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On the Capacity of the Two-User *M*-ary Multiple-Access Channel with Feedback

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***Abstract*—A study is made of the Cover–Leung capacity region for two specific noiseless two-user M-ary input multiple-access channels with complete feedback. It is shown that the symmetric rate point of the Cover–Leung capacity region is on the total cooperation line for large enough values of M. Thus feedback allows communication at the same rates as if both users had complete joint knowledge of intended messages.**

I. Introduction

We consider the communication situation where two senders transmit simultaneously to a single receiver via a discrete memoryless deterministic multiple-access channel. Both senders are synchronized and have complete knowledge of the channel output Y via a noiseless feedback link. The channels accept M-ary signals X_k, $k = 1,2$ and emit a single output Y in accordance with a prescribed conditional probability distribution $P(y|x_1, x_2)$.

Two specific channel models are considered. The output of the first channel indicates the subset of input symbols, i.e.,

$$Y = \{X_1, X_2\}, \qquad X_i \in \{0,1,\cdots, M-1\}.$$

The receiver does not know the origin of the two symbols if the inputs are different. There are $\binom{M}{2}$ of these ambiguous subsets. If both inputs are equal, then only one letter is detected. Hence, the cardinality of the output is $|\mathscr{Y}| = \binom{M}{2} + M = M(M+1)/2$. This channel is referred to as the M-ary erasure multiple-access channel (MAC).

The second channel model gives as an output the arithmetic sum of the input letters, i.e.,

$$Y = X_1 + X_2, \qquad X_i \in \{0,1,\cdots, M-1\}.$$

For this channel the output cardinality is $|\mathscr{Y}| = 2M - 1$. This channel is referred to as the M-ary adder MAC. In Fig. 1 we give the input/output relations for both channels when $M = 3$. For $M = 2$ both channels reduce to the well-known binary erasure MAC.

Cover and Leung [1] found an achievable rate region for the discrete memoryless MAC with feedback, using superposition

$X_2 \backslash X_1$	0	1	2
0	{0}	{0,1}	{0,2}
1	{0,1}	{1}	{1,2}
2	{0,2}	{1,2}	{2}

$Y = \{X_1, X_2\}$

$X_2 \backslash X_1$	0	1	2
0	0	1	2
1	1	2	3
2	2	3	4

$Y = X_1 + X_2$

Fig. 1. Input/output relation for ternary erasure MAC and ternary adder MAC.

coding. Subsequently, Willems [2] showed that the region found by Cover and Leung is the feedback capacity region for MAC's for which at least one input is a function of the output Y and the other input (MAC's in class D). The two families of aforementioned M-ary MAC's belong to the class D. The feedback capacity region for MAC's in class D is elaborated in [2] and is given by

$$\begin{aligned} R_{CL}^{D} \triangleq \{(R_1, R_2)\colon & 0 \le R_1 \le H(X_1|U) \\ & 0 \le R_2 \le I(X_2; Y|X_1, U) = H(X_2|U) \\ & R_1 + R_2 \le I(X_1, X_2; Y) = H(Y), \\ & \text{for } P(u, x_1, x_2, y) \\ & \quad = P(u)P(x_1|u)P(x_2|u)P(y|x_1, x_2) \\ & \text{and } |\mathscr{U}| \le \min\{|\mathscr{X}_1| \cdot |\mathscr{X}_2| + 1, |\mathscr{Y}| + 2\}\}, \end{aligned}$$

where $|\mathscr{U}|$, $|\mathscr{X}_1|$, and $|\mathscr{X}_2|$ are the alphabet cardinalities of the random variables U, X_1, and X_2, respectively.

The maximum value of $H(Y)$ is $\log(M(M+1)/2)$ and $\log(2M-1)$ for the erasure and the adder MAC, respectively. This sum rate can be obtained when both senders operate in total cooperation.

In the evaluation of the above region one needs the maximizing probability $P(u, x_1, x_2, y)$, which could be difficult to obtain for large values of M. For $M = 2$ Willems [3] showed that the symmetrical rate pair (0.79113, 0.79113) is on the boundary of the feedback capacity region. In this case $|\mathscr{U}| = 2$ was proved to be sufficient to achieve this rate pair. It seems to be difficult to prove optimality for asymmetric rate pairs even for $M = 2$.

