The Genesis of New Mathematical Knowledge as a Social Construction

Heinz Steinbring, Universität Dortmund, Germany

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Abstract. *Does the development of mathematical knowledge in everyday teaching depend primarily on the individual insights of singular students or does it require a social discourse and a reflexive discussion? This problematique is an instance of the contrast between an »individual psychological perspective« that emphasises the autonomous, cognitive development of the learning subject, and a »collective perspective« that understands learning as a process of socialisation into a culture of teaching and learning. From an ongoing research project exemplary teaching episodes are analysed expressing how individual learning strategies (in seat work and partner work) and social-interactive constructions of knowledge (in common discussion with the whole class) favour different forms of an epistemological development of new mathematical knowledge. During phases of individual or partner work one can first of all observe linear, step-by-step procedures (algorithmic construction of knowledge); the common interactive phases of reflection offer opportunities for students to interpret the new knowledge from a conceptual-structural point of view (structural-systemic construction of knowledge).*

1 The Problem of New and Old Mathematical Knowledge

Mathematical knowledge is subject to a conflict in the relation between new and old knowledge: On the one hand every mathematical knowledge is logically consistent and hierarchically organised and can be deduced from given fundaments and, therefore, is not really new. On the other hand there are really still unknown and new mathematical insights, for instance by solving problems or by proving mathematical assumptions and theorems.

The example of mathematical proof explains this conflict paradigmatically. Rotman gives the following description: "… a proof is a logically correct series of implications… Proofs are arguments and, as Peirce forcefully pointed out, every argument has an underlying idea – what he called a leading principle which converts what would otherwise be merely an unexceptionable sequence of logical moves into an instrument of conviction… It is perfectly possible to follow a proof, in the more restricted, purely formal sense of giving assent to each logical step, without such an idea… Nonetheless a leading principle is always present… without which [proofs] fail to *be* proofs" (Rotman, 1988, pp. 14/5).

In an extensive historical and philosophical study, Jahnke analyses the contradiction between *development* and *justification* of knowledge and derives educational consequences for the learning of mathematics. "According to the self-image of science … »logic« and »intuition« are completely separated. .The process of gaining knowledge is therefore essentially of an irrational character or at best to be explained as a psychological phenomenon, whereas a mathematical proof is only understood as a tautological chain of signs" (Jahnke 1978, p. 58/59). In the course of knowledge development new "ideal mathematical objects" are constructed that only permit to change the tautological justification on the basis of existing consistent knowledge into productive justification as regards mathematical content, a justification that argues from the future, i.e. it takes the new knowledge as its basis. "… the problem, to make students aware of the necessity of proving, shows that this only can happen with a look-ahead on more general points of view that make reasonable the possibility of an operative treatment of mathematical facts or of *idealised objects*" (Jahnke 1978, p. 251).

The distinction between the *logical structure* and the *(ideal) mathematical objects* becomes visible in interactions of every day mathematics teaching. The "logical consistency" is manifest in rules, laws of calculation and algorithmic procedures that are used more or less consistently as factual knowledge. In contrast, really new developed knowledge requires the interpretation of newly constructed mathematical relations. (cf. Steinbring 1999). This theoretical problem is an object of investigation in a research project ("Social and epistemological constraints of constructing new knowledge in the mathematics classroom"*), funded by the German Research Community, (DFG); cf. Steinbring 2000; Steinbring 1999); a central question is how children in primary grades are able to construct in an interactive way elementary new mathematical knowledge without having to use abstract notation or formal rules of production.

2 Theoretical Perspectives on Individual and Social Knowledge Construction

There is a controversial discussion whether new knowledge first of all emerges in form of individual contributions made by singular, competent students or whether it is the result of the social interactional context. The most prominent theoretical attempts within mathematics education trying to explain this contradiction are, on the one side, the social activity theory (according to Vygotsky) and the extension of the subjective constructivism to a social constructivism under an intersubjective perspective (cf. Lerman, 1996) and, on the other side, the radical constructivism (in its individualistic and interactionistic form). A common view is that in the subject dependent construction of new knowledge, both, the individual and the social environment are involved. But the "connection" between individual and social environment are differently judged.

As cultural tools for the acting subject, symbols have a central meaning for the relation between the individual and social position in the frame of activity theory. "Vygotsky's … hypothesis is that all development of the individual comes about through sign mediation in activity" (Meira & Lerman, 1999 p. 3). But these cultural signs do not transport automatically new knowledge into the students' heads. "Following Vygotsky, cultural tools and sign systems are not carriers of meaning …; that is, they do not carry their meaning to children. Instead, for any individual, cultural tools derive their meaning from his or her constructional activity" (Waschescio 1998, p. 234).

With regard to constructivism, Jörg Voigt makes the following distinction between an individualistic and an interactionistic perspective: ”Individualism views learning as structured by the subject’s attempts to resolve what the subject finds problematic in the world of her or his experience. Instead of asking how objective knowledge becomes internalised by a person or how subjective knowledge develops toward truth, interactionism and ethnomethodology study how intersubjectivity is achieved in the negotiation of meanings between persons” (Voigt 1998, p. 216). The main criticism of the constructivist positions first of all aim at the separation which, in the end, is assumed between an individual and a socially constructed meaning of knowledge or that – within radical constructivism – the social environment is seen as a mere formality and new knowledge is exclusively interpreted as a personal construction of an individual.

