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On the Capacity of the Asynchronous T-User M-Frequency Noiseless Multiple-Access Channel Without Intensity Information

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***Abstract*—We discuss an achievable rate region for the asynchronous T-user M-frequency multiple-access channel without intensity information. The problem is formulated in terms of frequencies, but the results are also applicable to Pulse Position Modulation (PPM) schemes. It will be shown that the achievable sum rate for T users reduces from $(M-1)$ bits per channel use in the synchronous multiple-access situation to at least $(M-1)\cdot\ln(2)$ bits per channel use in the asynchronous situation. In particular, the result shows that for instance, for multitone M-ary frequency shift keying multiple access, in asynchronous operation, multiple user interference reduces the capacity maximally by a factor $\ln(2)=0.693$ relative to the ideal TDMA system.**

***Index Terms*— Asynchronous multiple access, capacity, multitone MFSK.**

I. Introduction

Cohen, Heller, and Viterbi [1] presented a new approach to asynchronous multiple-access digital communications, which will be briefly described here. In asynchronous multiple access, one assumes that T individual users can access the system independently of the other users. Each user transmits by means of on–off signaling without regard to, or knowledge of the remaining $T-1$ users. Assuming that all users utilize the same duty cycle, determination of channel capacity reduces to the problem of calculating the capacity for a Z-channel where the transition probabilities are functions of the input distribution, see Fig. 1.

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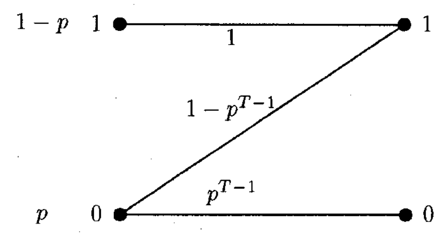

Fig. 1. The binary input–binary output asynchronous Z-channel.

In a synchronous Time-Division Multiple-Access (TDMA) system, each user would be assigned $1/T$ of the available dimensions and with on–off signaling the transmission rate can be 1 bit/dimension, yielding a capacity per user of $1/T$ bits per dimension. In [1] it has been shown that in the asynchronous system, multiple user interference reduces the total capacity for T users only by a factor of $\ln(2)=0.693$ relative to the ideal TDMA system. This efficiency can be achieved by using low-duty-cycle signaling.

In a synchronous Frequency-Division Multiple-Access (FDMA) system the available M dimensions are shared equally among the M users, leading to a transmission rate of 1 bit/dimension. In the situation that is under consideration here, the number of dimensions M is fixed while the number of users T goes to infinity. In addition, each particular user is allowed to transmit only one out of the M orthogonal frequencies (M-tone frequency shift keying).

Chang and Wolf [2] considered the synchronous (or coordinated as they call it) T-user M-frequency noiseless multiple-access channel without intensity information, and called it the A channel. Each component of each of the T channel input vectors is chosen from a common alphabet $\{f_1, f_2, \cdots, f_M\}$. At each time instant, the output of the A channel is a symbol which identifies which subset of frequencies were inputs to the channel at that time instant but *not* how many of each frequency occurred. Only one receiver decodes all users simultaneously. For a large number of users, the channel capacity approaches $M-1$ bits per signaling interval of M frequencies. The results were arrived at by a computer search.

In this correspondence, an achievable rate for the asynchronous T-user M-frequency noiseless multiple-access channel without intensity information will be derived. This channel will be referred to as the A' channel. We are interested in a large number of channel users and will therefore assume that $T\to\infty$. This will be realized by letting T go to infinity in an expression for an achievable rate with both M and T fixed. It will follow that the capacity reduces from $M-1$ bits per channel use in the synchronous multiple-access situation to at least $(M-1)\cdot\ln(2)$ bits per channel use in the asynchronous situation, when no coordination between users or one-to-one communication is assumed. For PPM, the signaling interval is partitioned into M subintervals or time slots. During a signaling interval, only one of the M subintervals is used to transmit a pulse. Another practical example, equivalent to PPM, of such a signaling is multitone (M-tone) frequency shift keying, where a specific user transmits *one* out of M orthogonal frequencies.

