambda(1 - \lambda))R_s$, and the error probability in U goes to zero for large values of N. For this value of $\lambda, 2h(\lambda) = 1.5819$. A more complex strategy could yield the optimum sum result of 1.584 bits/transmission, see Zigangirov [11].

We now give a suboptimal strategy, that operates beyond the point 1.5 bits/transmission. The coding method can be described as follows.

- Both users transmit a block of N pairs of symbols (i.e., $2N$ transmissions), each with probability $\frac{1}{3}$. For every pair, they select the same values for the binary variable U. For the first block they may select $u = 0$ for N pairs. The input/output relation is given in Figure 10. We have 1 bit of uncertainty at the receiver when we receive the symbols $11, 01, 10, 21$ or 12. This uncertainty occurs with probability $\frac{2}{3}$. Hence, a block generates on the average $\frac{2N}{3}$ bits of uncertainty.

<table>
<tr><td colspan="4" align="center">user 1</td><td></td><td colspan="4" align="center">user 1</td></tr>
</table>

	00	01	11				00	10	11
00	00	01	11			**00**	00	10	11
user 2 **01**	01	02	12		user 2	**10**	10	20	21
11	11	12	22			**11**	11	21	22

(a) $u = 0$ (b) $u = 1$

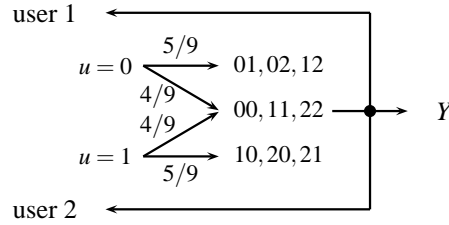**Figure 10.** Two different transmission input schemes

Figure 11. Artificial channel with feedback for jointly transmitting the uncertainty

- In the next block we transmit N *pairs* of symbols as follows. We again use the random variable U, which in our case is binary. The uncertainty that occurs in a block, *known at both* transmitters, can be described with the average of $\frac{2N}{3}$ bits. How do we transmit this uncertainty to the receiver? Depending on the uncertainty (which is now considered to be common information to be transmitted) both encoders select the same input scheme for $u = 0$, or the scheme for $u = 1$, see Figure 10. By doing this, the transmitters create a channel for transmitting the uncertainty to the receiver, see Figure 11. The capacity of the channel is $\frac{5}{9}$ bits/transmission.
- The decoder for block I first decodes the uncertainty for the previous block $I - 1$. Knowing this uncertainty, the decoder can reconstruct the transmitted messages for this particular block $I - 1$.

We can transmit the uncertainty for the previous block with $\frac{5}{9}$ bits/transmission via the artificial channel and $log_2 3$ bits in total cooperation after the end of each block. Since in a block of length $2N$, we have $\frac{2N}{3}$ bits of uncertainty and we resolve only $\frac{5N}{9}$ bits, the last $\frac{N}{9}$ bits are resolved in total cooperation, i.e., in $\frac{N}{9 \log_2 3}$ transmissions. The overall efficiency is

$$R = \frac{2N \log_2 3}{2N + \frac{N}{9 \log_2 3}} = 1.54 \text{ bits/transmission.}$$

As in the non-feedback situation, for the feedback situation we can also design coding methods with *zero error*. For this, we replace the two blocks in Figure 10 as follows

Since both users are assumed to have full feedback, they know the ambiguity of the receiver after every transmission (3 digits). This ambiguity is at maximum 1 bit of information. In the next transmission they can both decide on using the same $u = 0$
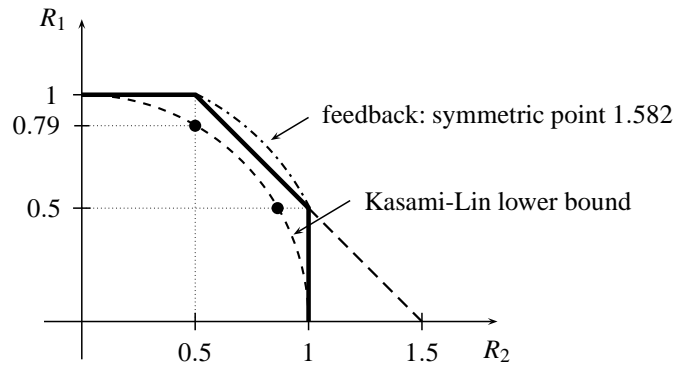
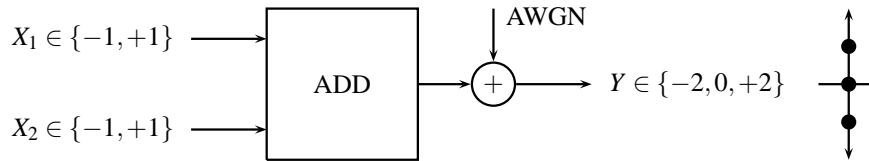**Figure 12.** Summary of the capacity regions for the binary erasure MAC

