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4-ary Codebook Design Using Reed Muller Codes for MIMO Beamforming Systems

Young Gil Kim, *Member, IEEE* and A. J. Han Vinck, *Fellow, IEEE*

***Abstract*—We design 4-ary codebooks using Reed Muller codes for multiple-input multiple-output (MIMO) beamforming systems. An upperbound on the maximum correlation magnitude metric of a 4-ary codebook is derived from the minimum Hamming distance of the corresponding binary linear block code. The 4-ary MIMO beamforming codebooks for the number of transmit antennas equal to 4, 8, 16, and 32 are tabulated. We show that the 4-ary codebook using Reed Muller codes provides the same symbol error probability (SEP) as the LTE codebook when the number of transmit antennas is four and the number of feedback bits is four.**

***Index Terms*— Grassmannian codebook, multiple-input multiple-output (MIMO) beamforming, Reed Muller code.**

I. Introduction

Among closed-loop transmit diversity schemes, multiple-input multiple-output (MIMO) beamforming systems with limited feedback have been considered to reduce the feedback information rate [1]- [15]. The codebooks adopted for recent standards demonstrate trends towards systematic finite alphabet codebooks [11, p. 63], [12, p. 457]. A binary codebook design for MIMO beamforming systems using limited feedback based on the Grassmannian beamforming criterion was investigated in [13]. We denote the number of transmit antennas and the number of feedback bits by M_{t} and F, respectively. Then, the number of weight vectors in a codebook is 2^F. In [1], the codebooks were designed only for the case of $M_{\mathrm{t}} = 4$ and 8. In [2], quadrature amplitude modulation (QAM) and phase shift keying (PSK) codebook design was investigated. A problem of QPSK codebook in [2] is that the number of feedback bits cannot be arbitrarily chosen. In [3], 4-ary rank-1 and rank-2 codebooks were designed for $M_{\mathrm{t}} = 4$ over correlated and dual polarized channels. In [5], codebooks for $M_{\mathrm{t}} = 2, 3, 4$ were tabulated. In [6], 4-ary codebooks for MIMO beamforming are tabulated only for $M_{\mathrm{t}} = 4, 6,$ and 8.

In this letter, we design 4-ary codebooks using Reed Muller codes for MIMO beamforming systems with $M_{\mathrm{t}} = 2^m$. The elements in a 4-ary codebook are $1, -1, j,$ or $-j$, where $j = \sqrt{-1}$. By using this 4-ary codebook, the average power transmission at the transmit antennas is constant, and computational complexity for finding the optimum weight vector and the storage requirement can be reduced. We tabulate 4-ary MIMO beamforming codebooks using Reed Muller codes for $M_{\mathrm{t}} = 4, 8, 16, 32$.

Since the generator matrix for Reed Muller code can be easily obtained, the 4-ary codebooks using Reed Muller codes can be obtained instantly. Thus, there is almost no design complexity for 4-ary codebook design using Reed Muller codes. An upperbound on the maximum correlation magnitude metric of a 4-ary codebook is derived from the minimum Hamming distance of the corresponding binary block code. It is shown that the 4-ary codebooks using Reed Muller codes have the same symbol error probability (SEP) performance as the codebooks in LTE standard in [11] for $M_{\mathrm{t}} = 2$ and 4.

This letter consists of six sections. In Section II, we describe the system model. In Section III, we design 4-ary codebooks using binary linear block codes and an upper bound of the maximum correlation magnitude metric for 4-ary codebooks is derived. In Section IV, design procedure of 4-ary codebooks using Reed Muller codes is described. In Section V, numerical results are presented. Finally, conclusions are drawn in Section VI.

