RECONSTRUCTING THE MATHEMATICAL IN SOCIAL DISCOURSE – ASPECTS OF AN EPISTEMOLOGY-BASED INTERACTION RESEARCH

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Abstract. *Mathematical knowledge depends on human thinking and social interaction. Neither symbols nor contexts alone provide the objective basic substance for mathematical existence. Mathematical knowledge is created by (mental and interactive) interpretation of signs with regard to possible reference contexts. With this theoretical perspective on knowledge in mind, mathematical interaction research is faced by a complementary difficulty: The object of research – mathematical communication – as well as its observation and scientific analysis are both "sign-interpreting-processes" that are constituted in social interaction. The analysing reconstruction of mathematical discourses requires the revelation of possible interactive interpretations of communicated signs and in this way the analysis reflects its "own" understanding of mathematical knowledge as a result of a social construction processes. This article presents crucial components of the epistemology-based research of mathematical interaction by using exemplary teaching episodes from an ongoing research project on "Social and epistemological constraints of constructing new knowledge in the mathematics classroom".*

1. Introduction

The social construction of new mathematical knowledge in teaching and learning processes depends on two important conditions: The special character of instructional communication and the specific epistemological nature of mathematical knowledge. In mathematics teaching at the primary level new knowledge cannot be constructed in a formal manner by a kind of preview technique, i.e. using algebra or formulas, but this construction is linked with the children's situated contexts of learning and of experience in a characteristic way. The young students have to learn –and they are able to do so by their personal means – to see the general in the particular. To better understand this problem is an important inquiry of the research project "Social and epistemological constraints of constructing new knowledge in the mathematics classroom" (funded by the German Research Society, DFG; see Steinbring et al. 1998). How are students of elementary grades able to grasp the new, general mathematical knowledge with their own conceptions and to describe it with their own words? And what factors support or hinder this generalising interactive knowledge construction?

2. What is the Specific Nature of Mathematical Concepts?

Mathematical concepts and mathematical knowledge are not given a priori in the "external" reality, neither as concrete, material objects, nor as independently existing (platonic) ideas. For the individual cognitive agent mathematical concepts are "mental objects" (Changeux & Connes 1995; Dehaene 1997); in the course of communication mathematical concepts are constituted as "social facts" (Searle 1997) or as "cultural objects" (Hersh 1997). From an evolutionary point of view mathematical concepts develop as cognitive and as social theoretical knowledge objects in confrontation with the material and social environment.

In contrast to objects constructed by humans as for instance a chair, a table, a knife or a screw-driver one cannot deduce the meaning of social facts, as for instance money, time or the number concept neither from their form nor from their material. There are no direct insights into the corresponding mathematical object when inspecting the "material" or the functional form of number signs as $\sqrt{2}$, -3.17 or π. The meaning of these theoretical, social respectively mental objects has to be constructed by the individual in interaction with experience based and abstract referential contexts. In a general way, mathematical concepts can be conceived as "symbolised, operational relations" between their formal codings and certain socially intended interpretation.

Mathematical knowledge can be looked at in two complementary ways: On the one side, each mathematical knowledge domain represents a consistent structural wickerwork, in which all elements are linked in an equivalent logical manner. On the other side, new concepts posing new questions and problems can be constructed in every mathematical knowledge structure, concepts that are not yet imbedded in the actual logical structure, and in this way producing new insights.

This distinction between the *logical structure* and the *mathematical objects* is in accordance with the distinction made in philosophy between a subjective ontology of reality and the subject independent structure of the world. "… the ontology of the world is created by the cognitive agent, the structure of the world depends on the mind-independent external reality. In this way, the experiential world can be seen as both created and mind-independent at once. As there cannot be a structure without an ontology, it is the cognitive agent's act of creating an ontology that endows external reality with a structure" (Indurkhya 1994, p. 106).

The "logical coherence" and consequently the "unique generativity" of mathematical knowledge often is taken as an irrefutable "proof" for the objective existence of mathematical knowledge independent of any cognitive agent (Changeux & Connes 1995, p. 12); but also this property – a specific, epistemological mechanism for the autopoietic development of mathematical knowledge – needs the cognitive as well as the social environment of the cognitive agent for its unfolding.

3. The Epistemological and Communicative Function of Signs

3.1 The Epistemological Dimension

The peculiar interrelation between "Signs / Symbols" and "Objects / Reference contexts" is central for the description and analysis of mathematical teaching as a specific culture. This relation also represents a basic item of the epistemologically based interaction analysis. All mathematical knowledge needs certain *systems of signs or symbols* for grasping and coding the knowledge in question. These signs themselves do not have an isolated meaning; their meaning must be constructed by the learning child. In a general sense, to endow mathematical signs with meaning, one needs an adequate *reference context*. Meanings of mathematical concepts emerge in the interplay between sign/symbol systems and objects/reference contexts (Steinbring 1993; or Maier & Steinbring 1998).

The interrelation between coding signs of knowledge and reference contexts can be structured in the *epistemological triangle* (cf. Steinbring 1989; 1991; 1998). The links between the corners in this epistemological triangle are not defined explicitly and invariably, they rather form a mutually supported and balanced system. In the course of further developing knowledge the interpretation of signs systems and their accompanying reference contexts will be modified and generalised.

