Isaac Newton

Auszug aus einem
Brief an Leibniz (bzw. Oldenburg)
(1676)

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Most worthy Sir,

I can hardly tell with what pleasure I have read the letters of those very distinguished men Leibniz(2) and Tschirnhaus.(3) Leibniz's method for obtaining convergent series is certainly very elegant, and it would have sufficiently revealed the genius of its author, even if he had written nothing else. But what he has scattered elsewhere throughout his letter is most worthy of his reputation—it leads us also to hope for very great things from him. The variety of ways by which the same goal is approached has given me the greater pleasure, because three methods of arriving at series of that kind had already become known to me, so(4) that I could scarcely expect a new one to be communicated to us. One of mine I have described before; I now add another, namely, that by which I first chanced on these series—for I chanced on them before I knew the divisions(5) and extractions of roots which I now use. And an explanation of this will serve to lay bare, what Leibniz desires from me, the basis of the theorem set forth near the beginning of the former letter.(6)

At the beginning of my mathematical studies,(7) when I had met with the works of our celebrated Wallis, on considering the series by the intercalation of which he himself exhibits the area of the circle and the hyperbola, the fact that, in the series of curves whose common base or axis is *x* and the ordinates

$$\overline{1-xx}|^{\frac{0}{2}},\ \overline{1-xx}|^{\frac{1}{2}},\ \overline{1-xx}|^{\frac{2}{2}},\ \overline{1-xx}|^{\frac{3}{2}},\ \overline{1-xx}|^{\frac{4}{2}},\ \overline{1-xx}|^{\frac{5}{2}} \text{ etc.,}$$

if the areas of every other of them, namely

$$x,\ x-\frac{1}{3}x^3,\ x-\frac{2}{3}x^3+\frac{1}{5}x^5,\ x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7 \text{ etc.}$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-x^2)^{\frac{1}{2}}$ is the circle: in order to interpolate these series I noted that in all of them the first term was *x* and that the second terms $\frac{0}{3}x^3.\ \frac{1}{3}x^3.\ \frac{2}{3}x^3.\ \frac{3}{3}x^3$ etc., were in arithmetical progression, and hence that the first two terms of the series to be intercalated(8) ought to be $x-\frac{1}{3}\left(\frac{1}{2}x^3\right),\ x-\frac{1}{3}\left(\frac{3}{2}x^3\right),\ x-\frac{1}{3}\left(\frac{5}{2}x^3\right)$, etc. To intercalate the rest I began to reflect that the denominators 1, 3, 5, 7, etc. were in arithmetical
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progression, so that the numerical coefficients of the numerators only were still in need of investigation. But in the alternately given areas these were the figures of powers of the number 11, namely of these $11^0, 11^1, 11^2, 11^3, 11^4$ that is, first 1; then 1, 1; thirdly, 1, 2, 1; fourthly 1, 3, 3, 1; fifthly 1, 4, 6, 4, 1, etc. And so I began to inquire how the remaining figures in these series could be derived from the first two given figures, and I found that on putting *m* for the second figure, the rest would be produced by continual multiplication of the terms of this series,

$$\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{m-4}{5}, \text{ etc.}$$

For example, let *m* = 4, and $4\times\frac{1}{2}(m-1)$, that is 6 will be the third term, and $6\times\frac{1}{3}(m-2)$, that is 4 the fourth, and $4\times\frac{1}{4}(m-3)$, that is 1 the fifth, and $1\times\frac{1}{5}(m-4)$, that is 0 the sixth, at which term in this case the series stops.

Accordingly, I applied this rule for interposing series among series,(9) and since, for the circle, the second term was $\frac{1}{3}\left(\frac{1}{2}x^3\right)$, I put $m=\frac{1}{2}$, and the terms arising were

$$\frac{1}{2}\times\frac{\frac{1}{2}-1}{2} \text{ or } -\frac{1}{8}, -\frac{1}{8}\times\frac{\frac{1}{2}-2}{3} \text{ or } +\frac{1}{16}, +\frac{1}{16}\times\frac{\frac{1}{2}-3}{4} \text{ or } -\frac{5}{128}$$

and so to infinity. Whence I came to understand that the area of the circular segment which I wanted was

$$x-\frac{\frac{1}{2}x^3}{3}-\frac{\frac{1}{8}x^5}{5}-\frac{\frac{1}{16}x^7}{7}-\frac{\frac{5}{128}x^9}{9} \text{ etc.}$$

And by the same reasoning the areas of the remaining curves, which were to be inserted, were likewise obtained: as also the area ofthe hyperbola and the other alternate curves in this series $(1+x^2)^{\frac{0}{2}}, (1+x^2)^{\frac{1}{2}}, (1+x^2)^{\frac{2}{2}}, (1+x^2)^{\frac{3}{2}}$ etc. And the same theory serves to intercalate other series, and that through intervals of two or more terms when they are absent at the same time. This was my first entry upon these studies, and it had certainly escaped my memory, had I not a few weeks ago cast my eye back on some notes.(10)

But when I had learnt this,(11) I immediately began to consider that the terms

$$(1+x^2)^{\frac{0}{2}}, (1+x^2)^{\frac{2}{2}}, (1+x^2)^{\frac{4}{2}}, (1+x^2)^{\frac{6}{2}} \text{ etc.}$$

that is to say,

$$1, 1-x^2, 1-2x^2+x^4, 1-3x^2+3x^4-x^6$$

could be interpolated in the same way as the areas genera1ted by them: and that nothing else was required for this purpose but to omit the denominators l, 3, 5, 7, etc., which are in the terms expressing the areas; this means that the coefficients of the terms of the quantity to be intercalated $(1-x^2)^{\frac{1}{2}}$, or $(1-x^2)^{\frac{3}{2}}$, or in general $(1-x^2)^m$, arise by the continued multiplication of the terms of this series

$$m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}\text{, etc.,}$$

so that (for example)

$$(1-x^2)^{\frac{1}{2}} \text{ was the value of } 1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6 \text{ etc.,}$$

$$(1-x^2)^{\frac{3}{2}} \text{ of } 1-\frac{3}{2}xx+\frac{3}{8}x^4+\frac{1}{16}x^6\text{, etc.,}$$

$$\text{and } (1-x^2)^{\frac{1}{3}} \text{ of } 1-\frac{1}{3}xx-\frac{1}{9}x^4-\frac{5}{81}x^6\text{, etc.}$$

So then the general reduction of radicals into infinite series by that rule, which I laid down at the beginning of my eardier letter(12) became known to me, and that before I was acquainted with the extraction of roots. But once this was known, that other could not long remain hidden from me. For in order to test these processes, I multiplied

$$1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6\text{, etc.}$$

into itself; and it became $1-x^2$, the remaining terms vanishing by the continuation of the series to infinity. And even so $1-\frac{1}{3}x^2-\frac{1}{9}x^4-\frac{5}{81}x^6$, etc., multiplied twice into itself also produced $1-x^2$. And as this was not only sure proof of these conclusions so too it guided me to try whether, conversely, these series, which it thus affirmed to be roots of the quantity $1-x^2$, might not be extracted out of it in an arithmetical manner.(13) And the matter turned out well.

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Your most devoted
IS. NEWTON.