II. System Model

We consider a MIMO beamforming system using limited feedback with M_{t} transmit antennas and M_{r} receive antennas shown in Fig. 1. We assume that maximal ratio combining (MRC) is used at the receiver. An M-ary symbol s with unit average energy is pre-encoded by a beamforming weight vector $\mathbf{w} = [w_1, w_2, ..., w_{M_{\mathrm{t}}}]^T$ from a 4-ary codebook $\mathcal{W}$, where $(\cdot)^T$ denotes transpose and $\sum_{i=1}^{M_{\mathrm{t}}} |w_i|^2 = M_{\mathrm{t}}$. Note that $w_i = 1, -1, j$ or $-j$ for all $i = 1, 2, ..., M_{\mathrm{t}}$ for a 4-ary codebook. The transmitted signal is given by $\sqrt{\frac{E_s}{M_{\mathrm{t}}}}\mathbf{w} \cdot s$, where E_s is the transmit symbol energy. Thus, the transmit energy per antenna is E_s/M_{t}. We denote the transmit bit energy which is equal to $E_s/\log_2 M$, by E_b. We use a 4-ary codebook that consists of 2^F weight vectors. The transmitter selects the weight vector $\mathbf{w}$ from the codebook $\mathcal{W}$ using the feedback signal from the receiver. The output signal of the maximal ratio combiner at the receiver is given by

$$r = \sqrt{\frac{E_s}{M_{\mathrm{t}}}}\mathbf{z}^{\mathrm{H}}\mathbf{H}\mathbf{w} \cdot s + \mathbf{z}^{\mathrm{H}}\mathbf{n} = \sqrt{\frac{E_s}{M_{\mathrm{t}}}}\mathbf{z}^{\mathrm{H}} \left(\sum_{l=1}^{M_{\mathrm{t}}} \mathbf{h}_l w_l \right) s + \mathbf{z}^{\mathrm{H}}\mathbf{n}$$

where $\mathbf{H}$ is the $M_{\mathrm{r}} \times M_{\mathrm{t}}$ channel gain matrix whose elements $h_{i,j}$ are independent and identically distributed (iid) circularly symmetric complex Gaussian fading gains with unit variance and $\mathbf{h}_l$ is the l^{th} column vector of $\mathbf{H}$. The gain from the jth

 Y. G. Kim is with the Department of Electrical & Computer Engineering, University of Seoul, Seoul, Korea 130-743 (e-mail: ygkim@ieee.org). A. J. Han Vinck is with University of Duisburg Essen, Duisburg, 47057, Germany (e-mail: han.vinck@uni-due.de).

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transmit antenna to the ith receive antenna is $h_{i,j}$, and $\mathbf{n}$ is a $M_{\rm r} \times 1$ additive white Gaussian noise (AWGN) vector with variance $N_0/2$ per dimension. The combining weight vector at the receiver, $\mathbf{z}$, is equal to $\mathbf{Hw}$. The instantaneous signal-to-noise ratio (SNR) at the receiver after MRC is given by $||\mathbf{Hw}||^2 \frac{E_s}{M_{\rm t} N_0}$. We use $(\cdot)^{\rm H}$ and $||\cdot||$ for conjugate transpose and vector two-norm, respectively.

III. 4-ary Codebook Design Using Binary Linear Block Codes

A. Design Procedure

In this subsection, we design 4-ary codebooks for MIMO beamforming from binary linear block codes based on the Grassmannian criterion.

The Grassmannian beamforming criterion for MIMO beamforming codebook design is to [15]

$$\max \min_{\mathbf{w} \neq \mathbf{v}} \sqrt{1 - \frac{1}{M_{\rm t}^2} |\mathbf{w}^{\rm H}\mathbf{v}|^2} \quad (1)$$

where $\mathbf{w}, \mathbf{v}$ are weight vectors in a codebook $\mathcal{W}$ and $\min_{\mathbf{w} \neq \mathbf{v}} \sqrt{1 - \frac{1}{M_{\rm t}^2} |\mathbf{w}^{\rm H}\mathbf{v}|^2}$ is called the minimum chordal distance. Thus, the Grassmannian beamforming criterion is to

$$\min \max_{\mathbf{w} \neq \mathbf{v}} |\mathbf{w}^{\rm H}\mathbf{v}| \quad (2)$$

where we call $|\mathbf{w}^{\rm H}\mathbf{v}|$ the correlation magnitude metric between weight vectors $\mathbf{w}$ and $\mathbf{v}$. By using the MIMO beamforming codebook that satisfies the Grassmannian criterion, the SNR at the receiver can be maximized [4].