In the next section we evaluate the Cover–Leung region for our families of MAC's. We show that in the equal rate case the Cover–Leung capacity *equals* total cooperation capacity for specific values of M. The given rate points are thus optimal.

II. Calculations

We first concentrate on the M-ary erasure MAC with feedback. For this channel we evaluate the Cover–Leung capacity in the symmetric case, i.e., $H(X_1|U) = H(X_2|U)$.

Let $|\mathscr{U}| = M$ and

$$P(U = j) = \frac{1}{M},$$

$$P(X_k = j|U = j) \triangleq \alpha,$$

$$P(X_k = i|U = j) \triangleq \frac{1-\alpha}{M-1}, \qquad (i \ne j)$$

where $k = 1,2$, and $i, j \in \{0,1,\cdots, M-1\}$. Then

$$P(Y = \{i\}) = \sum_{j=0}^{M-1} P(Y = \{i\}|U = j)P(U = j)$$

$$= \frac{1}{M}\left(\alpha^2 + (M-1)\cdot\left(\frac{1-\alpha}{M-1}\right)^2\right),$$

$$P(Y = \{i, j\}, i \ne j) = (1 - MP(Y = \{i\})) \Big/ \frac{(M^2 - M)}{2},$$

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and

$$H(X_k|U) = h(\alpha) + (1-\alpha)\log(M-1), \qquad k = 1,2.$$

For

$$\alpha = \frac{1}{M}\left(1 + \frac{M-1}{\sqrt{M+1}}\right)$$

the probability $P(Y = \{i\})$ equals $P(Y = \{i,j\})$ for all i, j. Hence, the output entropy is given by

$$H(Y) = \log\left(\frac{M(M+1)}{2}\right),$$

which equals the total cooperation rate.

For $M = 6$, $H(X_k|U) = 2.203$ b and $\frac{1}{2}H(Y) = 2.196$ b. In general, for $M \geq 6$, one can show that the above input probability distribution gives

$$H(X_k|U) \geq \tfrac{1}{2}H(Y), \qquad k = 1,2.$$

From this it follows that, for the M-ary erasure MAC with $M \geq 6$, the symmetric rate point of the Cover–Leung capacity region equals total cooperation capacity. Note that this is achieved with $|\mathcal{U}| = M$. As mentioned before, Willems [3] showed that for $M = 2$, $|\mathcal{U}| = 2$ gives an optimal result, although not equal to total cooperation. We conjecture that $|\mathcal{U}| = M$ also gives optimal results for $3 \leq M \leq 5$.

For the M-ary adder MAC we assume that $|\mathcal{U}| = 2$ for all M. The number of different outputs for this channel is $(2M - 1)$; hence, $H(Y)$ equals $\log(2M-1)$ when all outputs have equal probability. For $M \geq 3$ we give an input probability distribution for both senders yielding $H(Y) = \log(2M-1)$. Let $|\mathcal{U}| = 2$ and

$$P(U=0) = P(U=1) = \tfrac{1}{2}$$
$$P(X_k = i|U=0) = P_0(i)$$
$$P(X_k = i|U=1) = P_1(i)$$

where $k = 1,2$, and $i = 0,1,\cdots,M-1$. Let

$$P_0(i) = P_1(M-1-i).$$

Then

$$P(Y=i) = \sum_{k+l=i}\left(\frac{P_0(k)P_0(l)}{2} + \frac{P_1(k)P_1(l)}{2}\right).$$

In the Appendix we derive a probability distribution such that $P(Y = i) = 1/(2M-1)$. Here we only give the probabilities (see (17) in the Appendix).

$$P_0(i) = \sqrt{\frac{2}{2M-1}}\cos\frac{(4i+1)\pi}{4(2M-1)}\prod_{r=1}^{i}\frac{\sin\frac{2r-1}{2M-1}\pi}{\sin\frac{2r}{2M-1}\pi}, \qquad i = 0,1,\cdots,M-1.$$

By using the fact that $P_0(i) < P_0(i-1)$ for $0 < i \leq M-1$, one can show that

$$H(X_k|U) > \tfrac{1}{2}\log(2M-1), \qquad k = 1,2; \quad M \geq 3.$$

Hence for the adder MAC we achieve total cooperation for $M \geq 3$ with a binary random variable U.

