A Convolutional Single-Parity-Check Concatenated Coding Scheme for High-Data-Rate Applications

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Abstract—**The paper describes the possible implementation of concatenated coding scheme suitable particularly for high date rate applications. The proposed coding scheme uses a set of n convolutional codes multiplexed into an inner code and a $(n, n-1)$ single-parity-check code serving as the outer code. Each of the inner convolutional codes is decoded independently with maximum-likelihood decoding being achieved via n parallel implementations of the Viterbi algorithm. The Viterbi decoding is followed by additional outer soft-decision single-parity-check decoding. Considering $n = 12$ and the set of short constraint length $K = 3$, rate 1/2 convolutional codes, it is shown, that the performance of the concatenated scheme is comparable to the performance of the constraint length $K = 7$, rate 1/2 convolutional code with standard soft-decision Viterbi decoding. Simulation results are presented for the $K = 3$, rate 1/2 as well as for the punctured $K = 3$, rate 2/3 and rate 3/4 inner convolutional codes. Moreover, the performance of the proposed concatenated scheme using a set of $K = 7$, rate 1/2 inner convolutional codes is given.**

I. Introduction

ONE of the best-known convolutional codes is the constraint length $K = 7$, rate 1/2 code, with a free Hamming distance $d_{\text{free}} = 10$. Using this code for a transmission with antipodal signaling coherent detection in an additive white Gaussian noise channel (AWGNC) results in a coding gain of slightly more than 5 dB at a bit error rate of $P_b = 10^{-5}$. More detailed properties of the extensively studied convolutional code can be found, e.g., in [1]. Employing code puncturing techniques allows the operation of the basic convolutional code at different rates as rate 2/3 or rate 3/4 [1], [2]. Hardware-implemented 64-state Viterbi decoders are commercially available with a decoding speed up to about 20 Mb/s.

We consider a concatenated coding scheme that is characterized by a set of n multiplexed convolutional codes forming the inner code and an outer $(n, n-1)$ single-parity-check (SPC) code. The inner convolutional codes are maximum-likelihood (ML) decoded via n parallel, independently operating Viterbi decoders. The decoded overall parity check sequence is used to correct decoding errors resulting from Viterbi decoding. This outer soft-decision SPC decoding can be realized exploiting the analogue demodulator output used for Viterbi decoding before with only a small amount of additional hardware.

Using for the proposed scheme a set of $n = 12$ short constraint length $K = 3$, rate 1/2 inner convolutional codes results in a decoder for which the decoding speed is mainly determined by the speed of $n = 12$ multiplexed 4-state Viterbi decoders. Thus, a decoding speed much higher than 20 Mb/s can be expected employing a decoder with roughly the same hardware complexity as the standard 64-state Viterbi decoder. The performance of this scheme is shown to be within 0.5 dB the same as achieved with the $K = 7$, rate 1/2 convolutional code. Equivalent results can be concluded from the comparison of the performance of the concatenated scheme applying $n = 12$ punctured $K = 3$, rate 2/3 and rate 3/4 inner codes with the performance of the punctured $K = 7$, rate 2/3, rate 3/4 convolutional code. Significant performance improvements of the concatenated coding scheme are achieved simply substituting for the basic $K = 3$, rate 1/2 codes with the basic $K = 7$, rate 1/2 convolutional codes.

In Section II, the encoding and decoding structure of the concatenated scheme with the short constraint length $K = 3$, rate 1/2 inner convolutional codes is described. In Section III, the performance analysis is outlined and Section IV gives the simulation results.