To make a connection between a binary linear block code and a 4-ary codebook, we define QPSK mapping, $q(\cdot)$, as

$$\begin{aligned} q(00) &= 1 \\ q(01) &= j \\ q(11) &= -1 \\ q(10) &= -j. \end{aligned}$$

Then, we can define inverse QPSK mapping, $q^{-1}(\cdot)$, as

$$\begin{aligned} q^{-1}(1) &= 00 \\ q^{-1}(j) &= 01 \\ q^{-1}(-1) &= 11 \\ q^{-1}(-j) &= 10. \end{aligned}$$

Using the QPSK mapping, a codeword $\mathbf{c} = [c_1, c_2, ..., c_n]$ in a binary linear block code $\mathcal{C}$ is mapped to a weight vector $\mathbf{w} = [w_1, w_2, ..., w_{M_{\rm t}}]^T$ in a 4-ary codebook $\mathcal{W}$, where $w_i = q(c_{2i-1}, c_{2i})$ for $i = 1, 2, ..., M_{\rm t}$ and the block length $n = 2M_{\rm t}$. We use the mappings $q(\cdot)$ and $q^{-1}(\cdot)$ also for vectors. For example, $q([0,0,1,1,0,1,1,0]) = [1,-1,j,-j]^T$ and $q^{-1}([1,-1,j,-j]^T) = [0,0,1,1,0,1,1,0]$. Note that we represent a codeword and a weight vector by a row vector and a column vector, respectively. We assume that neither $[1,1,1,1,...,1,1]$, $[0,1,0,1,...,0,1]$, nor $[1,0,1,0,...,1,0]$ is a codeword in a binary linear block code $\mathcal{C}$. Then, we can have three cosets defined as

$$\begin{aligned} &[1,1,1,1,...,1,1] + \mathcal{C} = \{[1,1,1,1,...,1,1] + \mathbf{c} | \mathbf{c} \in \mathcal{C}\} \\ &[0,1,0,1,...,0,1] + \mathcal{C} = \{[0,1,0,1,...,0,1] + \mathbf{c} | \mathbf{c} \in \mathcal{C}\} \\ &[1,0,1,0,...,1,0] + \mathcal{C} = \{[1,0,1,0,...,1,0] + \mathbf{c} | \mathbf{c} \in \mathcal{C}\}. \end{aligned}$$

Since the modulo two addition by $[0,1]$ can be tabulated as

$$\begin{aligned} [0,0] + [0,1] &= [0,1] \\ [0,1] + [0,1] &= [0,0] \\ [1,1] + [0,1] &= [1,0] \\ [1,0] + [0,1] &= [1,1] \end{aligned}$$

the corresponding operation in a 4-ary codebook for adding $[0,1]$ to $[c_{2i-1}, c_{2i}]$ produces $(-jw_i)^* = jw_i^*$, where $w_i = q(c_{2i-1}, c_{2i})$.

Similarly, the corresponding operation in a 4-ary codebook for adding $[0,1]$ to $[c_{2i-1}, c_{2i}]$ produces $(jw_i)^* = -jw_i^*$. Also, the corresponding operation in a 4-ary codebook for adding $[1,1]$ to $[c_{2i-1}, c_{2i}]$ produces $-w_i$. Then, the three corresponding 4-ary codebooks for the three cosets $[1,1,1,1,...,1,1] + \mathcal{C}$, $[0,1,0,1,...,0,1] + \mathcal{C}$, and $[1,0,1,0,...,1,0] + \mathcal{C}$ are defined as

$$\begin{aligned} -\mathcal{W} &= \{-\mathbf{w} | \mathbf{w} \in \mathcal{W}\} \\ (-j\mathcal{W})^* &= \{(-j\mathbf{w})^* | \mathbf{w} \in \mathcal{W}\} \\ (j\mathcal{W})^* &= \{(j\mathbf{w})^* | \mathbf{w} \in \mathcal{W}\} \end{aligned}$$

respectively. Since

$$\begin{aligned} &|((j\mathbf{w})^*)^H (j\mathbf{v})^*| \\ &= |((-j\mathbf{w})^*)^H (-j\mathbf{v})^*| \\ &= |w_1 v_1^* + w_2 v_2^* + \cdots + w_{M_{\rm t}} v_{M_{\rm t}}^*| \\ &= |w_1^* v_1 + w_2^* v_2 + \cdots + w_{M_{\rm t}}^* v_{M_{\rm t}}| \\ &= |\mathbf{w}^H \mathbf{v}| \\ &= |(-\mathbf{w})^H (-\mathbf{v})| \end{aligned} \quad (3)$$