III. Conclusions

We have shown that for the M-ary erasure MAC ($M \geq 6$) and the M-ary adder MAC ($M \geq 3$) in symmetrical operation with feedback the Cover–Leung capacity equals total cooperation.

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Appendix

In this appendix we show that for the M-ary adder MAC there exists an input probability distribution such that

$$P(Y=0) = P(Y=1) = \cdots = P(Y=2n) = \frac{1}{2n+1}.$$

For notational convenience we set $n = M - 1$. The mathematical statement of the problem is as follows.

Problem

Let n be a positive integer. Find $n+1$ real positive numbers $x_0, x_1, \cdots, x_n$ satisfying the system of equations

$$\sum_{k+l=j}(x_k x_l + x_{n-k}x_{n-l}) = \frac{2}{2n+1}, \qquad j = 0,\cdots,n. \quad (1)$$

Solution

We introduce $n+1$ complex numbers $y_0,\cdots,y_n$ that are related to $x_0,\cdots,x_n$ by

$$x_k := \tfrac{1}{2}(1+i)y_k + \tfrac{1}{2}(1-i)y_{n-k}, \qquad k = 0,\cdots,n. \quad (2)$$

Then we observe that for all $k, l \in \{0,\cdots,n\}$

$$x_k x_l + x_{n-k}x_{n-l} = y_k y_{n-l} + y_{n-k}y_l, \quad (3)$$

so that (1) is transformed into the system of equations

$$y_0 y_n = \frac{1}{2n+1}$$
$$y_0 y_{n-1} + y_1 y_n = \frac{1}{2n+1}$$
$$y_0 y_{n-2} + y_1 y_{n-1} + y_2 y_n = \frac{1}{2n+1}$$
$$\cdots$$
$$y_0^2 + y_1^2 + y_2^2 \cdots + y_n^2 = \frac{1}{2n+1}. \quad (4)$$

These equations can be combined in the polynomial identity for the complex variable w that is given by

$$(y_0 + y_1 w + \cdots + y_n w^n)(y_n + y_{n-1}w + \cdots + y_0 w^n) \equiv \frac{1}{2n+1}\sum_{r=0}^{2n} w^r. \quad (5)$$

The right side of (5) vanishes if $w = \exp(2\pi i t/(2n+1))$ for some integer $t \not\equiv 0 \bmod (2n+1)$. Hence, if we define

$$\alpha := \exp\frac{2\pi i}{2n+1}, \quad (6)$$

then the product

$$(y_0 + y_1\alpha^t + \cdots + y_n\alpha^{nt})(y_n + y_{n-1}\alpha^t + \cdots + y_0\alpha^{nt})$$

must vanish for all $t = 1,\cdots,2n$.

In other words, for all $t = 1,\cdots,2n$ we have to satisfy one of the conditions

$$y_0 + y_1\alpha^t + \cdots + y_n\alpha^{nt} = 0 \quad (7a)$$

or

$$y_n + y_{n-1}\alpha^t + \cdots + y_0\alpha^{nt} = 0. \quad (7b)$$

At this point it should be mentioned that we need only consider values of t for which no pair has a sum equal to $2n+1$. This follows from the fact that for $t + u = 2n+1$, by (6), we have $\alpha^t = \alpha^{-u}$, so that

$$(y_0 + y_1\alpha^t + \cdots + y_n\alpha^{nt} = 0) \Leftrightarrow (y_n + y_{n-1}\alpha^u + \cdots + y_0\alpha^{nu} = 0).$$

Besides, from (1) and (2) we deduce that

$$y_0 + y_1 + \cdots + y_n = 1. \tag{8}$$

So we are left with 2^n possible systems of $(n+1)$ linear equations in $(n+1)$ unknown variables $y_0, \cdots, y_n$.