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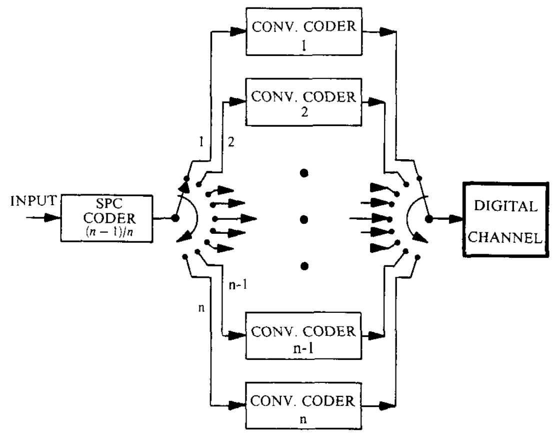

Fig. 1. Block diagram of the encoder.

II. System Description

The encoding structure of the concatenated coding scheme considered is given in Fig. 1. The $(n-1)$ information bits b_i; $i = 1, 2, \cdots, (n-1)$, are encoded in parallel by $(n-1)$ short constraint length ($K = 3$) rate 1/2 convolutional encoders. For each block of information bits an overall parity bit b_n is generated and encoded by the nth convolutional encoder. The $2n$ code symbols are subsequently multiplexed and transmitted. The code rate of this particular coding scheme is given by $R = (n-1)/2n$. Applying the concept of code puncturing for the inner convolutional code, other rates R can be obtained.

The encoded data is assumed to be transmitted over an additive white Gaussian noise channel using antipodal signaling with coherent detection.

At the receiver the demodulator output is demultiplexed and distributed over n parallel $K = 3$ soft-decision Viterbi decoders. The decoders operate fully independently of each other, and thus given a demodulator output sequence $\boldsymbol{r}_i$, the log-likelihood functions

$$L_i := -\log \Pr\left(\boldsymbol{r}_i \mid G(\boldsymbol{b}_i)\right) \tag{1}$$

are minimized with respect to all possible code sequences $G(\boldsymbol{b}_i)$ for each individual encoder/decoder pair i, $1 \le i \le n$. The notation

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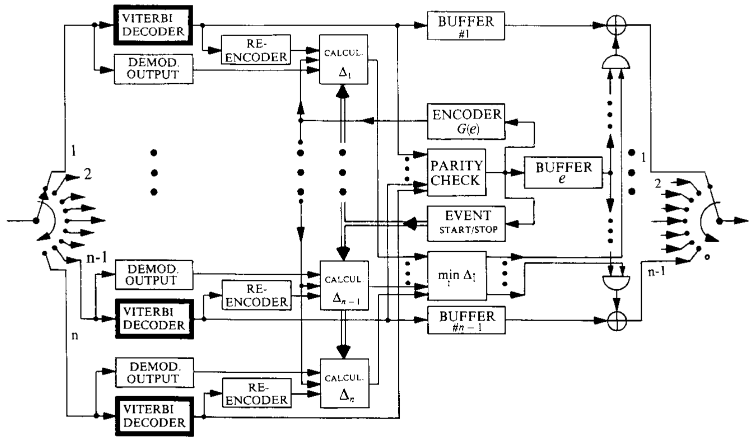

Fig. 2. Block diagram of the decoder.

$G(\boldsymbol{x})$ with the values $\{0, 1\}$ represents the convolutional encoder output sequence that corresponds to an input sequence $\boldsymbol{x}$.

Hence, the ith Viterbi decoder gives as an output that sequence $\hat{\boldsymbol{b}}_i$ for which the log-likelihood function is minimum, i.e.,

$$Lm_i := -\log \Pr\left(\boldsymbol{r}_i \mid G(\hat{\boldsymbol{b}}_i)\right). \tag{2}$$

However, all decoders operate independently, and thus the output of the set of decoders is not necessarily a possible transmitted SPC codeword. This is caused by an error event in at least one of the Viterbi decoder outputs. An error event, corresponding to an erroneous path in the trellis, always starts with a decoding error and ends whenever two (in general, $K - 1$) consecutive correct output bits appear.

The overall parity of the n Viterbi decoder outputs form the parity error sequence $\boldsymbol{e}$. In the decoding error-free case $\boldsymbol{e} = \boldsymbol{0}$.