where $(j\mathbf{w})^*, (j\mathbf{v})^* \in (j\mathcal{W})^*$, $(-j\mathbf{w})^*, (-j\mathbf{v})^* \in (-j\mathcal{W})^*$, $-\mathbf{w}, -\mathbf{v} \in -\mathcal{W}$, and $\mathbf{w}, \mathbf{v} \in \mathcal{W}$, the 4-ary beamforming codebooks $(-j\mathcal{W})^*$, $(j\mathcal{W})^*$, $(-\mathcal{W})$, and $\mathcal{W}$ have the same maximum correlation magnitude metric and the same error performance. Thus, we consider only the binary linear block code that has both $[0,1,0,1,...,0,1]$ and $[1,1,1,1,...,1,1]$ as the rows in the generator matrix. Removing $[0,1,0,1,...,0,1]$ and $[1,1,1,1,...,1,1]$ in the generator matrix, the new generator matrix G of the binary linear block code $\mathcal{C}$ is obtained. The corresponding 4-ary codebook $\mathcal{W}$ can be obtained by the QPSK mapping from the binary linear block code $\mathcal{C}$.

By removing the two rows $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$, we do not generate codewords that produces the equivalent weight vectors. If $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$ exist in a generator matrix of a linear block code $\mathcal{C}$,

$$\begin{aligned} &\mathbf{c} + [1,1,1,1,...,1,1] \in \mathcal{C} \\ &\mathbf{c} + [0,1,0,1,...,0,1] \in \mathcal{C} \\ &\mathbf{c} + [1,0,1,0,...,1,0] \in \mathcal{C} \end{aligned}$$

where $\mathbf{c} \in \mathcal{C}$, since all the linear combinations of the rows in the generator matrix are codewords in the binary block code $\mathcal{C}$. A binary block code $\mathcal{C}$ for 4-ary codebook design should have only one codeword among $\mathbf{c}$, $\mathbf{c}+[1,1,1,1,...,1,1]$, $\mathbf{c}+[0,1,0,1,...,0,1]$, and $\mathbf{c}+[1,0,1,0,...,1,0]$. Otherwise, the weight vectors are wasted since the four vectors are equivalent. For example, if we have the weight vector $[1,1,...,1]^T$ in a codebook $\mathcal{W}$, having $[j,j,...,j]^T$, $[-1,-1,...,-1]^T$, or $[-j,-j,...,-j]^T$ in the codebook $\mathcal{W}$ should be avoided since they are equivalent, i.e.

$$||\mathbf{H}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}|| = ||\mathbf{H}\begin{pmatrix}j\\j\\\vdots\\j\end{pmatrix}|| = ||\mathbf{H}\begin{pmatrix}-1\\-1\\\vdots\\-1\end{pmatrix}|| = ||\mathbf{H}\begin{pmatrix}-j\\-j\\\vdots\\-j\end{pmatrix}||. \tag{4}$$

B. Minimum Hamming Distance and Maximum Correlation Magnitude Metric

In this subsection, we derive an upperbound on the maximum correlation magnitude metric, $CM_{\max}$, of a 4-ary codebook $\mathcal{W}$ as a function of the minimum Hamming distance, $d_{\min}$, of the corresponding binary linear block code. We consider a binary linear block code $\mathcal{C}$ with the block length $n = 2M_{\mathrm{t}}$ and the minimum Hamming distance $d_{\min}$.

The squared correlation magnitude metric between the weight vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ is given by

$$\begin{aligned} |\mathbf{w}_1^H\mathbf{w}_2|^2 &= |w_{1,1}^*w_{2,1} + w_{1,2}^*w_{2,2} + \cdots + w_{1,M_{\mathrm{t}}}^*w_{2,M_{\mathrm{t}}}|^2 \\ &= (\alpha_0-\alpha_2)^2 + (\alpha_1-\alpha_3)^2 \end{aligned} \tag{5}$$

where α_0, α_1, α_2, and α_3 are the number of 1's, j's, -1's, and $-j$'s, respectively, in $\mathbf{w}_1^* \circ \mathbf{w}_2$, which is the Hadamard product (or the entrywise product) between $\mathbf{w}_1^*$ and $\mathbf{w}_2$. Note that $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = M_{\mathrm{t}}$.