For our purpose we choose $t = 2l - 1$ $(l = 1, \cdots, n)$ and refer to (8) and (7a), viz.,

$$\begin{aligned} y_0 + y_1 + y_2 + \cdots + y_n &= 1 \\ y_0 + y_1\alpha + y_2\alpha^2 + \cdots + y_n\alpha^n &= 0 \\ y_0 + y_1\alpha^3 + y_2\alpha^6 + \cdots + y_n\alpha^{3n} &= 0 \\ \cdots \\ y_0 + y_1\alpha^{2n-1} + y_2\alpha^{4n-2} + \cdots + y_n\alpha^{2n^2-n} &= 0. \end{aligned} \tag{9}$$

The system (9) is a Vandermonde system, and, by Cramer's rule, its solution can be expressed by the polynomial identity in z given by (10):

$$\sum_{k=0}^{n} y_k z^k \equiv \frac{V(z, \alpha, \alpha^3, \cdots, \alpha^{2n-1})}{V(1, \alpha, \alpha^3, \cdots, \alpha^{2n-1})}. \tag{10}$$

In this expression $V(\beta_0, \beta_1, \cdots, \beta_n)$ represents a Vandermonde determinant and has the value

$$V(\beta_0, \beta_1, \cdots, \beta_n) = \prod_{0 \le k < l \le n} (\beta_l - \beta_k),$$

so that (10) can be simplified to the form

$$\sum_{k=0}^{n} y_k z^k = \frac{\prod_{l=1}^{n} (\alpha^{2l-1} - z)}{\prod_{l=1}^{n} (\alpha^{2l-1} - 1)} = \frac{N}{D}. \tag{11}$$

We shall evaluate the denominator D and the numerator N of this form separately, always bearing in mind that $\alpha = \exp(2\pi i/(2n+1))$.

The denominator D can be evaluated as follows:

$$\begin{aligned} D &= \prod_{l=1}^{n} (\alpha^{2l-1} - 1) = \alpha^{n^2}(-1)^n \prod_{l=1}^{n} (\alpha^{1-2l} - 1) \\ &= \alpha^{n^2}(-1)^n \prod_{l=1}^{n} (\alpha^{2(n+1-l)} - 1) \\ &= \alpha^{n^2}(-1)^n \prod_{l=1}^{n} (\alpha^{2l} - 1), \end{aligned}$$

so that

$$\begin{aligned} D^2 &= \alpha^{n^2}(-1)^n \prod_{l=1}^{n} (\alpha^{2l} - 1) \prod_{r=1}^{n} (\alpha^{2r-1} - 1) \\ &= \alpha^{n^2}(-1)^n \prod_{s=1}^{2n} (\alpha^s - 1). \end{aligned}$$

Now

$$\{\alpha^s | 1 \le s \le 2n\} = \left\{ w \in \mathbb{C} \,\middle|\, \sum_{k=0}^{2n} w^k = 0 \right\},$$

so that clearly

$$\{\alpha^s - 1 | 1 \le s \le 2n\} = \left\{ w \in \mathbb{C} \,\middle|\, \sum_{k=0}^{2n} (w+1)^k = 0 \right\}.$$

The equation $\sum_{k=0}^{2n} (w+1)^k = 0$, by Newton's binomial formula, can be written in the form

$$w^{2n} + \cdots + w + (2n+1) = 0,$$

so that the product of its roots equals $2n+1$. In other words, we get $D^2 = \alpha^{n^2}(-1)^n(2n+1)$, so that $D = \pm\alpha^{n^2/2} i^n \sqrt{2n+1}$, and we only have to determine the sign. Recall that

$$\begin{aligned} D &= \prod_{l=1}^{n} (\alpha^{2l-1} - 1) = \alpha^{n^2/2} \prod_{l=1}^{n} (\alpha^{l-(1/2)} - \alpha^{-l+(1/2)}) \\ &= \alpha^{n^2/2} (2i)^n \prod_{l=1}^{n} \sin \frac{2\pi(l - \frac{1}{2})}{2n+1}. \end{aligned}$$

The sine product is clearly positive, so that we finally obtain

$$D = \alpha^{n^2/2} i^n \sqrt{2n+1}. \tag{12}$$

For the numerator N we write

$$N = \prod_{l=1}^{n} (\alpha^{2l-1} - z) = \alpha^{n^2} \prod_{l=1}^{n} \left(1 - \frac{z}{\alpha^{2l-1}}\right).$$