Figure 13. The two input interference channel model

$$X_1$$

			-1	1	1	1	-1	1	1	1	-1
-1	-1	-1	-2	0	0	0	-2	0	0	0	-2
1	1	1	0	2	2	2	0	2	2	2	0

X_2

Figure 14. A distance 2 code for the interference channel

or $u = 1$, depending on the ambiguity. The outputs of the block are chosen in such a way that the receiver can *always* detect the particular block used by the transmitters and thus solve the ambiguity of the previous triple. The overall sum rate of this scheme is $R = \frac{4}{3} = 1.33$ bits/transmission. It is a challenge to improve the idea of the presented scheme. For this, we have the freedom to increase $|U|$ and the length of the codewords. In [12] Zhang et al., constructed a more complicated method that achieves an overall rate of 1.38 bits/transmission.

In Figure 12 we summarize the capacity regions found for the 2-adder channel. For the zero-error region we use the Kasami-Lin lower bound [13].

It is difficult to find good applications for the 2-adder channel. An interesting idea is to see the 2-adder channel as an interference channel with additive white Gaussian noise. A problem is to design error correcting/detecting codes for this application. We give an example of a distance 2 block code in Figure 14. Research in this direction, using trellis codes, can be found in [14,15].
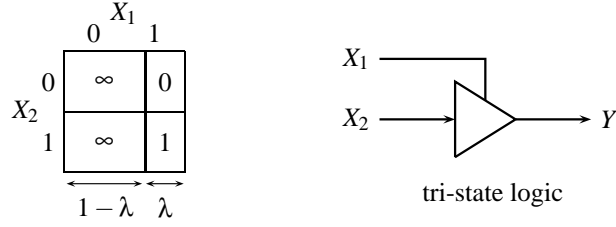
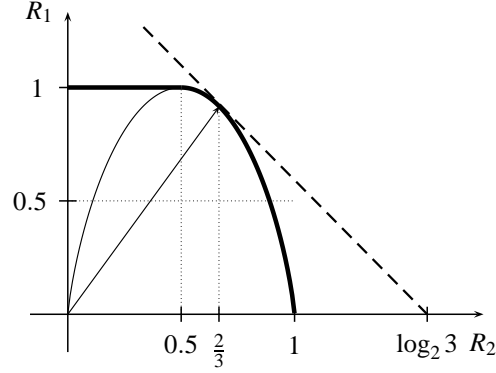**Figure 15.** Transition diagram for the switching channel

Figure 16. Capacity region for the switching channel

4. The Switching Channel

The binary input ternary output channel from Figure 15 is known as the switching channel. It was first described by Vinck in [16]. The transition diagram is given in Figure 15.

We first consider the capacity region as given in (1). Then we discuss the feedback capacity region, followed by some examples of error free access codes. For the two-adder channel, with the product input probability distribution, we can evaluate (1) as:

$$R_1 \leq I(X_1;Y \mid X_2) = H(X_1 \mid X_2) = H(X_1) = h(\lambda) \tag{1a}$$

$$R_2 \leq I(X_2;Y \mid X_1) = H(X_2 \mid X_1) - H(X_2 \mid Y,X_1) = \lambda H(X_2) \leq \lambda \tag{1b}$$

$$R_1 + R_2 \leq I(X_1,X_2;Y) = H(Y) - H(Y \mid X_1,X_2) = H(Y) \leq \lambda + h(\lambda) \tag{1c}$$

where $h(\cdot)$ is the binary entropy function. The capacity region is given in Figure 16.