Assuming that $\mathbf{w}_1 = q(\mathbf{c}_1)$ and $\mathbf{w}_2 = q(\mathbf{c}_2)$, the vector $\mathbf{w}_1^* \circ \mathbf{w}_2$ and the codeword $\mathbf{c}_2 - \mathbf{c}_1$ are related as

$$\begin{aligned} \alpha_0 &= \beta_0 \\ \alpha_2 &= \beta_2 \\ \alpha_1 + \alpha_3 &= \beta_1 + \beta_3 \end{aligned}$$

where β_0, β_1, β_2, and β_3 are the number of 00's, 01's, 11's, and 10's in the codeword $\mathbf{c}_2 - \mathbf{c}_1$, respectively, and $\beta_0+\beta_1+\beta_2+\beta_3 = M_{\mathrm{t}}$.

For example, if we assume that $\mathbf{c}_1 = [0,0,0,1,0,0,0,0]$ $\mathbf{c}_2 = [0,0,0,1,1,0,1,1]$, then $\mathbf{c}_2 - \mathbf{c}_1 = [0,0,0,0,1,0,1,1]$. Thus, $\beta_0 = 2$, $\beta_1 = 0$, $\beta_2 = 1$, and $\beta_3 = 1$. Since $\mathbf{w}_1 = [1,j,1,1]^T$ and $\mathbf{w}_2 = [1,j,-j,-1]^T$, then $\mathbf{w}_1^* \circ \mathbf{w}_2 = [1,1,-j,-1]^T$. Thus, $\alpha_0 = 2$, $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_3 = 1$ in this example.

If the Hamming distance between $\mathbf{c}_1$ and $\mathbf{c}_2$ is d, the Hamming weight of $\mathbf{c}_2 - \mathbf{c}_1$ is given by $d = 2\beta_2 + \beta_1 + \beta_3$. Then, from (5)

$$\begin{aligned} |\mathbf{w}_1^H\mathbf{w}_2|^2 &= (\alpha_0-\alpha_2)^2 + (\alpha_1-\alpha_3)^2 \\ &\le (\beta_0-\beta_2)^2 + (\alpha_1+\alpha_3)^2 \\ &= (\beta_0-\beta_2)^2 + (\beta_1+\beta_3)^2 \\ &\le (\beta_0-\beta_2)^2 + d^2 \\ &= (M_{\mathrm{t}}-d)^2 + d^2. \end{aligned}$$

Since we assume that $[1,1,1,1,...,1,1]$ is included in the generator matrix, $\mathbf{c}+[1,1,1,1,...,1,1]$ is a codeword if $\mathbf{c}$ is a codeword. As a result, $d_{\min} \le M_{\mathrm{t}}/2$. Then, the squared correlation magnitude metric is upperbounded as

$$|\mathbf{w}_1^H\mathbf{w}_2|^2 \le (M_{\mathrm{t}}-d_{\min})^2 + d_{\min}^2.$$

Thus, an upperbound of the maximum correlation magnitude metric, $CM_{\max}$, is given by

$$CM_{\max} \le (M_{\mathrm{t}}-d_{\min})^2 + d_{\min}^2. \tag{6}$$

This upperbound shows that a 4-ary codebook with a small value of $CM_{\max}$ can be designed by the corresponding binary linear code with the minimum Hamming distance $d_{\min}$.

IV. 4-ary Codebook Using Reed Muller Codes

In this section, we find 4-ary MIMO beamforming codebooks using Reed Muller codes. Reed Muller codes have $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$ in the generator matrix [17, p. 105]. Note that the weight vector $[1,1,...,1]^T$ is always included in the codebook $\mathcal{W}$ by QPSK mapping of the all-zero codeword $[0,0,0,0,...,0,0]$ in the binary block code $\mathcal{C}$. From (4), the binary codewords $[1,1,1,1,...,1,1]$, $[0,1,0,1,...,0,1]$, and $[1,0,1,0,...,1,0]$, which correspond to weight vectors $[-1,-1,...,-1]^T$, $[j,j,...,j]^T$, and $[-j,-j,...,-j]^T$, respectively, should be removed in the corresponding binary block code $\mathcal{C}$. By removing $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$ in the generator matrix, a generator matrix G is obtained for the binary linear block code $\mathcal{C}$ of which codewords are mapped to weight vectors by QPSK mapping.