First, let us assume that an arbitrary error event occurs for only **one** decoder. In this situation, the resulting parity error sequence $\boldsymbol{e}$ equals this particular error event. Therefore the beginning as well as the ending of the event can be correctly detected by the parity error sequence. In order to obtain regular transmitted outer SPC codewords for the duration of the error event, the output sequence $\hat{\boldsymbol{b}}_i$ of one of the decoders has to be changed according to the parity error sequence $\boldsymbol{e}$. Following the principle of ML-decoding, that particular Viterbi decoder output is selected which minimizes the "overall" log-likelihood function, that is

$$L_{\min} = \min_i \left[-\log \Pr\left(\boldsymbol{r}_i \mid G(\hat{\boldsymbol{b}}_i \oplus \boldsymbol{e})\right) + \sum_{\substack{j=1 \\ j \neq i}}^{n} Lm_j \right]; 1 \leq i \leq n. \tag{3}$$

We adopt decoding according to (3) as the decoding rule. This rule guarantees ML decoding as long as error events from different decoders do not share common decoding epochs.

Of course, two or more decoding error events might be overlapping, i.e., they share at least one common decoding epoch. The solution of (3) assumes that the parity error sequence $\boldsymbol{e}$ is caused by an error event from a single decoder. Obviously, in this case, the decoding according to (3) is not ML-decoding and outer SPC decoding errors occur. The decoding error rate of the suboptimal decoder considered is the subject of Section III.

For the determination of the minimum log-likelihood function in (3), given a parity error sequence $\boldsymbol{e}$, it is sufficient to calculate

$$\Delta_i = -\log \frac{\Pr\left(\boldsymbol{r}_i \mid G(\hat{\boldsymbol{b}}_i \oplus \boldsymbol{e})\right)}{\Pr\left(\boldsymbol{r}_i \mid G(\hat{\boldsymbol{b}}_i)\right)}; 1 \leq i \leq n \tag{4}$$

for each Viterbi decoder output. The decoder sequence $\hat{\boldsymbol{b}}_i$ for which Δ_i is minimum, is changed into $\hat{\boldsymbol{b}}_i \oplus \boldsymbol{e}$. For an AWGNC and antipodal signaling, (4) can also be written, neglecting common terms, as

$$\Delta_i = \sum_m r_{i,m} \cdot \left(G(\hat{\boldsymbol{b}}_i)_m - G(\hat{\boldsymbol{b}}_i \oplus \boldsymbol{e})_m\right)$$
$$= \sum_m r_{i,m} \cdot G(\boldsymbol{e})_m \left[2G(\hat{\boldsymbol{b}}_i)_m - 1\right] \tag{5}$$

where the index m is used to indicate the relevant time slots for which $G(\boldsymbol{e}) = 1$. The decoder structure is shown in Fig. 2. Summarizing the above, the task of the decoder is to find the minimum value of Δ_i in the case where an error event is detected, and subsequently to correct the respective Viterbi decoder output that gives the minimum. For the realization we need in addition to the n Viterbi decoders the following units:

- a reencoder to generate $G(\hat{\boldsymbol{b}}_i)$ for each Viterbi decoder output,
- a circuit that generates the parity error sequence from the decoder outputs,
- an encoder that generates $G(\boldsymbol{e})$,
- a buffer connected to each Viterbi decoder that stores the last l demodulator output values where l then denotes the length of the path memories of the Viterbi decoders,
- calculation of (5) for $1 \leq i \leq n$, during a detected error event,
- calculation of the min Δ_i at the end of a detected error event, and
- a buffer for each Viterbi decoder output that enables correction according to the parity error sequence. The length of the buffers is limited. The influence of the buffer lengths was considered in the simulations.