In Section II, the capacities of the two- and three-frequency A' channels will be discussed. The asymptotic optimizing input

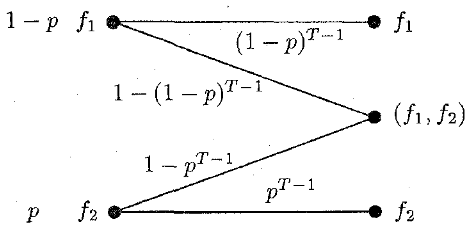

Fig. 2. The two-frequency A' channel.

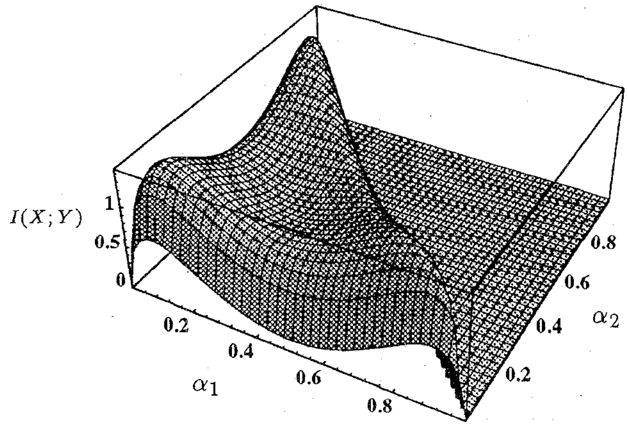

Fig. 3. Mutual information of the three-frequency A' channel for $T = 7$, where $\alpha_i = \mathbb{P}(X = f_i)$.

distribution is highly asymmetric, indicating that each of the users must transmit a low-duty-cycle signal. In Section III, the system will be extended to $M \geq 4$ frequencies and an input distribution will be given from which it follows that the asymptotic channel capacity (as $T \to \infty$) is lower-bounded by $(M-1) \cdot \ln(2)$ bits per frequency interval. Furthermore, it will be shown by using the asynchronous parallel channels, that the asymptotic channel capacity is upper-bounded by $M \cdot \ln(2)$, a reduction by a factor of $\ln(2)$ compared to the idealized FDMA situation. Since the channel transition probabilities are functions of the input distribution, the Kuhn–Tucker conditions cannot be used on a candidate (capacity achieving) input distribution.

II. The Two- and Three-Frequency Channels

First, two-tone signaling (Fig. 2) will be discussed. It is assumed that each user transmits by using one-out-of-two frequencies f_1 and f_2 without regard to, or knowledge of the remaining $T-1$ other users. All users utilize the same duty cycle, and $1-p$ is the probability that a user transmits the frequency f_1. Fig. 2 gives the channel for a particular user. The output is ternary $\{f_1, f_2, (f_1, f_2)\}$. Unfortunately, optimizing the mutual information with respect to p requires solving a transcendental equation. Therefore various numerical methods have to be used to find the maximum value of this mutual information. Of course, these methods give no actual proof what the maximal mutual information is, only an indication. Optimizing with respect to p and letting T go to infinity, all used numerical methods indicate that the capacity of the channel becomes $C(2,T) \to \ln(2)$ bits. For each individual user, the two-tone multiple-access channel is asymptotically, from a capacity point of view, equivalent to the binary input–binary output Z-channel as it has been discussed in [1]. Although the two-tone multiple-access channel has a ternary output, the asymptotic capacity is the same as if a hard decision were made in case of an ambiguous reception of two frequencies.

For the A' channel with $M = 3$, numerical methods indicate that the asymptotic channel capacity approaches $C(3,T) \to 2 \cdot \ln(2)$ bits (see Fig. 3, where $\alpha_i = \mathbb{P}(X = f_i)$). The limiting input distribution for both examples places the same amount of weight on $M-1$ frequencies and the remaining weight on the last frequency. For $M = 2$, this limiting input distribution is $\mathbb{P}_T(f_1) = \ln(2)/T$ and $\mathbb{P}_T(f_2) = 1 - \ln(2)/T$ and for $M = 3$ it is $\mathbb{P}_T(f_1) = \mathbb{P}_T(f_2) = \ln(2)/T$ and $\mathbb{P}_T(f_3) = 1 - 2 \cdot \ln(2)/T$, respectively.