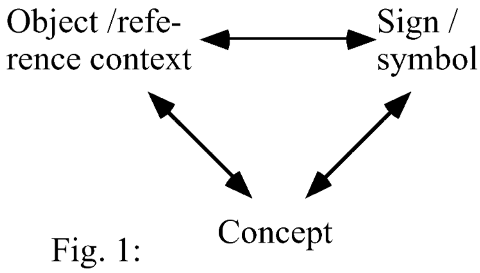


Fig. 1:
The epistemological triangle

Similar triangular schemes have been introduced in the philosophy of mathematics, in linguistics and the philosophy of language for analysing the semiotic problem of the relation between symbol and referent (Frege 1969; Ogden & Richards 1923).

Mathematical concepts are constructed as symbolic relational structures and are coded by means of *signs and symbols*, that can be combined logically in mathematical operations. With regard to the analysis of conditions for the construction of new mathematical knowledge in classroom interaction, mathematical signs and symbols are the central connecting links between the epistemological and the communicative dimension of interactive construction processes; on the one hand, signs and symbols are the *carriers of mathematical knowledge*, and on the other hand, they contain at the same time the *information of the mathematical communication*.

3.2 The Communicative Dimension

The sociologist Niklas Luhmann characterises »communication« as the constitutive concept of sociology: "… when communication shall come about, … an autopoietic system has to be activated, that is a social system, that reproduces communications by communications and makes nothing else but this" (Luhmann 1996, p. 279).

The concept of "autopoietic system" has been introduced by Maturana and Varela (cf. i. e. 1987); it characterises self-referential systems, that exist and develop autonomously on the basis of this self-referential relation. These systems consist of components that are permanently re-produced within the system for its maintenance. With the concept of "autopoietic system" not only biologic processes are investigated but it is also applied to social and psychic processes.

What is the essential difference between a social and a psychic process? The psychic process is based on consciousness and the social system is based on communication. "A social system cannot think, a psychological system cannot communicate. Nevertheless, from a causal view there are immense, highly complex interdependencies" (Luhmann

1997, p. 28). How these interdependencies can be understood? "Communication systems and psychic systems (or consciousness) form two clearly separated autopoietic domains; … But these two kinds of systems are linked in a special narrow relation and they form reciprocally a »portion of necessary environment«: Without the participation of consciousness systems there is no communication, and without the participation of communication there is no development of consciousness" (Baraldi, Corsi & Esposito 1997, p. 86).

Language is a central "linking mean" between communication and consciousness. Within languageone has to distinguish between »sound« and »sense«; accordingly within written language one has to distinguish between »sign« (more exactly »signifier«) and »sense«. This distinction between sign and intended meaning is the starting point – the take off (Luhmann, 1997, p. 208) – for the autopoiesis of communicative systems.

For the analysis of the conditions of the auto-poiesis Luhmann refers among others to the work of de Saussure, who made the following distinction between signifier (signifiant), signified (signifié) and sign (signe). Luhmann writes: "Signs are also forms, that means marked distinctions. They distinguish, following Saussure, the signified (signifiant) from the signifier (signifié). In the form of the sign, that means in the relation between signifier and signified, there are referents: The signifier signifies the signified. But the form itself (and only this should be named sign) has no reference; it functions only as a distinction, and that only when it is actually used as such" (Luhmann 1997, p. 208f.).

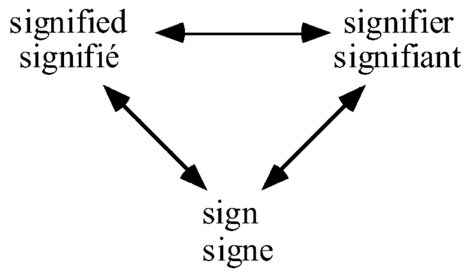


Fig. 2: The semiotic triangle

How the autopoiesis of the social, of communication, is possible? According to Luhmann, in the course of interaction or in the communication system the participants provide with their "conveyances" (or communicative actions) mutually "signifiers" which may signify certain "information" (signifieds). "Decisive might be…, that speaking (and this imitating gestures) elucidates an intention of the speaker, hence forces a distinction between information and conveyance with likewise linguistic means" (Luhmann 1997, p. 85).

The conveyor only can convey a signifier, but the signified intended by the conveyor, which alone could lead to an understandable sign, remains open and relatively uncertain; in principle, it can be constructed only by the receiver of the conveyance, in a way that he himself articulates a new signified. Luhmann explains this in the following way: "We do not start with the speech action, which will happen only when one expects, that it is expected and understood, but we start with the situation of the receiver of the conveyance, hence the person who observes the conveyor and who ascribes to him the conveyance, but not the information. The receiver of the conveyance has to observe the conveyance as the designation of an information, hence both together as a sign (as a form of the distinction between signifier and signified) ….” (Luhmann 1997, p. 210). The receiver must not ascribe the possible signified strictly to the conveyor of the conveyance but he/she has to construct the signified himself/herself; the signified and hence the sign is constituted within the process of communication.

The possible detachment of the information belonging to the conveyance from the conveyor is the starting point of the autopoiesis of the communicative system. By this "mechanism" that describes the autopoietic functioning of the communicative system as an ongoing conveyance of signifiers which are simultaneously transformed into signs by the contrasting conveyance of other, new signifiers, general properties of the functioning of *mathematical communication* are explained, too. In a first approximation, the epistemological triangle can be seen as analogue to the semiotic triangle (according to de Saussure); in addition, the epistemological triangle contains very specific features with regard to the particularities of mathematical communication.