As can be seen from Fig. 2, the decoder structure is characterized by its high parallelism being achieved via n multiplexed and thus completely independently operating regular Viterbi decoders. Moreover, the basic speed of each of the above units is the same as the decoding speed of an individual Viterbi decoder, i.e., a fraction $1/(2n)$ of the channel rate. The additional hardware that is required to realize the outer decoding depends mainly on the calculation of Δ_i and the search for the minimum value. The determination of Δ_i requires a summation over the quantized demodulator outputs during the detected error event. If the demodulator output is quantized to 3 b and the length of the parity error sequence is restricted to be less than 19 the maximum sum can be represented by 8 b. The search of the minimum value might be done serially requiring only one comparator for two 8 b words.

III. Decoding Error Rate

For the coding scheme considered decoding errors occur if:

A) an error event in only **one** Viterbi decoder output appears but is assigned to another decoder output;
B) two or more Viterbi decoder outputs have overlapping error events and thus always give rise to decoding errors.

As a consequence of the above distinction the decoding error rate is written as

$$P_b = {}_AP_b + {}_BP_b. \quad (6)$$

Case A) Assume that only Viterbi decoder i, $1 \le i \le n$ has an error event of Hamming weight d and corresponding Δ_i. Thus, the outer SPC decoder makes decoding errors if for Viterbi decoder j, $1 \le j \le n$, $j \ne i$, $\Delta_j < \Delta_i$. The probability for this event is denoted by P_d. For an AWGNC with a one-sided noise density N_0 the probability P_d can be upper bounded by the probability that the **sum of two**—out of n—statistically independent Gaussian random variables each having mean $-d\sqrt{2E_s/N_0}$ and variance d is greater than zero whereas $(n-2)$ variables are less than zero. Hence,

$$P_d \le \binom{n}{2} Q\left(\sqrt{4dE_s/N_0}\right) \quad (7)$$

where E_s represents the energy per coded symbol. The E_s is related to the energy per information bit E_b via $E_s = RE_b$. The function $Q(x)$ is given by

$$Q(x) = 1/(2\pi)^{1/2} \int_x^\infty \exp(-t^2/2)\, dt.$$

Using the union bound argument one gets

$${}_AP_b \le \frac{1}{n} \sum_{d=d_{\text{free}}}^{\infty} 2w_d P_d \le (n-1) \sum_{d=d_{\text{free}}}^{\infty} w_d Q\left(\sqrt{4dE_s/N_0}\right). \quad (8)$$

Adopting the notations in [1], w_d gives the total information weight of all weight d paths in the trellis of the inner convolutional code. Note, that in (8) the factor $2w_d$ appears since always two Viterbi decoder outputs are in error.

Case B) Let more than one Viterbi decoder output have an error event such that the different events share common decoding epochs. We denote by s_k a specific configuration of $k \ge 2$ overlapping error events and by S_k the set of all possible configurations of k error events. Furthermore, Ns_k represents the total number of erroneous information bits for a particular s_k, i.e., the sum of the number of errors in the k individual error events. If the outer SPC decoder, changing the output of only one Viterbi decoder according to the resulting parity error sequence $\boldsymbol{e}$, makes the maximum number $2Ns_k$ of decoding errors the contribution ${}_BP_b$ to the decoding bit error rate can be union bounded by

$${}_BP_b \le \frac{1}{n} \sum_{k=2}^{\infty} \sum_{s_k \in S_k} \Pr(s_k) \cdot 2Ns_k \quad (9)$$

where $\Pr(s_k)$ denotes the probability for a configuration s_k.

For sufficiently high signal-to-noise ratios the $k = 2$ term in (9) can be expected to dominate.

Let v_1 and v_2 be the two specific overlapping error events of length Lv_1 and Lv_2 and number of errors Nv_1 and Nv_2, respectively. Then, using the union bound argument

$$\sum_{s_2 \in S_2} \Pr(s_2) 2Ns_2 \le \binom{n}{2} \sum_{v_1} \Pr(v_1) \sum_{v_2} \Pr(v_2) \cdot (Lv_1 + Lv_2 - 1) 2(Nv_1 + Nv_2). \quad (10)$$