A rth order binary Reed Muller code with a parameter m, denoted by RM(r,m), has the block length $n = 2^m$, the dimension $k = 1 + \binom{m}{1} + \cdots + \binom{m}{r}$, and the minimum Hamming distance $d_{\min} = 2^{m-r}$ [17, p. 105], where $n = 2M_{\mathrm{t}}$. The number of rows in the generator matrix for the rth order (n,k) Reed Muller code is k. Since $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$ are always included in the generator matrix, the available rows of the generator matrix for the beamforming codebooks is $k-2$. Since we select the first F rows from the available $k-2$ rows for designing the beamforming codebook with 2^F weight vectors, $F \le k-2$. Thus,

$$F \le -1 + \binom{m}{1} + \binom{m}{2} + \cdots + \binom{m}{r}.$$

The procedure of finding 4-ary beamforming codebooks with the number of transmit antennas M_{t} and the number of feedback bits F from Reed Muller codes is following.

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1) Find the generator matrix of Reed Muller code with the block length $n = 2M_\mathrm{t}$ and the parameter r. The parameter r is the minimum of r' that satisfies the following inequality

$$F \le -1 + \binom{m}{1} + \cdots + \binom{m}{r'}.$$

Select the first $F+2$ rows from the generator matrix of the rth order binary Reed Muller code with block length $n = 2M_\mathrm{t}$. This $(F+2) \times 2M_\mathrm{t}$ matrix is denoted by G_0.
2) Removing the first two rows, which are $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$, from the matrix G_0, we obtain the $F \times 2M_\mathrm{t}$ generator matrix G.
3) Obtain a binary linear block code $\mathcal{C}$ by taking all the linear combinations of the rows in G, i.e. $\mathcal{C}$ is the row space of the $F \times 2M_\mathrm{t}$ generator matrix G.
4) Apply QPSK mapping to $\mathcal{C}$, and obtain the transpose of the codebook $\mathcal{W}$.

An example for $M_\mathrm{t} = 4$ and $F = 3$ is following.

- $M_\mathrm{t} = 4, F = 3$

$$G_0 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

After removing the two rows $[1,1,1,1,...,1,1]$ and $[0,1,0,1,...,0,1]$, the generator matrix for $M_\mathrm{t} = 4$ and $F = 3$ is given as

$$G = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

All the linear combinations of the rows from the generator matrix G produce the 2^F codewords. The codebook $\mathcal{W}$ can be obtained by QPSK mapping to the 2^F codewords.

In Table I, III, IV, and V, generator matrices for 4-ary codebooks using Reed Muller codes for $M_\mathrm{t} = 4, 8, 16$, and 32 are tabulated, respectively. Due to the width limit, in Table V for $M_\mathrm{t} = 32$, the $(2m-1)^{\mathrm{th}}$ row and the $(2m)^{\mathrm{th}}$ row are for the transmit antenna $1, 2, ..., 16$ and the transmit antenna $17, 18, ..., 32$, respectively, where $m = 1, 2, ..., F$. Thus, the $(2m-1)^{\mathrm{th}}$ row and $(2m)^{\mathrm{th}}$ row in Table V corresponds to the weight vector $\mathbf{w}_m$ in a codebook $\mathcal{W}$. In Table II, minimum chordal distances, $c_{\min}$, minimum Hamming distances, $d_{\min}$, correlation magnitude metrics, $CM_{\max}$, and upperbounds (6) on $CM_{\max}$ of the 4-ary codebooks using Reed Muller codes for $M_\mathrm{t} = 4, 8, 16$, and 32 are tabulated. An advantage of the 4-ary codebooks using Reed Muller codes is that design procedure is extremely simple and does not require any searching algorithm. As a result, 4-ary codebooks for massive MIMO system with $M_\mathrm{t} = 16$ and 32 can be found without any difficulties.

V. Numerical results and discussion

In this section, the SEP performance of 4-ary codebooks are compared with discrete Fourier Transform (DFT) codebooks and LTE codebooks in [11] by computer simulation.