4. Open and Superimposed Discourses in Mathematics Teaching – Analysis of Exemplary Teaching Episodes

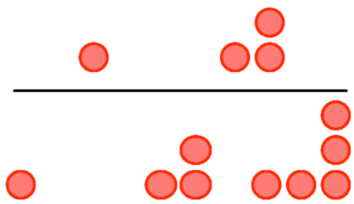
In the following, interactive patterns in two different teaching episodes are analysed in the course of constructing and justifying new mathematical knowledge. In the first episode students work within a learning environment about figurative numbers, where *geometric* reference contexts are offered for the interpretation of mathematical signs. The second episode is part of a teaching unit about special number squares, where the new mathematical signs have to be interpreted with the help of *structured arithmetical* reference contexts.

4.1 What is the "Correct" Representation of the Third Triangular Number?

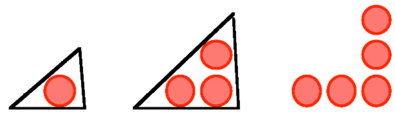
The content of the observed lesson in a 4th grade can be summarised in the following way. The teacher has placed a pattern of magnetic chips (little disks with one red and one blue side) on the black board. This pattern obviously should show the first two triangular numbers. The children are asked to construct the next pattern in this sequence. They offer different interesting proposals. The teacher guides the interactive construction process, and she asks for a justification of the last pattern she had accepted.

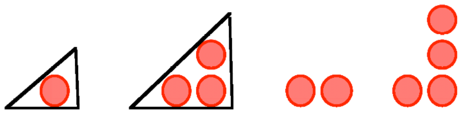
4.1.1 The Children's Proposals for Continuing the Pattern

The teacher places two patterns on the blackboard. How to continue? She emphasises that special numbers are involved having to do with the chips. Dennis continues the pattern in the following way. His proposal can be seen indeed as a possible correct continuation, in which the hook is extended by placing down left and right above one chip each time. The teacher comments this proposal: "Is this already correct? ... One could have the impression, but it is not yet quite right"(5). She refers to the shape and draws rectangular triangles

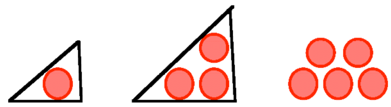


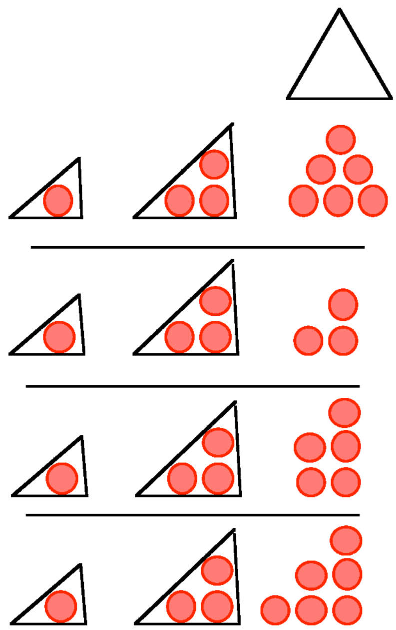
around the two first patterns. Then the teacher points to Dennis' pattern and she says that one could not draw such an triangle around it: "This could not yet be made here."(7). With her finger she goes around his pattern and in this way she outlines the shape of a hook or an angle. The teacher seems to have in mind that only one chip is still missing at the correct place and she tries to focus the children's attention to this fact. By asking the question: "Who could place this now, or use something else?" (7), the teacher expects that the one missing chip will be inserted now. But Lisa answers by making a completely different proposal and constructs the following pattern. She seems to take the initial patterns as one single figure and looks for a possible continuation. Her proposal is a plausible continuation in which to each part of the complete initial figure each time one chip is added, once in the horizontal line, and once in the vertical line. The teacher refuses Lisa's proposal by referring to the shape of the triangles. She points to the base line and to the inclined line of the triangle.





Kai takes away all new chips from the blackboard and starts to place his configuration. With the help of his classmates he inserts the last missing chip. He has produced an isosceles triangle, as the teacher then accentuates: "Well, first I have to look here. We have had such a form there but now we have seemingly this [*she draws an isosceles triangle above the constructed configuration*]" (14). The teacher poses the question whether this is the same (16); in this way she refutes Kai's proposal. Kai takes his chips off from the blackboard. Once again the teacher tries to focus his attention to the drawn shape of the two first patterns and she says that the new figure should look the same as the two already existing figures, only with more chips (18). First Kai places exactly the same pattern as the second one. Then he adds two chips in the following way. The teacher confirms: "Ooh, he is very close!" (22). Also Kai's classmates make supporting comments that only one chip is still missing. But Kai is not able to finish the proposal. Tugba places the missing chip on the left side in the base row.



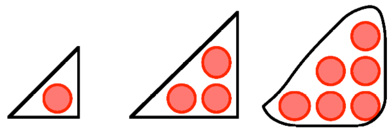


4.1.2 An Empirical Justification for the Correctness of the Third Pattern

Let us look more closely at the following short interaction phase wherein the justification of the correctness for the third pattern is negotiated.