III. The M-Frequency Channel

For the A' channel (or the asynchronous T-user M-frequency noiseless multiple-access channel without intensity information), the symbol $(f_{i_1}, f_{i_2}, \cdots, f_{i_l})$ will be used to identify the occurrence of the frequencies f_{i_1} through f_{i_l} as inputs of the channel. Note that the symbol gives no information on how many of each frequency have been transmitted. Since all users utilize the same duty cycle, the channel can be represented by T parallel channels, one channel per user. These channels will be referred to as the single transmitter channels. Due to the multiple user interference, the transition and output probabilities of these single transmitter channels are functions of the input distribution of the users. If Y is the random variable that gives the output distribution of the channel and X_n is the random variable that gives the input distribution of the nth user, it follows that

$$\mathbb{P}(Y = (f_{i_1}, \cdots, f_{i_l}) | X_n = f_{i_m}) = \mathbb{P}(\{X_1, \cdots, X_T\} = \{f_{i_1}, \cdots, f_{i_l}\} | X_n = f_{i_m}).$$

Since all users utilize the same duty cycle, a general formula for the transition probabilities of the nth single-transmitter channel can be given in terms of the numbers $\alpha_i = \mathbb{P}(X_n = f_i)$. The following theorem gives the formula for the transition probability

$$\mathbb{P}(Y = (f_{i_1}, \cdots, f_{i_l}) | X_n = f_{i_m}).$$

Let W be defined by $W = \{i_1, i_2, \cdots, i_l\} \setminus \{i_m\}$.

Theorem 1 (Transition Probabilities)

$$\mathbb{P}(Y = (f_{i_1}, f_{i_2}, \cdots, f_{i_l}) | X_n = f_{i_m}) = \sum_{V \subseteq W} (-1)^{|V|} \left(\alpha_{i_m} + \sum_{j \in W \setminus V} \alpha_j \right)^{T-1}.$$

The proof of this theorem, that will be given by means of the mathematical principle of inclusion and exclusion, can be found in the Appendix. In exactly the same way, the principle of inclusion and exclusion can be used to give a general formula for the output probabilities. The following theorem gives the formula for the output probability $\mathbb{P}(Y = (f_{i_1}, \cdots, f_{i_l}))$ and the proof of this theorem will be omitted. Let $\tilde{W}$ be defined by $\tilde{W} = \{i_1, i_2, \cdots, i_l\}$.

Theorem 2 (Output Probabilities):

$$\mathbb{P}(Y = (f_{i_1}, f_{i_2}, \cdots, f_{i_l})) = \sum_{V \subseteq \tilde{W}} (-1)^{|V|} \left(\sum_{j \in \tilde{W} \setminus V} \alpha_j \right)^{T}.$$

From Section II, one might guess that in general for $M \geq 4$, the capacity achieving input distribution puts a probability $1 - (M-1) \cdot \ln(2)/T$ on one frequency and a probability $\ln(2)/T$ on each of the remaining $M-1$ frequencies. With the formulas for the transition and output probabilities, the mutual information $I(X;Y)$ of a single transmitter channel can be expressed as a function of the input distribution of the users. The following theorem will give the asymptotic mutual information of the entire channel, using the postulated input distribution.

Theorem 3: The mutual information $I(X;Y)$ of the M-frequency A' channel for the input distribution that puts the probability $1-(M-1)\cdot\ln(2)/T$ on one frequency and the probability $\ln(2)/T$ on the remaining $M-1$ frequencies, tends to $(M-1)\cdot\ln(2)$, as $T\rightarrow\infty$.

This theorem can be proven by means of a Taylor series. A sketch of this proof is given in the Appendix. The complete proof can be found in [4]. The mutual information, achieved by the postulated input distribution, can be used as a lower bound on the capacity of the A' channel. As mentioned in Section II, numerical methods indicate that the lower bound is tight for the two- and three-frequency channels.