The capacity region can be achieved in the same way as for the 2-adder channel. User 1 transmits with rate $h(\lambda)$, and is uniquely decodable by the receiver. For User 2 we then have a binary erasure channel with transition probability $(1-\lambda)$ and capacity λ. This together gives the sum rate $\lambda + h(\lambda)$. Note that for $\lambda = \frac{2}{3}$, the sum rate equals the total cooperation value $\log_2 3$. For the switching channel from Figure 15 feedback *does not* increase the capacity region, since

$$R_1 \leq H(X_1 \mid U) \leq \sum_u P_i \alpha_i h(\lambda_i) \leq \overline{\alpha},$$

$$R_2 \leq H(X_2 \mid U) \leq \sum_u P_i h(\alpha_i) \leq h(\overline{\alpha}),$$

$$R_1 + R_2 \leq I(X_1, X_2; Y) = H(Y) \leq \overline{\alpha} + h(\overline{\alpha}),$$

where $\alpha_i = P(X_2 = 1 \mid u = i)$, $\lambda_i = P(X_1 = 1 \mid u = i)$, $P_i = P(u = i)$. This is an example, where the feedback does not help to improve the maximum transmission rate.

Example 2. *P. Vanroose designed code constructions for the switching channel in [17]. These constructions can be explained as follows. Suppose that user 1 specifies up to $(n-k)$ positions in a binary length n vector to be 0. User 2 transmits a vector of length n from a linear code. At the receiver, the 0-positions are recognized as the erased positions from the ∞ symbol. Hence, if the remaining non-erased part uniquely identifies the transmitted vector from user 2, then the decoding problem at the receiver can be solved. For error correcting codes with minimum distance $n-k+1$ (Singleton bound), $n-k$ erasures can be decoded. The total efficiency would then be*

$$R = \frac{k}{n} + \frac{1}{n} \log_2 2^{nh\left(\frac{n-k}{k}\right)} := \lambda + h(\lambda),$$

which equals the capacity boundary. The proof can also be given using combinatorial arguments.

Suppose that we have a $k \times n$ binary matrix $\mathbf{K}$. We let X_1 specify $(n-k)$ columns by selecting at the output the ∞ symbol. User two transmits a linear combination of the k rows of $\mathbf{K}$. If the specified columns are such that the remaining part of the matrix has rank k, then the input of X_2 can be reconstructed from Y. The question is: what is the maximum number of selections X_1 can make for a particular matrix, such that the remaining part has rank k? The answer can be given using the following steps:

1. the number F of invertible $k \times k$ matrices: $F = (2^k - 1)(2^k - 2) \cdots (2^k - 2^{k-1})$;
2. a specification allows: $2^{(n-k)k} \cdot F$ matrices;
3. $F \geq 0.28 \cdot (2^k)^k$, see Tolhuizen [18];
4. for large n, the average number of allowed sequences per matrix is

$$\frac{2^{nh\left(\frac{n-k}{n}\right)} 2^{(n-k)k} F}{2^{nk}} \geq 0.28 \cdot 2^{nh\left(\frac{n-k}{n}\right)}.$$

From this it follows that at least one matrix must have more than the average number of allowable sequences and thus the normalized sum rate

$$R_1 + R_2 \geq \frac{k}{n} + \frac{1}{n} \log_2 0.28 \cdot 2^{nh\left(\frac{n-k}{n}\right)} \stackrel{n \to \infty}{\to} \frac{k}{n} + h\left(\frac{n-k}{n}\right),$$

can be obtained with zero error probability.

From the above we see that we have an example where the ε-error, the feedback and the 0-error capacity regions are the same!

5. Extensions

Extensions of the two user Multiple Access Channel go into several directions. We can increase the number of users, the size of the input alphabet and also the channel transition diagram can be extended.

5.1. Extension 1: Binary input T-user

Example 3. *For the binary input adder MAC with $T = 3$, we have the following code with efficiency 1.5 bits/transmission. Length 2 code for the $T = 3$ binary adder channel:*

$$x_1 \in \{00, 11\}, \ x_2 \in \{10, 01\}, \ x_3 \in \{00, 10\},$$
$$y = x_1 + x_2 + x_3 \in \{10, 20, 01, 11, 21, 31, 12, 22\}.$$

Example 4. *The following code is again for the binary input adder MAC, $T = 3$. The sum rate is $\frac{2+\log_2 6}{3} = 1.53$ bits/transmission. Length 3 code for the $T = 3$ binary adder channel:*

$$x_1 \in \{000, 111\}, \ x_2 \in \{110, 001\}, \ x_3 \in \{000, 001, 010, 100, 011, 101\}.$$
$$y = x_1 + x_2 + x_3 \in \{001, 110, 112, 221, 002, 111, 113, 223, 011, 120, 122, 231, 101,$$
$$210, 212, 321, 012, 121, 123, 232, 101, 211, 213, 322\}.$$

The code is uniquely decodable. More results can be found in Bross et al. [14].