Rearranging the summations in (10) with respect to the weight d and using the generating function $T(D, L, N)$, [1] of the inner convolutional code one gets

$$\sum_{s_2 \in S_2} \Pr(s_2) 2Ns_2 \le 4\binom{n}{2} \left\{ T(D,1,1) \cdot \left.\frac{dT(D,L,N)}{dL\,dN}\right|_{\substack{N=1\\L=1}} + \left.\frac{dT(D,L,1)}{dL}\right|_{L=1} \cdot \left.\frac{dT(D,1,N)}{dN}\right|_{N=1} - T(D,1,1) \cdot \left.\frac{dT(D,1,N)}{dN}\right|_{N=1} \right\}$$
$$\le 4\binom{n}{2} \sum_{l=2d_{\text{free}}}^{\infty} g_l D^l;\ D^l = Q\left(\sqrt{2lE_s/N_0}\right) \quad (11)$$

with the coefficients g_l which are determined by $T(D, L, N)$ and derivations. Note, that configurations in S_k for $k \ge 3$ lead to terms in the series expansion of the order $D^{3d_{\text{free}}}$ and higher.

Combining (6), (8), (9), and (11) we may write P_b as

$$P_b \le (n-1) \cdot \left\{ \left(2g_{2d_{\text{free}}} + w_{d_{\text{free}}}\right) D^{2d_{\text{free}}} + \sum_{l=2d_{\text{free}}+1}^{\infty} g_l' D^l \right\};\ D^l = Q\left(\sqrt{2lE_s/N_0}\right) \quad (12)$$

where g_l' includes the contributions to D^l from (8), (11) and (9), respectively. As can be seen from (12), the lowest order term in the series expansion is $D^{2d_{\text{free}}}$. Thus, the concatenated code can be regarded as a code with **twice** the free distance of the inner convolutional codes used. For the $K = 3$, rate 1/2 inner convolutional code, (12) is given by

$$P_b \le 11(n-1) Q\left(\sqrt{10 \cdot E_b/N_0 \cdot (n-1)/n}\right) + (n-1) \sum_{l=11}^{\infty} g_l' Q\left(\sqrt{l \cdot E_b/N_0 \cdot (n-1)/n}\right) \quad (13)$$

where we substituted $E_s = E_b(n-1)/2n$. Comparing the first term in (13) for $n = 12$ to the first term in the bit error performance bound of the $K = 7$, rate 1/2 code, which is $36Q(\sqrt{10E_b/N_0})$, we expect an asymptotic difference in the coding gains of only 0.38 dB.

IV. Simulation Results

The simulation results to be presented are based on the use of antipodal signaling, coherent detection and additive white Gaussian noise. We consider the unquantized channel.

Figs. 3 and 4 show the postdecoding bit error rates as a function of the signal-to-noise ratio E_b/N_0 that were measured in simulations.

The simulations were carried out for the concatenated coding scheme with $n = 12$, and an inner convolution code with:

- constraint length $K = 3$, rate 1/2, $d_{\text{free}} = 5$, (overall code rate $R = 0.46$); constraint length $K = 7$, rate 1/2, $d_{\text{free}} = 10$, (overall code rate $R = 0.46$);
- constraint length $K = 3$, punctured to rate 2/3, $d_{\text{free}} = 3$, (overall code rate $R = 0.61$); constraint length $K = 3$, punctured to rate 3/4, $d_{\text{free}} = 3$, (overall code rate $R = 0.69$).

The path memory length of the Viterbi decoders as well as the length of the buffers connected with the parity error sequence were set equal to 5K and for punctured versions 7K; no significant improvements were obtained by going beyond these parameter settings.