Unfortunately, since the channel transition probabilities are functions of the input distribution, the Kuhn–Tucker conditions cannot be used on a candidate (capacity-achieving) input distribution as given above. For the same reason, the mutual information of a single-transmitter channel is not a concave function (see Fig. 3). Since the Kuhn–Tucker conditions cannot be used, an upper bound on the capacity will be given that is close to the lower bound.

Consider the extended asynchronous system, where instead of using only one of M frequencies, each user can transmit any subset of the set of M frequencies. In the asynchronous situation, all users are assumed to have the same input distribution. The receiver simply detects the presence or nonpresence of a frequency for each of the M frequencies. This system can be considered as M independent parallel channels, one "frequency channel" for each frequency. The output for each particular frequency channel is statistically related only through the input to that channel. Each frequency channel is thus equivalent to the previously described Z-channel with capacity $\ln(2)$ as $T\rightarrow\infty$. It follows from [3] that the capacity of the M parallel frequency channels approaches $M\ln(2)$ as $T\rightarrow\infty$. By allowing each user to transmit only one out of M frequencies, only a subset of the possible inputs of this extended system is used. Hence, the capacity of the A' channel can be bounded as

$$(M-1)\cdot\ln(2)\leq\lim_{T\rightarrow\infty}C_{A'}(M,T)\leq M\cdot\ln(2).$$

IV. Conclusions

The asymptotic capacity (as $T\rightarrow\infty$) for the A' channel approaches $\ln(2)$ bits per frequency, as $M\rightarrow\infty$.

An input distribution of which the mutual information approaches the asymptotic capacity, places a large amount of weight on one frequency and distributes the remaining weight equally over the remaining $M-1$ frequencies.

If instead of transmitting any combination of M frequencies, users are only allowed to transmit one out of M frequencies, the asymptotic capacity reduces maximally from $M\cdot\ln(2)$ to $(M-1)\cdot\ln(2)$. Using only a single frequency from an M-frequency interval, is the advantage of M-tone frequency shift-keying systems.

Appendix

In this Appendix, the proof of Theorem 1 and a sketch of the proof of Theorem 3 will be given.

Some events will be defined and used to prove the formula for the transition probabilities. The events will involve certain discrete random variables that will be defined first. Consider the nth single-transmitter channel. For every $i\in\{1,2,\cdots,M\}$, a discrete random variable N_i is defined as the number of users that transmitted frequency f_i, where the nth user is excluded. From this definition it may be concluded that

- $N_i\in\{0,1,\cdots,T-1\}$, for $i=1,2,\cdots,M$ and
- $\sum_{i=1}^{M}N_i=T-1$.

The notation $E\{\cdot\}$ will be used to denote an event. The following events will be defined:

$$A=E\begin{Bmatrix}N_k\geq 0, & \text{for } k\in\{i_1,\cdots,i_l\}\\ N_k=0, & \text{for } k\in\{1,\cdots,M\}\backslash\{i_1,\cdots,i_l\}\end{Bmatrix}$$

and for every $j\in W$

$$A_j=E\begin{Bmatrix}N_k\geq 0, & \text{for } k\in\{i_1,\cdots,i_l\}\backslash\{j\}\\ N_k=0, & \text{for } k\in\{1,\cdots,M\}\backslash\{i_1,\cdots,i_l\}\\ N_j=0 & \end{Bmatrix}.$$

By these definitions, it follows that the events $A\backslash A_j$ and

$$E\begin{Bmatrix}N_k\geq 0, & \text{for } k\in\{i_1,\cdots,i_l\}\backslash\{j\}\\ N_k=0, & \text{for } k\in\{1,\cdots,M\}\backslash\{i_1,\cdots,i_l\}\\ N_j\geq 1 & \end{Bmatrix}$$

are identical for all $j\in W$. Also a notation is required. For every subset $V\subseteq W$ the event A_V is defined by

$$A_V=\bigcap_{j\in V}A_j.$$

These definitions provide the necessary tools to prove the theorem.