Let us define $g(z)$ by

$$\begin{aligned} g(z) &:= \left(1 - \frac{z}{\alpha}\right)\left(1 - \frac{z}{\alpha^3}\right)\left(1 - \frac{z}{\alpha^5}\right) \cdots \left(1 - \frac{z}{\alpha^{2n-1}}\right) \\ &= \sum_{k=0}^{n} g_k z^k. \end{aligned} \tag{13}$$

One easily checks that

$$\left(1 - \frac{z}{\alpha^{2n-1}}\right) g(\alpha^2 z) \equiv (1 - \alpha z) g(z)$$

or

$$(1 - \alpha^2 z) g(\alpha^2 z) \equiv (1 - \alpha z) g(z)$$

so that

$$g_k \alpha^{2k} - g_{k-1}\alpha^{2k} = g_k - \alpha g_{k-1}, \qquad 1 \le k \le n.$$

In other words, we can write

$$g_k = \frac{\alpha^{2k} - \alpha}{\alpha^{2k} - 1} g_{k-1} = \alpha^{1/2} \cdot \frac{\alpha^{k-(1/2)} - \alpha^{-k+(1/2)}}{\alpha^k - \alpha^{-k}} g_{k-1}$$

and get the expressions

$$g_k = \exp \frac{\pi i k}{2n+1} \prod_{r=1}^{k} \sin \frac{(2r-1)\pi}{2n+1} \Big/ \sin \frac{2r\pi}{2n+1}, \qquad 0 \le k \le n \tag{14}$$

$$N = \alpha^{n^2} \sum_{k=0}^{n} z^k \exp \frac{\pi i k}{2n+1} \cdot \prod_{r=1}^{k} \sin \frac{(2r-1)\pi}{2n+1} \Big/ \sin \frac{2r\pi}{2n+1}. \tag{15}$$

It is immediately seen that

$$\alpha^{n^2/2} = \exp \frac{n^2}{2n+1}\pi i = \exp\left(\frac{1}{2}n - \frac{1}{4} + \frac{1}{4}\frac{1}{2n+1}\right)\pi i$$

or

$$\alpha^{n^2/2} = i^n \exp\left(-\frac{\pi i}{4}\right) \exp \frac{\pi i}{4(2n+1)}.$$

Now we can combine (11), (12), and (15) to obtain

$$y_k = \frac{1}{\sqrt{2n+1}} \exp\left(\frac{k}{2n+1} - \frac{1}{4} + \frac{1}{4(2n+1)}\right)\pi i \cdot \prod_{r=1}^{k} \sin \frac{(2r-1)\pi}{2n+1} \Big/ \sin \frac{2r\pi}{2n+1}. \tag{16}$$

Observe that

$$\prod_{r=1}^{n-k} \sin \frac{(2r-1)\pi}{2n+1} \Big/ \sin \frac{2r\pi}{2n+1} = \prod_{r=1}^{k} \sin \frac{(2r-1)\pi}{2n+1} \Big/ \sin \frac{2r\pi}{2n+1}$$

and

$$x_k = \frac{1}{\sqrt{2}}\left(y_k \exp \frac{\pi i}{4} + y_{n-k} \exp \frac{-\pi i}{4} \right).$$

Next we find

$$\exp\left(\frac{k}{2n+1} - \frac{1}{4} + \frac{1}{4(2n+1)} + \frac{1}{4} \right)\pi i + \exp\left(\frac{n-k}{2n+1} - \frac{1}{4} + \frac{1}{4(2n+1)} - \frac{1}{4} \right)\pi i$$

$$= \exp \frac{4k+1}{4(2n+1)}\pi i + \exp \frac{-4k-1}{4(2n+1)}\pi i$$

$$= 2\cos \frac{(4k+1)\pi}{4(2n+1)},$$

so that finally

$$x_k = \sqrt{\frac{2}{2n+1}} \cos \frac{(4k+1)\pi}{4(2n+1)} \cdot \prod_{r=1}^{k} \sin \frac{2r-1}{2n+1}\pi \Big/ \sin \frac{2r}{2n+1}\pi, \qquad k = 0,\cdots,n. \quad (17)$$

The numbers $x_0, x_1, \cdots, x_n$ are clearly positive and satisfy (1).

