Potential Performance of PLC Systems Composed of Several Communication Links

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***Abstract*— We consider data transmission over a bus system composed of several communication links. All users of the system simultaneously receive transmitted signals with different reliabilities depending on their distance from the sender. Two possible ways of organizing communication are analyzed. The first algorithm includes transmission in a few steps, where each step is devoted to a reliable delivery of the message from one intermediate user to another one. All other users exclude themselves from the system for this step. The second algorithm assumes that all users of the system are active all the time, and they receive side information from communication of intermediate users, which allows them to reduce uncertaincy about the original message. Lower bounds on the time needed to deliver a message over the system that follow from the capacity arguments for both methods are given. These bounds are illustrated by numerical examples.**

Keywords : Power–line communication, channel capacity, reliability.

I. Introduction

Power–line data transmission can be considered as communication over a very unstable channel whose characteristics essentially depend on the distance between sender and receiver. A model studied in [1], [2], and other papers describes the power–line channel as a binary symmetric channel (BSC) having the crossover probability

$$\varepsilon^{(\mathrm{PL})}(\lambda) \triangleq \frac{1-\mathrm{erf}(\sqrt{\eta^* 10^{-0.01\lambda}})}{2}, \qquad (1)$$

where η^* is the signal-to-noise ratio, λ denotes the distance between sender and receiver in meters, and

$$\mathrm{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad x \geq 0.$$

Let us consider a data transmission system consisting of $L+1$ users called User 0, ..., User L (see Fig. 1). At each time instant, one of users can send a bit over the channel. Noisy versions of this bit are simultaneously received by all other users. This scheme will be referred to as data transmission over a bus system. The channel between User j and User j' is a BSC whose crossover probability depends on $\ell = |j - j'|$. This channel will be considered as a channel composed of ℓ links and its capacity will be denoted by C_ℓ.

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By (1) and an assumption that users are located at distance Δ from each other, we conclude that the bit received by User L is the result of transmission over a BSC whose crossover probability is expressed as $\varepsilon^{(\mathrm{PL})}(\Delta L)$ Examples of the function $\varepsilon^{(\mathrm{PL})}(\Delta L)$ are given in Fig. 2. One can conclude that, for all η^* and Δ, there are two values, L_0 and $L_{1/2}$, such that the channel is almost noiseless for $L < L_0$ and completely noisy for $L > L_{1/2}$. For example, if $\eta^* = 15\mathrm{dB}, \Delta = 20$, then $L_0 \approx 15$ and $L_{1/2} \approx 50$. Thus, communication of users at distance less than 300 meters can be easily organized, and it is almost impossible if the distance becomes equal to 800 meters. Communication within the distance from 300 to 800 meters requires coding.

Denote $\mathrm{SNR}(\Delta L) \triangleq \eta^* 10^{-0.01\Delta L}$. The value of the parameter η^* is computed as

$$\eta^* = \frac{P_{\mathrm{tr}}}{10^6 B N_0},$$

where P_{tr} is the power of transmitted signals, B is the bandwidth in MHz, and N_0 is the one-sided power spectral density. Suppose that $N_0 = 10^{-8}$W/Hz. Then

$$10 \lg \mathrm{SNR}(\Delta L) = 10 \lg P_{\mathrm{tr}} + 20 - 10 \lg B - 0.1\Delta L.$$

A typical assignment for the indoor PLC is $B = 10$MHz, and $10 \lg \mathrm{SNR}(\Delta L) = 10 \lg P_{\mathrm{tr}} + 5\mathrm{dB}$, while a typical assignment for the CENELEC type of communication from transformer to the house (outdoor PLC) is $B = 0.1$MHz and $10 \lg \mathrm{SNR}(\Delta L) = 10 \lg P_{\mathrm{tr}} + 30\mathrm{dB} - \Delta L$. In particular, the latter case brings $10 \lg \mathrm{SNR}(300) = 14$dB and $10 \lg \mathrm{SNR}(400) = 4$dB.