Proof of Theorem 1

From the definition of the random variables N_i for $i=1,2,\cdots,M$, it follows that

$$\begin{aligned}&\mathbb{P}(Y=(f_{i_1},\cdots,f_{i_l})|X_n=f_{i_m})\\ &=\mathbb{P}(\{X_1,\cdots,X_T\}=\{f_{i_1},\cdots,f_{i_l}\}|X_n=f_{i_m})\\ &=\mathbb{P}\begin{pmatrix}N_k\geq 1, & \text{for } k\in\{i_1,\cdots,i_l\}\backslash\{i_m\}\\ N_{i_m}\geq 0 & \\ N_k=0, & \text{for } k\in\{1,\cdots,M\}\backslash\{i_1,\cdots,i_l\}\end{pmatrix}.\end{aligned}$$

Using the notations given above, this equation can be written as

$$\mathbb{P}(Y=(f_{i_1},\cdots,f_{i_l})|X_n=f_{i_m})=\mathbb{P}(A\backslash A_W).$$

The principle of inclusion and exclusion for chances yields that

$$\mathbb{P}(Y=(f_{i_1},\cdots,f_{i_l})|X_n=f_{i_m})=\sum_{V\subseteq W}(-1)^{|V|}\mathbb{P}(A_V).$$

The probability $\mathbb{P}(A_V)$ that event A_V occurs is equal to

$$\begin{aligned}&\mathbb{P}\begin{pmatrix}N_j\geq 0, & \text{for } j\in\{i_1,\cdots,i_l\}\backslash V\\ N_j=0, & \text{for } j\in\{1,\cdots,M\}\backslash\{i_1,\cdots,i_l\}\\ N_j=0, & \text{for } j\in V\end{pmatrix}\\ &=\left(\sum_{j\in\{i_1,\cdots,i_l\}\backslash V}\alpha_j\right)^{T-1}=\left(\alpha_{i_m}+\sum_{j\in W\backslash V}\alpha_j\right)^{T-1}.\end{aligned}$$

So, the transition probability equals

$$\begin{aligned}&\mathbb{P}(Y=(f_{i_1},\cdots,f_{i_l})|X_n=f_{i_m})\\ &=\sum_{V\subseteq W}(-1)^{|V|}\left(\alpha_{i_m}+\sum_{j\in W\backslash V}\alpha_j\right)^{T-1}.\quad\square\end{aligned}$$

Sketch of the Proof of Theorem 3

Since all single-transmitter channels are identical and asynchronous, the mutual information $I(X;Y)$ of the A' channel is equal to $T\cdot I'(X;Y)$, where $I'(X;Y)$ is the mutual information of one single-transmitter channel. To determine the asymptotic behavior of the mutual information for $T\rightarrow\infty$, the asymptotic behavior of $T\cdot I'(X=a;Y=b)$ will be determined for every possible transition from a to b, where $I'(X=a;Y=b)$ is defined as

$$\mathbb{P}(X=a)\mathbb{P}(Y=b|X=a)\log\left(\frac{\mathbb{P}(Y=b|X=a)}{\mathbb{P}(Y=b)}\right).$$

Transitions from f_{i_m} to $(f_{i_1}, f_{i_2}, \cdots, f_{i_l})$ will be considered, where a distinction between three different types of transitions will be made, namely

I. $M \notin \{i_1, i_2, \cdots, i_l\}$,
II. $M \in \{i_1, i_2, \cdots, i_l\}$ and $i_m \neq M$,
III. $M \in \{i_1, i_2, \cdots, i_l\}$ and $i_m = M$.

I The Case $M \notin \{i_1, i_2, \cdots, i_l\}$: Using Theorems 1 and 2, it follows that as $T \to \infty$

$$\begin{aligned} &T \cdot I'(X = f_{i_m}; Y = (f_{i_1}, f_{i_2}, \cdots, f_{i_l})) \\ &\quad = T\left(\frac{\ln(2)}{T}\right)^T \sum_{n=0}^{l-1} (-1)^n \binom{l-1}{n} (l-n)^{T-1} \\ &\qquad \cdot \log \left(\frac{\sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} (l-k)^{T-1}}{\frac{\ln(2)}{T} \sum_{k=0}^{l-1} (-1)^k \binom{l}{k} (l-k)^T} \right) \to 0. \end{aligned} \tag{1}$$

So, transitions of type I do not contribute to the asymptotic mutual information of the channel.