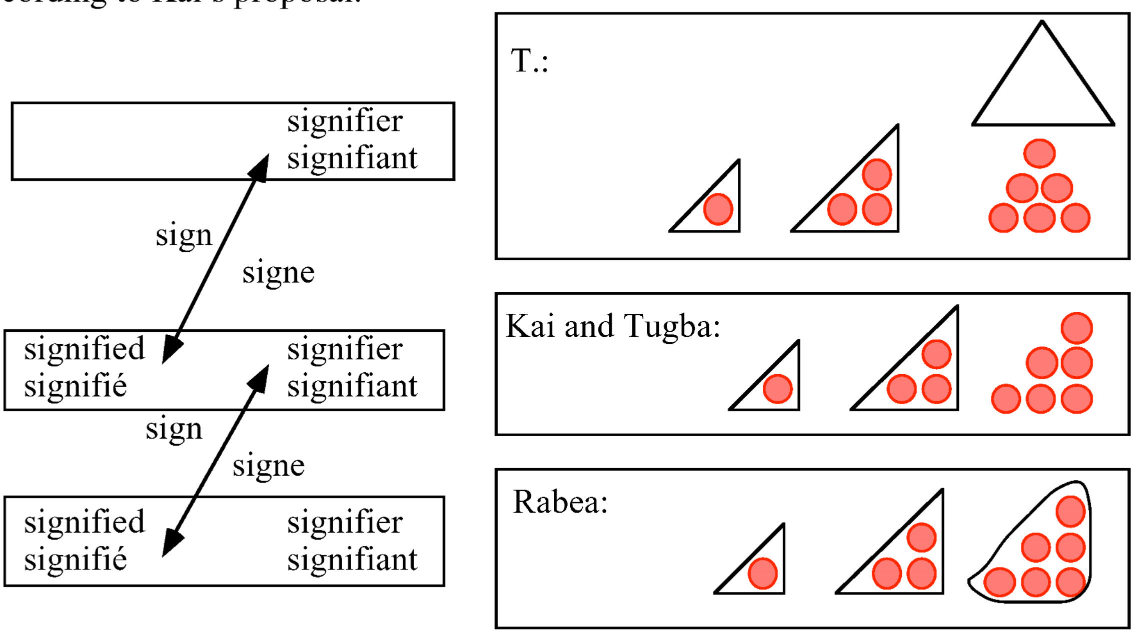
24 T … Who could now explain why this is correct now? That is correct, you must know. Rabea.

25 R Because it is again the same pattern.
26 L Mhm. Could you come and draw the pattern around it? [*Tugba goes to her place and Rabea comes to the blackboard*] … Draw first the pattern! Go once around it! [*Rabea draws with chalk a triangle around the third configuration*] Aha.



In principle, this justification consists of one statement: “Because it is again the same pattern.”, which then is illustrated by Rabea as the “same pattern”. How could this short statement gain the status of a justification? This justification function is only possible on the basis of the earlier interaction process. We have seen that the continuations of the teacher’s two initial patterns as proposed by the students could have been possible and reasonable. But the teacher refused them one after the other and at the same time she explicated the conditions of “similarity” in the patterns. The children’s proposed continuations are excluded until in the end the teacher’s intended unique, correct third pattern is produced.

With the scheme of the communicative analysis the final interactive justification with Kai and Tugba can be described in the following manner. The teacher emphasises the rectangular property of the figures and as a contrast she draws the isosceles triangle according to Kai’s proposal.



Kai changes his signifier, and after Tugba has completed the pattern, the teacher accepts this continuation of the series of patterns. By drawing the triangular shape of the pattern Rabea makes the “similarity” between the different shapes more explicit and in this way the “conformity” of the patterns becomes the justifying argument and this is legitimised and agreed upon interactively.

The functioning of the autopoiesis of the communication system requires that a given signifier is not directly linked with the signified intended by the conveyor of the message. This openness is essential for the communicative process. During the analysed episode one can observe that the teacher's denials of the children's proposed signifiers aim at identifying a definite relation between the given signifier (the two first triangular patterns) and a fixed signified (the rectangular shaped pattern). The elaboration of this definite third pattern takes place by a kind of negative delimitation in the course of this interaction.

This mathematical interaction is dominated by the idea that there exists one single correct third pattern, and this idea is made explicit step by step. The teacher stresses this point at different occasions: "Is this already correct? ... One could have the impression, but it is not yet quite right"(5); "… mh, this is not yet quite correct." (10); "… he is very <u>close</u>!" (22); "… why this is correct now? That is correct, you must know." (24).

The communication analysis shows that the interaction is used to point out the teacher's a priori correct relation between the presented signifier (the two patterns) and the appropriate fixed signified (the shape of a rectangular triangle). To give an acceptable justification in this situation means to identify the correct relation between signifier and signified. The basis is the dogma that in mathematics there always exists one single correct relation between signifiers and signifieds. From an epistemological point of view, the new sign "third pattern" is interpreted with regard to a reference context of fixed rectangular shapes for triangles – all other possible shapes are excluded. News signs / symbols and corresponding elements in the reference context are strictly fixed with one another; the signs become names for observed empirical objects (in this case for chip configurations).