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Multiple Detection of a Slowly Fluctuating Target

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***Abstract*—A "slowly" fluctuating target is assumed to keep its radar cross section constant for the duration of several (M) dwells on target. To resolve multiple range and/or Doppler ambiguities, the received signal, which is presumably coherently processed (i.e., predetection integrated or matched filtered) over each dwell, must often be tested against a threshold, *independently* of those on other dwells. Such a procedure is referred to as *multiple detection*. A technique for the evaluation of a tight lower bound on the multiple-detection probability P_M, under Swerling case I statistics for the cross section, is presented in term of an infinite series and worked out in detail for P_2 and P_3. Estimates on the computation error due to the truncation of the series are derived. Numerical results indicate that P_3 comes much closer to P_1 than to P_1^3 or even to P_1P_2; at an expected signal-to-noise ratio of 13 dB and at $P_1 = 0.51$, it obtains that $P_3 \geq 0.40$, whereas $P_1P_2 = 0.23$ and $P_1^3 = 0.17$.**

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I. INTRODUCTION

The probability of detecting a signal of random phase in additive white Gaussian noise is given by Marcum's Q-function [1]

$$Q(a,y) = \int_y^\infty u \exp\left(-\frac{1}{2}(u^2+a^2)\right) I_0(au)\, du, \qquad (1)$$

where a and y are the amplitude and the comparison threshold, respectively, normalized to the standard deviation of the noise, and I_0 is the modified Bessel function. If the signal amplitude and the noise variance are constant during M successive tests, the probability of M successes is $Q^M(a,y)$. Under the Swerling case I model, a is assumed to be Rayleigh distributed, so that the expected multiple-detection probability $P_M(\bar{x}, y)$ is

$$P_M(\bar{x},y) = \int_0^\infty a\bar{x}^{-1} \exp\left(-\frac{1}{2}a^2\bar{x}^{-1}\right) Q^M(a,y)\, da, \qquad (2)$$

where $\bar{x}$ is the expected signal-to-noise ratio (SNR). P_1 has been extensively tabulated [1]–[3]. Here we present a technique for evaluating P_M in terms of a truncated series and for bounding the truncation error. The technique ensures that all the terms of the series are positive and therefore that the values obtained are a strict lower bound. Numerical results are obtained for P_2 and P_3. P_2 has been calculated before using a different summation technique [4].

II. AN EVALUATION TECHNIQUE

Using (1) in (2) yields

$$P_M(\bar{x},y) = \bar{x}^{-1}\int_0^\infty da \int_y^\infty \cdots \int_y^\infty du_1 \cdots du_M u_1 \cdots u_M \cdot \exp\left\{-\frac{1}{2}(u_1^2 + \cdots + u_M^2)\right\} \cdot a \exp\left\{-\frac{1}{2}a^2(M+\bar{x}^{-1})\right\} \cdot I_0(au_1)\cdots I_0(au_M). \qquad (3)$$

Denoting $K_M \triangleq (M+\bar{x}^{-1})/2$ and using

$$I_0(x) = \sum_{i=0}^{\infty} (x/2)^{2i}(i!)^{-2}, \qquad (4)$$

one obtains a new form in which the integrands are decoupled,

$$P_M(\bar{x},y) = \sum_{i_1=0}^{\infty} \cdots \sum_{i_K=0}^{\infty} \left[\bar{x}(i_1!)^2 \cdots (i_M!)^2\right]^{-1} \cdot \int_y^\infty du_1 u_1 (u_1/2)^{2i_1} \exp\left(-\frac{1}{2}u_1^2\right) \cdots \int_y^\infty du_M u_M (u_M/2)^{2i_M} \exp\left(-\frac{1}{2}u_M^2\right) \cdot \int_0^\infty daaa^{2(i_1+\cdots+i_M)} \exp(-K_M a^2). \qquad (5)$$

Using [5, eqs. (6.1.1), (6.1.15), (6.5.22)], one obtains

$$P_M(\bar{x},y) = (2\bar{x}K_M)^{-1} \sum_{i_1=0}^{\infty} \cdots \sum_{i_M=0}^{\infty} G(i_1,y)\cdots G(i_M,y) \cdot (i_1+\cdots+i_M)!\, K_M^{-(i_1+\cdots+i_M)}, \qquad (6)$$

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