The bus system in Fig. 1 is a general data transmission scheme constructed when the communication channel is composed of several links. This system is characterized

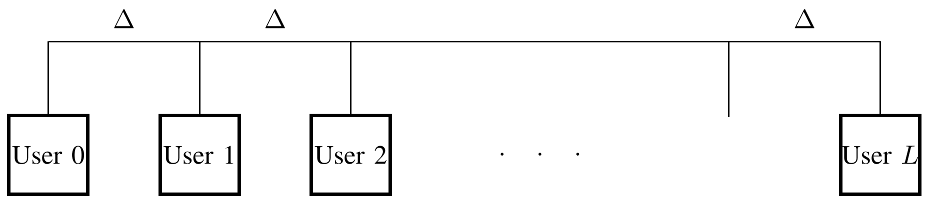

Fig. 1. Data transmission scheme over a bus system.

by the point that *all* users receive noisy versions of bits transmitted over the channel. Therefore each of them also has some side information when some user transmits the message to another user. In other words, scheme in Fig. 1 represents a broadcast channel [3], [4]. If this fact is ignored by users according to a data transmission protocol, we say that data are transmitted without using side information. Otherwise, we say that data are transmitted with the use of side information. Certainly, the second variant is at least not worse than the first one, but it requires more complex decoding algorithms. Therefore the potential gain obtained by the protocol that uses side information should be analyzed.

II. Transmission rates achievable in bus systems composed of binary symmetric channels

II-A. Data transmission without using side information

Data transmission over a bus system can be realized in several steps. Suppose that one of M messages of User 0 has to be delivered to User L. Choose parameters $\ell_1, \ldots, \ell_k \geq 1$ in such a way that $\ell_1 + \ldots + \ell_k = L$. At the first step, User 0 transmits the message to User ℓ_1. At the second step, User ℓ_1 transmits the message to User $(\ell_1 + \ell_2)$, etc., at the k-th step, User $(\ell_1 + \ldots + \ell_{k-1})$ transmits the message to User L. As a result, the problem is solved by using the channel $n_1 + \ldots + n_k$ times, where n_i is the length of a codeword that was sent over the channel at the i-th step, $i = 1, \ldots, k$. If all channels would be noiseless, then this sum is equal to $k \log M$. However, such an assumption is not true, and the sum when reliable communication is possible is expressed as the product of $\log M$ and a coefficient $\beta_1(\ell_1, \ldots, \ell_k) > k$.

Let us restrict ourselves to the analysis of a potential behavior of data transmission systems and understand the requirement for a reliable communication as inequalities

$$n_i \geq C_{\ell_i}^{-1} \log M, \quad i = 1, \ldots, k.$$

Then

$$n_1 + \ldots + n_k \geq \beta_1(\ell_1, \ldots, \ell_k) \log M, \tag{2}$$

where

$$\beta_1(\ell_1, \ldots, \ell_k) \stackrel{\triangle}{=} \sum_{i=1}^{k} C_{\ell_i}^{-1}. \tag{3}$$

The problem of assignment of a parameter k and lengths $(\ell_1, \ldots, \ell_k) \in \mathcal{L}_k^{(L)}$ minimizing the function $\beta_1(\ell_1, \ldots, \ell_k)$ defined in (3), where $\mathcal{L}_k^{(L)}$ is the set of k-tuples such that each component is a positive integer and their sum is equal to L. This problem is a rooting problem, and it can be easily solved.

Proposition. *Determine parameter $\beta_1^{(L)}$ by the following algorithm : set*

$$\beta_1^{(L)} \stackrel{\triangle}{=} \min_{\ell \in \{1, \ldots, L\}} \Big[\beta_1^{(L-\ell)} + C_\ell^{-1} \Big],$$

where $\beta_1^0 \stackrel{\triangle}{=} 0$ and

$$\ell_{\min} \stackrel{\triangle}{=} \arg \min_{\ell \in \{1, \ldots, L\}} \Big[\beta_1^{(L-\ell)} + C_\ell^{-1} \Big].$$

Then

$$\min_{k \geq 1} \min_{(\ell_1, \ldots, \ell_k) \in \mathcal{L}_k^{(L)}} \beta_1(\ell_1, \ldots, \ell_k) = \beta_1^{(L)}.$$

Moreover,

$$\arg \min_{k \geq 1} \min_{(\ell_1, \ldots, \ell_k) \in \mathcal{L}_k^{(L)}} \beta_1(\ell_1, \ldots, \ell_k) = k^{(L)},$$

where

$$k^{(L)} \stackrel{\triangle}{=} \begin{cases} k^{(L-\ell_{\min})} + 1, & \textit{if } \ell_{\min} < L, \\ 1, & \textit{if } \ell_{\min} = L. \end{cases}$$