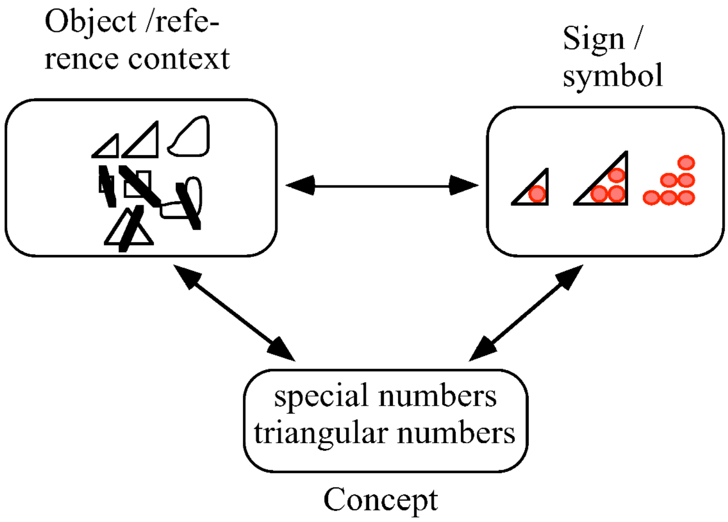


Fig. 3: The epistemological triangle

The justification of the correctness for the third pattern can be characterised in this way: The proposed patterns are compared with one invisible fixed pattern and differences or similarities are remarked until the new pattern is in agreement with the teacher's intended pattern. The last pattern is an admissible one, but it is accentuated in a special social manner as the only correct pattern. No specific reasons are provided for the choice of this pattern; the sole basis is the teacher's authority.

4.2 How is it Possible to Recover a Lost Number in the Number Square?

During this lesson in a mixed class of grade 3 and 4 the children had to work on the following problem: How could one recover a lost number in a certain number square, in such a way that this number reproduces the former arithmetical structure? (cf. Fig. 4).The special number squares as used in this class can be constructed in the following way: First one adds some given numbers in the border row and border column of a table (cf. Fig. 5). The squares thus created have the following property: You can choose (circle) in a (3 · 3) number square three numbers arbitrarily such that in every row and in every column there is one and only one circled number. The sum of three numbers chosen is always constant – independent of its choice (cf. Fig. 6). Such squares are called "crossing out number squares", because when circling a certain number in the square, all

15	16	17
14	15	16
13	14	

Fig. 4

+	5	6	7
10	15	16	17
9	14	15	16
8	13	14	

Fig. 5

~~15~~	~~16~~	(17)
~~14~~	(15)	~~16~~
(13)	14	~~ ~~

Fig. 6

other numbers in the same row and in the same column have to be crossed out. The children called such a square »magical square« and the constant sum the »magical number«. In this episode, the children reproduced the lost number with three different strategies.

4.2.1 First Strategy: Using Structures in the Arithmetical Pattern

By using the arithmetical pattern of the given numbers in the square, Kevin argues that 15 is the missing number: "… because here is the fifteen, sixteen, seventeen [*points at the first row of the magical square*]. There is the fif-, fourteen, fifteen, sixteen [*points at the second row of the magical square*]. And here is the thirteen, fourteen [*points at the two numbers in the third row of the magical square*]. And then comes there the fifteen [*points at the empty field in the third row*]" (12). Kevin completes his argument by refering to the arithmetical regularities in the columns, too.

4.2.2 Second Strategy: Reconstructing the Missing Number with Numbers in the Border Lines

Some students reconstruct possible numbers in the border column and border row from which the magical square could have been built up. They start with the additive decomposition of 15 = 10 + 5 (cf. Fig. 7) and they calculate further numbers in the border row (cf. Fig. 8) and finally in the border column (cf. Fig. 9). On this basis the children determine 15 as the missing number; this is justified by checking all calculations.

+	5		
10	15	16	17
	14	15	16
	13	14	

Fig. 7

+	5	6	7
10	15	16	17
	14	15	16
	13	14	

Fig. 8

+	5	6	7
10	15	16	17
9	14	15	16
8	13	14	

Fig. 9

4.2.3 Third Strategy: Reconstructing the Missing Number by Using the Magical Number

Already earlier in the course of this lesson Kim has sketched her idea. Later she explains her plan in detail. First she calculates the magical number 45 by adding the numbers 13, 15 and 17 in the diagonal. With this proposal she expresses that one can determine the magical number in an incomplete magical square. Then her argumentation starts.

147	K	And then one could already make it this way. One circles the fifteen [*points at 15 in the first row*] and this fifteen [*points at 15 in the second row*] and adds it up. And then one still calculates, how much there must be up to forty-five.

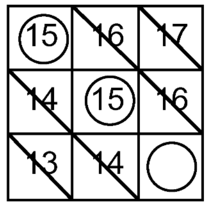
The signifier "One circles the fifteen and this fifteen and adds it up." denotes the intention to apply the known procedure for calculating the magical number to two numbers in the diagonal. The second signifier "And then one still calculates, how much there must be up to forty-five." could be understood in this way: One has to calculate how much is left from the sum of 15 + 15 up to 45 (one has to calculate the difference); seemingly, this number has to be placed into the empty field.

At this moment, several classmates object that nothing could be really calculated here. "Well, that really leads nowhere … Where you would like calculate up to? … Exactly. After all, you do not at all know which number is the result here!" (152, 153).

Kim formulates further explanations.