In Fig. 2, we show the SEP as a function of E_b/N_0 for the codebook using Reed Muller codes, DFT codebook, and LTE codebook [11, p. 63] employing QPSK signaling with $(M_\mathrm{t}, F) = (2,2)$, $(4,4)$, $(16,5)$ and $(32,6)$. For $M_\mathrm{t} = 2$ and $F = 2$, the codebook using Reed Muller codes whose weight vectors are $[1,1]^T$, $[1,j]^T$, $[1,-1]^T$, and $[1,-j]^T$, is the same as the DFT codebook and the LTE codebook. There is no LTE codebook available for $M_\mathrm{t} = 16$ and $M_\mathrm{t} = 32$. For $M_\mathrm{t} = 16$ and $M_\mathrm{t} = 32$, it is shown that the codebook using Reed Muller codes has almost the same SEP performance as the DFT codebook. The weights in the DFT codebook for $M_\mathrm{t} = 4$ and $F = 4$ are $e^{-j2kl\pi/16}$, where $l = 0,1,2,3$ and $k = 0,1,...,15$. For $M_\mathrm{t} = 4$ and $F = 4$, it is shown that the codebook using Reed Muller codes has the same SEP performance as the LTE codebook. For $M_\mathrm{t} = 2$ and $F = 2$, the codebook using Reed Muller codes provides 0.4 dB gain over the DFT codebook. Note that the LTE codebook for $M_\mathrm{t} = 4$ and $F = 4$ includes $\frac{\pm 1 \pm j}{\sqrt{2}}$. In a practical implementation, the weights in the DFT and LTE codebooks must be quantized. Thus, the performance of the DFT and LTE codebooks in a practical implementation would be poorer than our simulation results. Furthermore, 4-ary codebooks have reduced complexity for finding the optimum weight vector. The optimum weight vector $\mathbf{w}_{\mathrm{opt}}$ is found at the receiver by

$$\begin{aligned} \mathbf{w}_{\mathrm{opt}} &= \arg\max_{\mathbf{w}\in\mathcal{W}} ||\mathbf{H}\mathbf{w}||^2 \\ &= \arg\max_{\mathbf{w}\in\mathcal{W}} \sum_{i=1}^{M_\mathrm{r}} \left| \sum_{j=1}^{M_\mathrm{t}} h_{i,j} w_j \right|^2 . \end{aligned} \tag{7}$$

Using a 4-ary codebook rather than an infinite alphabet codebook, computation of $h_{i,j}w_j$ in (7) is simplified from multiplication to sign change and permutation of the real and the imaginary parts of complex numbers.

VI. Conclusion

4-ary codebook design for MIMO beamforming systems using Reed Muller codes has been investigated. The 4-ary codebooks using Reed Muller codes were tabulated for $M_\mathrm{t} = 4, 8, 16$, and 32. It was shown that the 4-ary codebook using Reed Muller codes provided the same SEP as the LTE codebook for $M_\mathrm{t} = 4$ and $F = 4$ by computer simulations.

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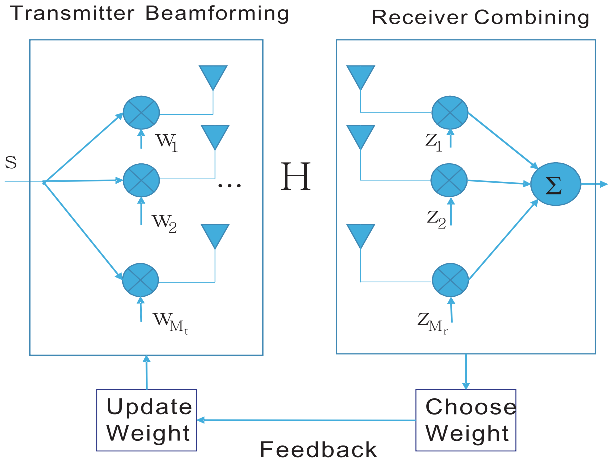

Fig. 1. MIMO beamforming system with limited feedback.