II-B. Data transmission with the use of side information

Theorem 1. *Let* $\mathbf{C}$ *be a left triangular* $k \times k$ *matrix whose entries are defined as*

$$c_{ij} \triangleq \begin{cases} C_{\ell_j+\ldots+\ell_i}, & \text{if } i \geq j, \\ 0, & \text{if } i < j \end{cases}$$

for all $i, j = 1, \ldots, k$. *If*

$$n_1 + \ldots + n_k = \beta(\ell_1, \ldots, \ell_k) \log M, \tag{4}$$

where $n_1, \ldots, n_k$ *are determined by the system of linear equations*

$$\mathbf{C} \cdot \begin{bmatrix} n_1 \\ n_2 \\ \cdot \\ \cdot \\ n_k \end{bmatrix} = \begin{bmatrix} \log M \\ \log M \\ \cdot \\ \cdot \\ \log M \end{bmatrix}, \tag{5}$$

then reliable data transmission of one of M *messages from User* 0 *to User* $(\ell_1 + \ldots + \ell_k)$ *is possible using a* k*-step protocol where the length of a codeword transmitted by User* $(\ell_1 + \ldots + \ell_{i-1})$ *at the* i*-th step is equal to* n_i, $i = 1, \ldots, k$.

We present a sketch of the proof of Theorem 1 for $k = 2$ when $\ell_1 = \ell_2 = 1$. In this case we assume that there is a coloring of the set of messages with $2^{n_1(C_1 - C_2)}$ colors. User 1 having received a noisy version of a codeword of length n_1 for $2^{n_1 C_1}$ messages, correctly decodes the message. After that he transmit the color of the decoded message by using a code of length n_2. User 2, having received a noisy version of the codeword of User 0 constructs a list of size $2^{n_1(C_1 - C_2)}$ containing messages that are the most likely to be transmitted. Having received a noisy version of the color sent by User 1, he selects only one message from this list. The total transmission length is equal to $n_1 + n_2$ and if

$$\begin{bmatrix} C_1 & \\ C_2 & C_1 \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \log M \\ \log M \end{bmatrix},$$

then there exist corresponding codes and a coloring such that all decoding steps are reliable from the point of view developed in our approach.

Let us consider the following data transmission algorithm for a bus system composed of $L = 2$ binary symmetric channels having crossover probability p where channels User 0 – User 1 and User 1 – User 2 are $\mathrm{BSC}(p)$, while channel User 0 – User 2 is a $\mathrm{BSC}(2pq)$, $q \triangleq 1 - p$. This bus system will be referred to as a $\mathrm{BSC}_2(p)$. We present a statement for the $\mathrm{BSC}_2(p)$, which is more general than the statement of Theorem 1, since it gives insight to the exponents of the decoding error probability. Moreover, the approach used in our scheme simultaneously introduces a structure to the coloring procedure.

Suppose that messages are binary vectors $\mathbf{m} \in \{0,1\}^{nR}$. Introduce a parameter γ in such a way that $n\gamma$ is an integer. If $\mathbf{m}$ is the message of User 0 to be delivered to User 2, then let User 0 to encode this message by using a binary linear code of length n_1 specified by generator matrix $\mathbf{G}_1$ of dimension $nR \times n\gamma$. The codeword $\mathbf{m}\mathbf{G}_1$ is sent over the channel. Let User 1, having received a noisy version of the transmitted codeword, to decode the message as $\hat{\mathbf{m}}_1$ and send a codeword $\hat{\mathbf{m}}_1\mathbf{G}_2$, where $\mathbf{G}_2$ is a generator matrix of dimension $nR \times n(1-\gamma)$. User 2, having received a noisy version of the codeword $\mathbf{m}\mathbf{G}_1$ at the 1–st step and a noisy version of the codeword $\hat{\mathbf{m}}\mathbf{G}_2$ at the 2–nd step, decodes the message as $\hat{\mathbf{m}}_2$. The decoding error is defined as the event that either $\hat{\mathbf{m}}_1 \neq \mathbf{m}$ or $\hat{\mathbf{m}}_2 \neq \mathbf{m}$. Thus, the decoding error probability taken in the average over uniform distribution on messages to be sent, $\Lambda(\mathbf{G}_1, \mathbf{G}_2)$, is bounded from above as