161	K	First one calculates, one first calculates these numbers, that I have, which are there, what is their result. And then …, and then one calculates …
165	K	These three, oh, yes, this, this and then afterwards one calculates fifteen [*circles 15 in the second row*], one takes this way. Cross out that, and that. And cross out that and that [*crosses the other numbers in the same column and the same row*]. Then one also takes the fifteen [*circles 15 in the first row*]. Crosses out then the seventeen and the thirteen [*crosses out the still uncrossed numbers in the same column and the same row*]. And then one circles this here, this here [*circles the empty field*]. And then one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five.

The signifier "And then one circles this here, this here." indicates the application of the crossing algorithm for calculating the magical number to a missing third number – to the empty field. On the one hand, the second signifier "…one has to calculate, fifteen and fifteen this makes thirty, how much is left up to forty-five." intends the calculation of the magical number from three circled positions: 15 and 15 makes 30. But with the third circled position one cannot calculate in the known standard way. On the other hand, one should now calculate in a “reversal manner” with the empty field: Here one has to place the number that represents the difference between thirty and forty-five (the magical number). Later, the addition task is written as a complementary task “15 + 15 + __ = 45”. This task displays the form of an addition task with three terms, but in an unusual way. It reflects a generalised structure allowing to express the calculation with a partially unknown number. In this way, new mathematical signs or symbols are created. By applying the calculation scheme for the magical number to the empty field and by writing the complementary task with an unknown term, new mathematical signs are expressed and are represented as abstract icons.



13 + 15 + 17 = 45

15 + 15 + __ = 45

Fig. 10

The epistemological analysis verifies that Kim constructs genuine new knowledge when including the empty field into the mathematical operation to determine the magical number. She argues that one cannot calculate with concrete numbers only, but the algorithm for the magical number can be extended also to arbitrary fields –with or without numbers. The new mathematical knowledge constructed in Kim’s argumentation can be described with the help of the epistemological triangle. The new relation (resp. “unknown number” or “variable”) is symbolised in two ways; once as a “circled number” and then as a missing term in the addition task. In this domain of representation and of mathematical operation we can observe how Kim works with the “unknown number” in a specific situated manner. Kim places the unknown number into a new mathematical relation with other numbers and in this way she constructs new knowledge; the new mathematical object is created as a relation in the extended and generalised operational structure of the number square.

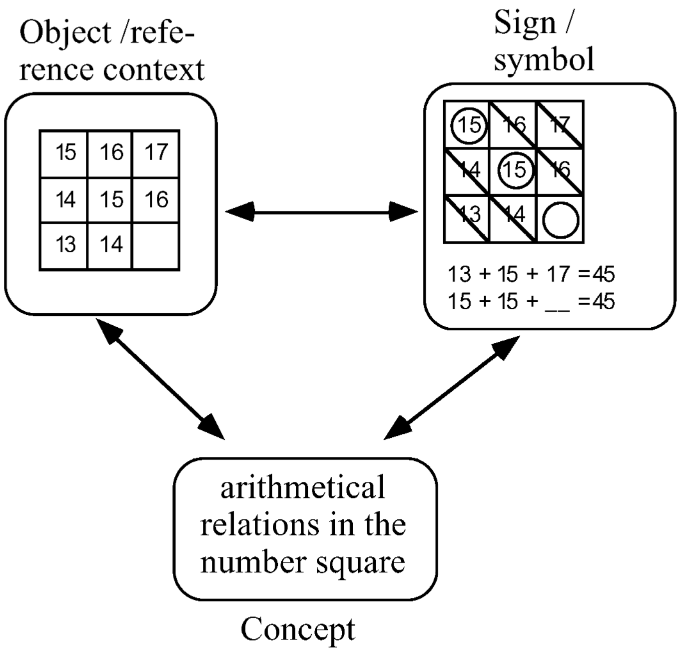


Fig. 11: The epistemological triangle

4.2.4 Eva Repeats the Strategy Using the Magical Number

During the discussion of Kim’s strategy, the teacher encourages other children to explain how this “trick” works. Eva argues in the following way.

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193  E    Well. Ohm, one has to take three numbers out of the
          magical square. Add them up. But not the empty field,
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          there is not much to calculate, all right?
196  E    Yes. And then you get the magical number. Then one must,
          ohm, that, ohm take again three numbers, but now also the
          empty field must be therein. And then you have to … from
          this number, you get then … from these two, one has still
          to go to forty-five.
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With the first signifier, Eva intends the calculation of the magical number taking three concrete arbitrary numbers from the square. She adds: "But not the empty field, there is not much to calculate, all right?". One cannot and should not use this field for the calculation of the magical number.

The second signifier indicates that the scheme of the algorithm for the magical number shall be transferred to three other numbers: "… take again three numbers, but now also the empty field must be therein." Here, the empty field is identified in a way with a number; Eva characterises a mathematical unknown in a situation specific manner. When transferring the algorithm the following calculation cannot be done in the usual direct manner, because the third term is missing; consequently Eva modifies the calculation procedure: "And then …from these two, one has still to go to forty-five".

In her description, Eva makes the distinction between the *impossibility* to calculate a sum by using the empty field as one of the three terms and the *possibility* to operate with the empty field (as a kind of pre-variable) in the same way as with a known concrete number, provided that the result of the operation is already known. Her argument expresses a fundamental epistemological dialectic between old and new mathematical knowledge. In the frame of understanding subtraction as »taking away« –an example for old knowledge – the task "5 –7 " cannot be calculated; later, with a relational understanding of the number concept –an example for new knowledge –the same task can be calculated by developing and using the new concept of "negative numbers".