5.2. Extension 2: M-ary input two-user access channel

Two classes of two user channels with M-ary inputs are given in [16]. The output of the first channel indicates the subset of input symbols, i.e.,

$$Y = \{X_1, X_2\}, \ X_1, X_2 \in \{0, 1, \ldots, M-1\}.$$

The receiver does not know the origin of the two symbols if the inputs are different. There are $\binom{M}{2}$ of these ambiguous subsets, see also Figure 17 (a). If both inputs are equal, then only one letter is detected. Hence, the cardinality of the output is $|Y| = \binom{M}{2} + M = \frac{M(M+1)}{2}$. This channel is referred to as the M-ary frequency detecting channel. The second M-ary channel model B gives as output the arithmetic sum of the input letters, i.e.,

$$Y = X_1 + X_2, \ X_1, X_2 \in \{0, 1, \ldots, M-1\}.$$

For this channel the output cardinality is $|Y| = 2M - 1$. This channel is referred to as the adder channel. In Figure 17 we give the input/output relations for both channels when $M = 3$. For $M = 2$ all channels reduce to the well-known 2-adder MAC from Figure 4. The channels (c) and (d) are the erasure and the collision channel, respectively.

The channels in Figure 17 A and B are in class D and the maximum value of $R_1 + R_2 \leq I(X_1, X_2; Y) = H(Y)$ is $\log_2 \frac{M(M+1)}{2}$ and $\log_2(2M - 1)$, respectively. This sum rate can be obtained for $M > 5$, $|U| = M$ for the A channel and $M > 2$, $|U| = 2$ for the (b) channel, respectively. In the evaluation of (2), one needs the maximizing probability $P(u, x_1, x_2, y)$, which could be difficult to obtain for even small values of M. The capacities for the channels in Figure 17 (c) and (d) are open problems.

	X_1					X_1		
	0	1	2			0	1	2
0	{0}	{0,1}	{0,2}		0	0	1	2
X_2 1	{0,1}	{1}	{1,2}	X_2 1		1	2	3
2	{0,2}	{1,2}	{2}		2	2	3	4

(a) Frequency detector   (b) Adder MAC

	X_1					X_1		
	0	1	2			i	1	2
0	0	E	E		i	i	1	2
X_2 1	E	1	E	X_2 1		1	C	C
2	E	E	2		2	2	C	C

(c) Erasure channel   (d) Collision channel

Figure 17. Examples of two user M-ary input channels

5.3. Extension 3: M-ary input T-user access channel

In [19] Chang and Wolf introduce the T-user M-frequency noiseless multiple access channel with (A) and without (B) intensity information. Both models use the same input alphabet for each of the T users.

For the B channel, the output at each time instant is a symbol which identifies which subset of integers occurred as inputs to the channel, but not how many of each integer occurred. Chang and Wolf showed that capacity of the cooperative M-ary OR channel approaches $M - 1$ bits per transmission. A simple time division method achieves the capacity of $(M - 1)$ bits/transmission for large T.

Assume that we divide the T users into groups of size $(M - 1)$. Within one group the users are numbered from 1 up to $M - 1$. User i, $1 \leq i \leq M - 1$, uses as the transmitting frequencies f_0 and f_i. At the receiver, we are able to detect whether user i uses frequency f_0 or f_i and thus we transmit $M - 1$ bits per transmission. Note that here, we have a "*central*" frequency f_0.

Example 5. *For $M = 3$, $\frac{T}{2}$ groups of users use the pairs (f_0, f_1) and (f_0, f_2), respectively. The output of the channel is $f_0, (f_0, f_1), (f_0, f_2)$, or (f_1, f_2). Note that the input of the channel is uniquely decodable.*

For the A channel the output indicates which subset of integers was transmitted and also how many of each integer were transmitted.

6. Conclusions

We learn that the multiple access channel leads to interesting coding problems. An open problem is the development of fixed length coding for the feedback situation. In addition, zero error codes are to be improved and new applications to be developed.

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