$$\Lambda(\mathbf{G}_1, \mathbf{G}_2) \leq \Lambda^{(1)}(\mathbf{G}_1) + \Lambda^{(2)}(\mathbf{G}_1, \mathbf{G}_2),$$

where

$$\begin{aligned} \Lambda^{(1)}(\mathbf{G}_1) \;&\triangleq\; 2^{-nR} \sum_{\mathbf{m}} \sum_{\mathbf{y}_1 \in \{0,1\}^{n\gamma}} W(\mathbf{y}_1|\mathbf{m}\mathbf{G}_1) \\ &\quad \cdot \chi\{\exists \mathbf{m}_1 \neq \mathbf{m} : W_1(\mathbf{y}_1|\mathbf{m}_1\mathbf{G}_1) \geq W_1(\mathbf{y}_1|\mathbf{m}\mathbf{G}_1)\}, \\ \Lambda^{(2)}(\mathbf{G}_1, \mathbf{G}_2) \;&\triangleq\; 2^{-nR} \sum_{\mathbf{m}} \sum_{\mathbf{y}'_1 \in \{0,1\}^{n\gamma}} \sum_{\mathbf{y}_2 \in \{0,1\}^{n(1-\gamma)}} \\ &\quad W_2(\mathbf{y}'_1|\mathbf{m}\mathbf{G}_1) W_1(\mathbf{y}_2|\mathbf{m}\mathbf{G}_2) \chi\{\exists \mathbf{m}_2 \neq \mathbf{m} : \\ &\qquad W_2(\mathbf{y}'_1|\mathbf{m}_2\mathbf{G}_1) W_1(\mathbf{y}_2|\mathbf{m}_2\mathbf{G}_2) \\ &\quad \geq W_2(\mathbf{y}'_1|\mathbf{m}\mathbf{G}_1)\} W_1(\mathbf{y}_2|\mathbf{m}\mathbf{G}_2)\}, \end{aligned}$$

χ denotes the indicator function, i.e., $\chi\{S\} = 1$ if the statement S is true and $\chi\{S\} = 0$ otherwise.

Let

$$\begin{aligned} \overline{\Lambda_\gamma^{(1)}}(R) \;&\triangleq\; 2^{-n^2\gamma R} \sum_{\mathbf{G}_1} \Lambda^{(1)}(\mathbf{G}_1), \\ \overline{\Lambda_\gamma^{(2)}}(R) \;&\triangleq\; 2^{-n^2 R} \sum_{\mathbf{G}_1, \mathbf{G}_2} \Lambda^{(2)}(\mathbf{G}_1, \mathbf{G}_2) \end{aligned}$$

denote the expected values of probabilities $\Lambda^{(1)}(\mathbf{G}_1)$ and $\Lambda^{(2)}(\mathbf{G}_1, \mathbf{G}_2)$ computed over the ensemble where each component of generator matrices $\mathbf{G}_1, \mathbf{G}_2$ is an i.i.d. random variable taking values 0 and 1 with probability $1/2$.

The statement formulated below can be proved using the well–known technique introduced by Gallager [6].

Theorem 2. *Exponents of the decoding error probability achievable in a* $\mathrm{BSC}_2(p)$ *under decoding with the use of side information can be bounded from below as*

$$\frac{-\log \overline{\Lambda_\gamma^{(i)}}(R)}{n} \;\geq\; e_\gamma^{(i)}(R), \quad i = 1, 2,$$