5. Interrelations between the Epistemological and Communicative Dimension of Mathematical Signs

Mathematical interactions are social systems, being at the same time characterised by very specific intentions. On the one hand they are *educational* or *instructional* communications, on the other hand *mathematical knowledge* is in some special way the object of these communication processes.

• Interactions between teacher and students are pedagogically intended social communications with the aim to mediate knowledge. This implies a superposition in the autopoietic development of the social communication with an "additional sense", which is the result of the teacher's intention linked with his *teaching* and *instruction*. This intention dominates the educational communication for all participants (teacher and students). When trying to realise their instructional intentions teachers often unconsciously make the assumption that the separation between the social and the psychic system could be bridged directly and the meaning conveyed in the communicative process could be transported instantly and unchanged into the student's consciousness (as it is tried with the well known interactive funnel patterns; cf. i. e. Bauersfeld 1978; Krummheuer & Voigt 1991, p. 18; Wood 1994, pp. 153ff.; 1998).

•Mathematical communication copes with mathematical knowledge; this implies for the (external) observer (the researcher) to analyse the knowledge which is generated in the course of communication interactively from an epistemological perspective. The analysis of the specific status of school mathematics and its interactively constituted meaning shows that it can be interpreted as a "symbolically generalised communication medium" (Luhmann 1997, p. 316ff.); in analogy to "scientific truth" (Baraldi, Corsi & Esposito 1997, p. 190) one can speak here of "school mathematical correctness".

In the course of the first episode on "triangular numbers", one can observe a kind of "compensating communicative strategy". During the interactive process of constructing new knowledge up to the short phase of the accepted justification, the teacher assists, comments and refutes the children's proposals. She uses a number of similar key words: "…trying to place the next …" (1); "Is this already correct? ... One could have the impression, but it is not yet quite right" (5); "…this is not yet the right." (10); "Is this the same?" (16); "Ooh, he is very close!" (22); "… Who could now explain why this is correct now? That is correct, you must know." (24). With these descriptions, the teacher indicates that she starts from a very definite idea about the third pattern. In the course of the episode, the students are led to find out what, according to the teacher's opinion, is the only correct pattern. After having given their own, correct proposals for continuing the pattern, now the students do no longer interpret the conveyed signifiers with reference to other *mathematical* signifiers for constructing in this way new epistemological mathematical relations, but they start to seek for the intended signified belonging to the teacher's signifier for attaining in this way the correct solution.

This episode shows in an exemplary manner that the necessary condition for starting the autopoietic behaviour of the *mathematical* communication is "destroyed". First of all, the receiver of the conveyance (a student) can ascribe the given conveyance only to the conveyor (the teacher or another student). The possibility to detach the information of the conveyance that the conveyor "attached" to his conveyance implies the possibility of the autopoiesis of the social system. The students are more and more urged to deduce directly from the teacher's signifiers the intended signified. Such a type of communication takes place with the tacit assumption that mathematical conveyances (the signifiers; or the mathematical signs) possess *unequivocal* information (definite referential signifieds), which can be derived in a communication *about* the conveyances. In this way, the referential links of the signifiers are shifted. The new signifiers no longer refer to mathematical referential contexts immediately but they refer to the interpretation that is postulated by the teacher in the existing reference context. The conveyances become the new, proper object of communication. The signifiers are no longer conveyed in communication for relating them to other *mathematical* signifiers or *mathematical* signs mediated in interaction and thus constituting an interpretation.

During the second episode, the communication has a different character. The attempts to reconstruct the missing number are not dominated by explicit aims of the teacher. Different proposed strategies are allowed: Kevin uses the number pattern of the given number square; then several children reconstruct the missing number by reproducing possible numbers in the border row and border column. Subsequently, Kim presents her ideas to calculate the missing number with the help of the magical number; moreover, three children explain this strategy with their own descriptions. The teacher does not confront these proposals with her own fixed ideas about the correct solution; she moderates and supports the children's construction of new knowledge that develops in the course of interaction. Instead of pushing ahead her own ideas, the teacher places the students' solution strategies into the foreground of the communication process, and this makes possible to maintain a "true" mathematical discourse in this classroom.

The autopoiesis of the communication system is possible only if the signifiers are not connected with the intention of the conveyor strictly but always have to be interpreted in the course of communication in a new manner. For a given signifier there is no definite, fixed signified in communication, and therefore there is no unique universal sign, but different interactively evolving interpretations. Accordingly, a successful authentic mathematical interaction requires that a communicated (new) mathematical sign is not fixed a priori by a given referential object, but the participants have to develop their own different and multiple readings of the communicated sign. Such multiple, evolving interpretations of mathematical signs are possible only if these signs are not explained by linking them with concrete properties of pre given empirical objects but if the referents of mathematical signs are seen as relational structures.

An open referential interpretation of communicated signifiers or of mathematical signs is indispensable for the progression of a *mathematical* communication process. A successful mathematical discourse requires not to fix the mathematical signs definitely and once for all, but to respect a rather open relation between mathematical signs and referential contexts the learner permanently has to establish in new ways in interaction. The realization of an open interpretation of mathematical signs strongly depends on the acceptance of mathematical objects as »symbolic relational structures« in interaction. When mathematical knowledge is reduced to its formal terminology and its logical consistency with reference to fixed referents then the mathematical discourse is in danger to turn into a communication about the definte "correct" interpretation of mathematical signs what in the end is decided by the teacher's authority.