The measured results in Figs. 3 and 4 can be compared to the bit error performance bounds as given, e.g., in [3] for the $K = 7$, rate 1/2 convolutional code as well as for the punctured rate 2/3 and

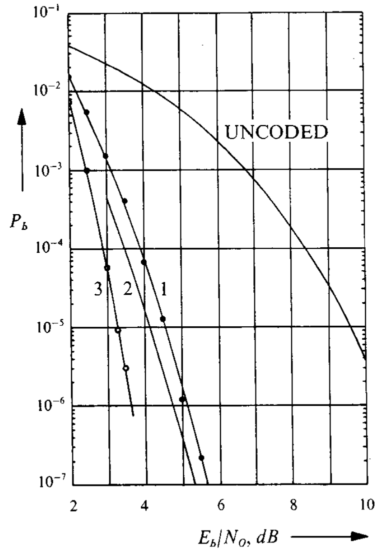

Fig. 3. Decoding bit error rate in an AWGNC (antipodal signaling with coherent detection) versus signal-to-noise ratio E_b/N_0. 1) Concatenated (12, 11) SPC outer code, $K = 3$, rate 1/2 inner convolutional code; rate $R \cong 0.46$. 2) Bit error performance bound for the $K = 7$, rate 1/2 convolutional code. 3) Concatenated (12, 11) SPC outer code, $K = 7$, rate 1/2 inner convolutional code; rate $R \cong 0.46$.

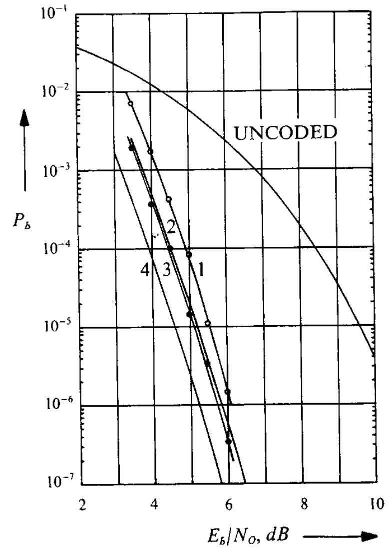

Fig. 4. Decoding bit error rate in an AWGNC (antipodal signaling with coherent detection) versus signal-to-noise ratio E_b/N_0. 1) concatenated (12, 11) SPC outer code, punctured $K = 3$, rate 3/4 inner convolutional code; rate $R \cong 0.61$. 2) bit error performance bound for punctured $K = 7$, rate 3/4, $d_{\text{free}} = 5$ convolutional code. 3) concatenated (12, 11) SPC outer code, punctured $K = 3$, rate 2/3 inner convolutional code; rate $R \cong 0.61$. 4) bit error performance bound for punctured $K = 7$, rate 2/3, $d_{\text{free}} = 6$ convolutional code.

rate 3/4 codes. The comparison shows, that the performance of the concatenated coding scheme using the (12, 11) SPC code and the $K = 3$, rate 1/2 inner convolutional code is within 0.5 dB of that expected for the $K = 7$, ML-decoded convolutional code.

Furthermore, Fig. 3 shows the measured decoding error rate for a scheme using a $K = 7$, rate 1/2 inner code and the outer (12, 11) SPC code. The coding gain measured at $P_b = 10^{-5}$ is slightly more than 6 dB.

V. Conclusion

We described a concatenated coding scheme that can be advantageously used in communication links with high data rates. The high decoding speed capability is achieved via multiplexing a set of regular Viterbi decoders. The outer soft-decision SPC decoding, applied to correct decoding errors resulting from the inner decoding requires only a small amount of additional hardware, not likely to limit the decoding speed.

The performanc of the proposed scheme using the basic $K = 3$, rate 1/2 inner convolutional code and the outer (12, 11) SPC code is shown to be very close to the results obtained for the $K = 7$ code with Viterbi decoding. The hardware complexity of the required overall decoder is in this case roughly the same as for a 64-state Viterbi decoder, however, the achievable decoding speed might be expected to be about a factor $n = 12$ higher. This depends of course on the speed of the mux/demux implementation.

Since the proposed concatenated coding scheme is shown to be independent of the constraint length of the inner convolutional codes, a set of $K = 7$, rate 1/2, coder/decoders can also be used. This results in a concatenated code with a performance of a $d_{\text{free}} = 20$ code.

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