II. The Case $M \in \{i_1, i_2, \cdots, i_l\}$ and $i_m \neq M$: Using Theorems 1 and 2, it follows that as $T \to \infty$

$$\begin{aligned} &T \cdot I'(X = f_{i_m}; Y = (f_{i_1}, f_{i_2}, \cdots, f_{i_l})) \\ &\quad \to \ln(2) \left(\sum_{k=0}^{l-2} (-1)^k \binom{l-2}{k} e^{-(M-l+k)\ln(2)} \right) \\ &\qquad \cdot \log \left(\frac{\sum_{k=0}^{l-2} (-1)^k \binom{l-2}{k} e^{-(M-l+k)\ln(2)}}{\sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} e^{-(M-l+k)\ln(2)}} \right) \\ &\quad = \ln(2) \left(\frac{1}{2}\right)^{M-2}. \end{aligned} \tag{2}$$

The contribution of transitions of type II to the asymptotic mutual information of the channel equals

$$\begin{aligned} &\sum_{i_m=1}^{M-1} \sum_{l=2}^{M} \sum_{\substack{\{i_1, i_2, \cdots, i_l\} \subseteq \{1,2,\cdots,M\} \\ i_1 \le i_2 \le \cdots \le i_l = M}} \ln(2) \left(\frac{1}{2}\right)^{M-2} \\ &\quad = (M-1) \cdot \ln(2) \left(\frac{1}{2}\right)^{M-2} \sum_{l=2}^{M} \binom{M-2}{l-2} \\ &\quad = (M-1) \cdot \ln(2). \end{aligned}$$

III. The Case $M \in \{i_1, i_2, \cdots, i_l\}$ and $i_m = M$. Using Theorems 1 and 2, it follows that as $T \to \infty$

$$\begin{aligned} &T \cdot I'(X = f_{i_m}; Y = (f_{i_1}, f_{i_2}, \cdots, f_{i_l})) \\ &\quad \to -\sum_{k=0}^{l-1} (-1)^k \binom{l-1}{k} (M-l+k) e^{-(M-l+k)\ln(2)} \\ &\quad = -(M-l)\left(\frac{1}{2}\right)^{M-2} + (M-1)\left(\frac{1}{2}\right)^{M-1}. \end{aligned} \tag{3}$$

The contribution of transitions of type III to the asymptotic mutual information of the channel equals

$$\begin{aligned} &\sum_{l=1}^{M} \sum_{\substack{\{i_1, i_2, \cdots, i_l\} \subseteq \{1,2,\cdots,M\} \\ i_1 < i_2 < \cdots < i_l = i_m = M}} \left(\frac{1}{2}\right)^{M-1} (2l - M - 1) \\ &\quad = \left(\frac{1}{2}\right)^{M-1} \sum_{l=1}^{M} \binom{M-1}{l-1} (2l - M - 1) \\ &\quad = M + 1 - M - 1 = 0. \end{aligned}$$

Hence, it holds that as $T \to \infty$

$$I(X;Y) \to (M-1)\ln(2).$$

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Bounds on the Aliasing Error in Multidimensional Shannon Sampling

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***Abstract*—We present a pair of sharp lower and upper bounds on the 2-norm of the aliasing error in general multiband sampling representations for not necessarily bandlimited multidimensional functions. These bounds improve and generalize previous bounds. They also complement a recent uniform upper bound due to Higgins.**

***Index Terms*— Shannon sampling, aliasing error, multidimensional, multiband, Poisson summation, error energy.**

I. Introduction

The Whitaker–Shannon–Kotelnikov (WSK) sampling theorem has been extended to multivariate functions bandlimited on sets in n-dimensional Euclidean space [1] (see also [2], [3], and the references therein). In many applications, and in particular in multiple dimensions where the signal is usually spatially limited, the signal is only approximately bandlimited. This leads to an *aliasing error* in the sampling series expansion. While bounds on the aliasing error in the one-dimensional (1-D) case have also been studied extensively for the past three decades [2]–[7], their extensions to n dimensions

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