The successful functioning of authentic mathematical interaction requires "open" mathematical objects. A central implication is that every theoretical analysis of possible reasons for the success or the failure of everyday mathematical interaction has to take into account the very specific epistemological nature of mathematical knowledge as "symbolised relations" and it has to reconstruct the social epistemology of mathematics in interaction, i.e. the characteristic forms and situated descriptions for this knowledge. The particularity of the mathematical in classroom interaction –and also in any reconstructing analysis of such interaction –is basically established by the symbolic, relational character of mathematical concepts; these concepts represent open interrelations between formal sign systems and situated referential structures that have to be negotiated in mathematical interaction.

Consequently, the manner of interpreting mathematical knowledge influences the mathematical communication process essentially: Strictly fixed readings of mathematical signs may cause a paralysis of mathematical communication and they also may lead to a transformed, a ritualised communication. Open readings of mathematical signs with regard to multi relational, structural reference contexts are preconditions for any authentic mathematical interaction. The development of a successful mathematical communication requires to take into account the epistemological particularities of mathematical knowledge and at the same time the specificity of instructional interactions between teacher and students. Direct, intentional teaching and interactive construction of new mathematical knowledge often constitute a fundamental dilemma that cannot be dissolved easily in mathematical discourse.

References

Baraldi, C., Corsi, G., & Esposito, E. (1997). GLU. Glossar zu Niklas Luhmanns Theorie sozialer Systeme. Frankfurt am Main: Suhrkamp.

Bauersfeld, H. (1978). Kommunikationsmuster im Mathematikunterricht - Eine Analyse am Beispiel der Handlungsverengung durch Antworterwartung. In H. Bauersfeld (Ed.), Fallstudien und Analysen zum Mathematikunterricht, (pp. 158-170). Hannover: Schroedel.

Changeux, J.-P., & Connes, A. (1995). Conversations on Mind, Matter, and Mathematics. Princeton: Princeton University Press.

Dehaene, S. (1997). The Number Sense. How the Mind Creates Mathematics. London: Allen Lane The Penguin Press.

Frege, G. (1969). Funktion, Begriff, Bedeutung Fünf logische Studien. Göttingen: Vandenhoeck & Ruprecht.

Hersh, R. (1997). What is Mathematics, Really? London: Jonathan Cape.

Indurkhya, B. (1994). Metaphor as Change of Representation: An Interaction Theory of Cognition and Metaphor. In J. Hintikka (Ed.), Aspects of Metaphor, (pp. 95-150). Dordrecht: Kluwers Academic Publishers.

Krummheuer, G., & Voigt, J. (1991). Interaktionsanalysen von Mathematikunterricht - Ein Überblick über Bielefelder Arbeiten. In H. Maier & J. Voigt (Eds.), Interpretative Unterrichtsforschung, (pp. 13-32). Köln: Aulis.

Luhmann, N. (1996). Takt und Zensur im Erziehungssystem. In N. Luhmann & K.-E. Schorr (Eds.), Zwischen System und Umwelt. Fragen an die Pädagogik., (pp. 279 - 294). Frankfurt am Main: Suhrkamp.

Luhmann, N. (1997). Die Gesellschaft der Gesellschaft. Frankfurt am Main: Suhrkamp.

Maier, H., & Steinbring, H. (1998). Begriffsbildung im alltäglichen Mathematikunterricht – Darstellung und Vergleich zweier Theorieansätze zur Analyse von Verstehensprozessen. Journal für Mathematik-Didaktik, 292-329.

Maturana, H. R., & Varela, F. J. (1987). Der Baum der Erkenntnis. Die biologischen Grundlagen des menschlichen Erkennens. Bern, München: Scherz.

Ogden, C. K., & Richards, F. A. (1923). The Meaning of Meaning. London: Routledge and Kegan.

Searle, J. R. (1997). Die Konstruktion der gesellschaftlichen Wirklichkeit. Zur Ontologie sozialer Tatsachen. Hamburg: Rowohlt.

Steinbring, H. (1989). Routine and Meaning in the Mathematics Classroom. For the Learning of Mathematics, 9(1), 24-33.

Steinbring, H. (1991). Mathematics in teaching processes - The disparity between teacher and student knowledge. Recherches en Didactique des Mathématiques, 11(1), 65 – 107.

Steinbring, H. (1993). Die Konstruktion mathematischen Wissens im Unterricht – Eine epistemologische Methode der Interaktionsanalyse. Journal für Mathematik-Didaktik, 14(2), 113 - 145.

Steinbring, H. (1998). Elements of Epistemological Knowledge for Mathematics Teachers. Journal of Mathematics Teacher Education, 1(2), 157-189.

Steinbring, H. (in collaboration with Kratzin, C. & Scherer, P.) (1998). Epistemologische und sozial-interaktive Bedingungen der Konstruktion mathematischer Wissensstrukturen (im Unterricht der Grundschule). (Zwischenbericht zu einem DFG–Projekt). Dortmund: Universität Dortmund.

Wood, T. (1994). Patterns of Interaction and the Culture of Mathematics Classrooms. In S. Lerman (Ed.), Cultural Perspectives on the Mathematics Classroom, (pp. 149 – 168). Dordrecht: Kluwer Academic Publishers.

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