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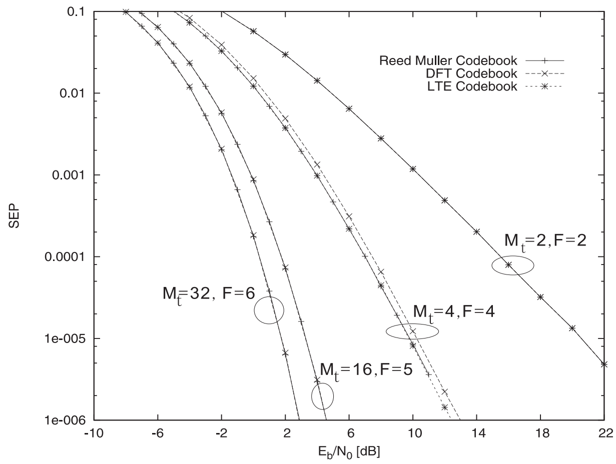

Fig. 2. The SEP as a function of E_b/N_0 with $(M_\mathrm{t}, F) = (2, 2)$, $(4, 4)$, $(16, 5)$ and $(32, 6)$ for QPSK signaling. Reed Muller codebook stands for the codebook using Reed Muller codes.

TABLE I

GENERATOR MATRICES FOR 4-ARY CODEBOOKS USING REED MULLER CODES FOR $M_\mathrm{t} = 4$ AND $F = 4, 5$.

- $M_\mathrm{t} = 4$, $F = 4$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $M_\mathrm{t} = 4$, $F = 5$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

TABLE II

MINIMUM CHORDAL DISTANCES, $c_{\min}$, MINIMUM HAMMING DISTANCES, $d_{\min}$, CORRELATION MAGNITUDE METRICS, $CM_{\max}$, AND UPPERBOUNDS (6) ON $CM_{\max}$ OF THE 4-ARY CODEBOOKS USING REED MULLER CODES

M_t	F	$c_{\min}$	$d_{\min}$	$CM_{\max}$	Upperbound
4	3	0.866025	2	2	$2\sqrt{2}$
4	4	0.707107	2	$2\sqrt{2}$	$2\sqrt{2}$
4	5	0.707107	2	$2\sqrt{2}$	$2\sqrt{2}$
8	4	0.866025	4	4	$4\sqrt{2}$
8	5	0.866025	4	4	$4\sqrt{2}$
8	6	0.866025	4	4	$4\sqrt{2}$
8	7	0.707107	4	$4\sqrt{2}$	$4\sqrt{2}$
8	8	0.707107	4	$4\sqrt{2}$	$4\sqrt{2}$
8	9	0.707107	4	$4\sqrt{2}$	$4\sqrt{2}$
16	5	0.866025	8	8	$8\sqrt{2}$
16	6	0.866025	8	8	$8\sqrt{2}$
16	7	0.866025	8	8	$8\sqrt{2}$
16	8	0.866025	8	8	$8\sqrt{2}$
16	9	0.866025	8	8	$8\sqrt{2}$
16	10	0.866025	8	8	$8\sqrt{2}$
32	6	0.866025	8	16	$8\sqrt{10}$
32	7	0.866025	8	16	$8\sqrt{10}$
32	8	0.866025	8	16	$8\sqrt{10}$

TABLE III

GENERATOR MATRICES FOR 4-ARY CODEBOOKS USING REED MULLER CODES FOR $M_{\mathrm{t}} = 8$ AND $F = 4, 5, 6, 7, 8, 9$.

- $M_{\mathrm{t}} = 8, F = 4$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 8, F = 5$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 8, F = 6$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 8, F = 7$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 8, F = 8$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 8, F = 9$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

TABLE IV

GENERATOR MATRICES FOR 4-ARY CODEBOOKS USING REED MULLER CODES FOR $M_{\mathrm{t}} = 16$ AND $F = 5, 6, 7, 8, 9, 10$.

- $M_{\mathrm{t}} = 16$, $F = 5$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 16$, $F = 6$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 16$, $F = 7$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 16$, $F = 8$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 16$, $F = 9$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \end{pmatrix}.$$

- $M_{\mathrm{t}} = 16$, $F = 10$

$$\begin{pmatrix} 0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \end{pmatrix}.$$

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TABLE V

GENERATOR MATRICES FOR 4-ARY CODEBOOKS USING REED MULLER CODES FOR $M_{\mathrm{t}} = 32$ AND $F = 6, 7, 8$.

- $M_{\mathrm{t}} = 32$, $F = 6$

$$\begin{pmatrix}
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1
\end{pmatrix}.$$

- $M_{\mathrm{t}} = 32$, $F = 7$

$$\begin{pmatrix}
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1
\end{pmatrix}.$$

- $M_{\mathrm{t}} = 32$, $F = 8$

$$\begin{pmatrix}
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1&0&0&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1
\end{pmatrix}.$$