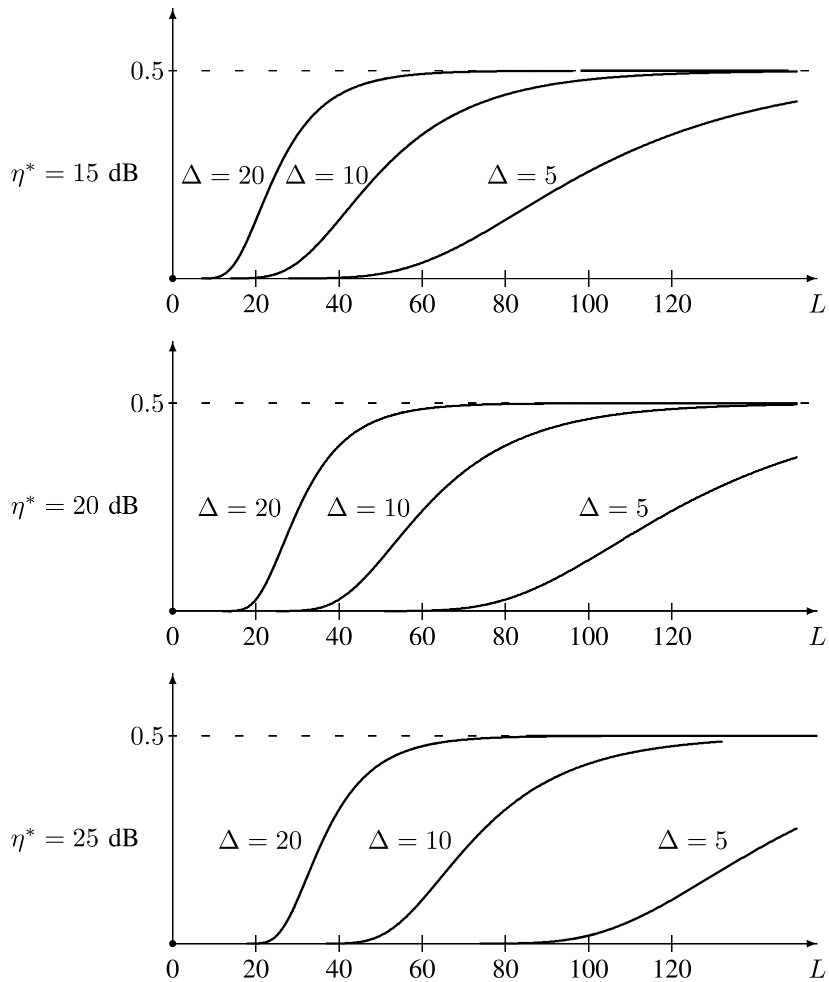

Fig. 2. The functions $\varepsilon^{(\mathrm{PL})}(\Delta L)$ for different signal-to-noise ratios η^*.

where

$$\begin{aligned} e_\gamma^{(1)}(R) &\triangleq \max_{s\in[0,1]}\Big[\gamma E_s(p) - sR\Big], \\ e_\gamma^{(2)}(R) &\triangleq \max_{s\in[0,1]}\Big[\gamma E_s(2pq) + (1-\gamma)E_s(p) - sR\Big], \\ E_s(p) &\triangleq s - (1+s)\log(p^{\frac{1}{1+s}} + q^{\frac{1}{1+s}}). \end{aligned}$$

Example. Suppose that the $(7,4)$ binary Hamming code is used and $t = t' = 1$. Then User 1 has to send one of 8 helping messages : "decode $\mathbf{y}'$ as it is received" and "invert the i-th position of $\mathbf{y}'$ and then decode", where $i = 1, \ldots, 7$. If the helping message can be noiselessly delivered to User 2, then the bus system is used $7+3 = 10$ times. Furthermore, the same performance can be obtained when the helping message depends on the message of User 0 and does not depend on the vector $\mathbf{y}$. Namely, assuming that the Hamming code is introduced by the generator matrix in the systematic form with the parity symbols $u_2 \oplus u_3 \oplus u_4$, $u_1 \oplus u_3 \oplus u_4$, $u_1 \oplus u_2 \oplus u_4$, we assign the same helping message to codewords $\mathbf{uG}$ and $\mathbf{u'G}$, where $\mathbf{u} = (u_1, u_2, u_3, u_4)$ and $\mathbf{u}' = (u_1', u_2', u_3', u_4')$ if and only if $u_i' = u_i \oplus 1$ for all $i = 1, \ldots, 4$. This can be done by the assignment : $(u_1 \oplus u_4, u_2 \oplus u_4, u_3 \oplus u_4)$. Moreover, correct decoding in this case is also attained when $t' = 2$, i.e., the decoder of User 2 has to make a decision which of two codewords located at the Hamming distance 7 is more likely to be transmitted after a particular vector $\mathbf{y}'$ of length 7 is received.

II-C. Comparison of performances of transmission protocols

Notice that the solution presented in (2), (3) can be viewed as a special case of the solution (4), (5), which is obtained after the matrix $\mathbf{C}$ is replaced with the matrix containing entries $C_{\ell_1}, \ldots, C_{\ell_k}$ at the diagonal and 0's at all other positions.

Table I contains coefficients β and β_1 obtained from (2), (3) and (4), (5) for the PLC when the bus system for $L = 128$ users is uniformly partitioned in k pairs (sender,receiver). For example, if $k = 4$, then the coefficients β_1 and β in corresponding lines can be obtained if the message is sent first from User 0 to User 32, then from User 32 to User 64, then from User 64 to User 96, and finally from User 96 to User 128 (for the protocol does not use side information) and when only users 32, 64, 96, 128 are active meaning that all of them finally decode the transmitted message (for the protocol where side information is used). One can conclude that β essentially decreases when k increases, and it is minimized when all users participate in decoding. The smallest coefficient β_1 is attained for a certain k depending on L, η^*, and Δ. Notice also that the smallest value of β_1 is, for example, about 200 times smaller if $\Delta = 20, \eta^* = 15$dB than the largest value, i.e., a proper choice of k is very important.

Table II contains ratios of smallest possible coefficients β and β_1 as a function of L, i.e., β_1 is attained with the best partitioning constructed using the Proposition, and β is attained when all users are active and each channel has only 1 link. Data of Table II illustrate the point that if the number of users is small, then the relative effectiveness of the protocol that uses side information essentially depends on a specific L, and this dependence can be irregular.

TABLE I

COEFFICIENTS β AND β_1 FOR PLC WHEN $L = 128$.

$\eta^* = 15$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
k	β	β_1	β	β_1	β	β_1
128	2.16	128	3.82	128	7.18	128
64	2.17	64	3.85	64	7.33	64
32	2.18	32	3.92	32	7.66	32
16	2.20	16	4.06	16	8.48	16
8	2.25	8	4.41	8.02	10.8	11.5
4	2.37	4.01	5.46	5.75	84.8	84.8
2	2.78	2.87	42.4	42.4		
1	21.2	21.2				
$\eta^* = 20$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
k	β	β_1	β	β_1	β	β_1
128	1.86	128	3.04	128	5.63	128
64	1.86	64	3.06	64	5.72	64
32	1.86	32	3.09	32	5.90	32
16	1.86	16	3.17	16	6.31	16
8	1.87	8	3.34	8	7.23	8.18
4	1.88	4	3.67	4.09	28.2	28.2
2	1.90	2.05	14.1	14.1		
1	7.06	7.06				
$\eta^* = 25$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
k	β	β_1	β	β_1	β	β_1
128	1.62	128	2.62	128	4.68	128
64	1.62	64	2.63	64	4.73	64
32	1.62	32	2.65	32	4.83	32
16	1.62	16	2.69	16	5.07	16
8	1.62	8	2.76	8	5.91	8
4	1.62	4	3.06	4	10.4	10.4
2	1.62	2	5.21	5.21		
1	2.61	2.61				

TABLE II

RATIOS OF THE SMALLEST VALUES OF β AND β_1 FOR PLC.

$\eta^* = 15$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
L	β^*/β_1^*	k	$\beta^*\beta_1^*$	k	β^*/β_1^*	k
25	1	1	0.992	1	0.841	2
50	0.992	1	0.841	2	0.709	4
75	0.746	1	0.747	3	0.679	5
100	0.841	2	0.706	4	0.664	7
125	0.768	2	0.682	5	0.653	9
$\eta^* = 20$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
L	β/β_1	k	β/β_1	k	β/β_1	k
25	1	1	1	1	0.768	2
50	1	1	0.768	2	0.810	3
75	0.987	1	0.882	2	0.779	4
100	0.768	2	0.807	3	0.737	5
125	0.904	2	0.758	3	0.729	7
$\eta^* = 25$ dB						
	$\Delta = 5$		$\Delta = 10$		$\Delta = 20$	
L	β/β_1	k	β/β_1	k	β/β_1	k
25	1	1	1	1	0.981	1
50	1	1	0.981	1	0.882	2
75	1	1	0.925	2	0.845	3
100	0.981	1	0.880	2	0.814	4
125	0.785	2	0.845	3	0.789	5

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