3 The Learning of New Knowledge as Collective Argumentation

The sociologist Max Miller (1986) makes the problem of the genesis of new knowledge the central question of what role the social, collective learning process has for the cognitive development of the individual. “From a theory of learning or development… one can legitimately expect, that it provides an answer to the question of how the *New can emerge* in development. … Every answer to this question … is subject to … the following criterion for validity: it has to be shown that the New in development presupposes the Old in development and at the same time exceeds systematically the Old, otherwise there cannot be a New or the New already is an Old, and then the concept of »learning« and »development« looses any sense” (Miller 1986, p. 18).

Then Miller puts three important questions for learning: “How for the individual … the validity of his or her already acquired (old) knowledge … can be shaken or relativized? How the individual can make new, his or her actual knowledge systematically exceeding experiences relevant for learning? And, how … can there be for the individual a compulsion to further develop his or her knowledge?” (Miller 1986, p. 18/9).

Genetic individualism cannot give answers to these questions. “Because within the paradigm of genetic individualism processes of learning are exclusively limited on mental processes of the single individual, also the constitution of experiences relevant for learning … can only be understood as a mental performance of the single, monological subject” (Miller 1986, p. 19). This theoretical position cannot explain how really new knowledge emerges. “In case that the learning subject already has to know somehow the New in the development for being able to get to know it first of all, then the New in development is already identical with the Old in development; …. The genetic individualism founders on a dissolution of the basic problems of a genetic epistemology” (Miller 19986, p. 20).

Individualistic orientations on “learning processes” concentrate – when taking a mathematical perspective – on consistent elements of knowledge which are already known and not really new, for example factual knowledge. Only collective processes make a potential development of new knowledge possible by contrasts, contradictions and re-interpretation. “What genetic interactionism substantially distinguishes from genetic individualism is the basic assumption that the collective and symbolically mediated application of mental capacities that are limited for the single individual might lead for the participating individuals to a process of experience constitution helping the single individual to solve the dialectic of knowledge and experience – of course not on the level of a theoretical (philosophical) reconstruction but on the level of an actual execution. Only within the social group and due to social processes of interaction between members of a group the single individual can make those experiences enabling fundamental steps of learning” (Miller 1986, p. 20/21).

Not every form of communication induces a learning process. “Only a type of discourse in which the principle goal is to find collective solutions to interindividual problems of coordination has a built-in capacity to release processes of collective learning. There is only one type of discourse that fulfils this condition: collective argumentation. Collective argumentations constitute the very basic method for jointly solving problems of interpersonal coordination” (Miller 1987, p. 231).

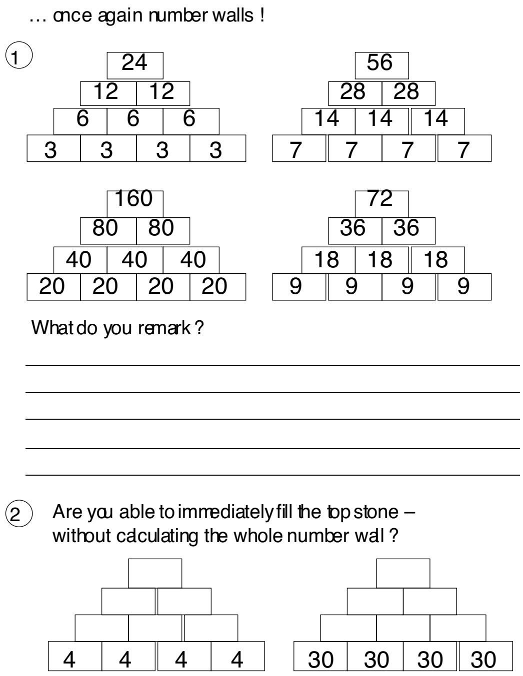
Collective argumentation cannot be reduced to the individual. ”.. the method of collective argumentation cannot .. be described or explained by a reduction to a method of a mere individual argumentation. Also collective argumentations happen in the head of an involved subject, but the constitutive properties of a collective argumentation can only be understood adequately within a interaction theoretical frame” (Miller 1986, p. 25).

4 Differences in Interactive Knowledge Construction in Phases of Single and Partner Work or in Common Discussion – Analysis of Examplary Episodes

The detailed analysis of a number of teaching episodes in the mentioned research project (cf. Steinbring 2000) reveals that in the course of single and partner work students primarily concentrate on aspects of factual knowledge and re-construct consistent connections in already existing knowledge. With the perspective of unequivocal readings of mathematical signs learning is individualised in the sense of being seemingly simply the overtaking of ready, pre-fabricated mathematical knowledge and definite meanings. In contrast, the analysis shows that rather in common, social reflective discourses there are possibilities for open interpretations of mathematical signs and relational structures whose meaning still has to be established interactively in the course of discussion – dependent on the communicative style between teacher and students. The contrarieties in communicated readings of different symbols play an important condition for the construction of really new mathematical knowledge in reflective discussion.

4.1 Sonja and Julia Shorten her Calculation Process

In the course of this teaching unit (grade 4) the students dealt with the learning environment of“number walls”. During this lesson, the children worked on an exercise sheet with empty number walls in which the base row of stones was filled with four equal numbers. They had to calculate the missing numbers and to notice remarkable facts; another aim was to discover that the top number is the eightfold of a base number.

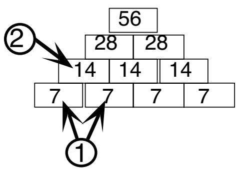


In the meantime, Julia and Sonja have filled the four number walls above; they want to report her observation to the teacher. First they simply remark: "There we have no longer really to calculate. There you have only three times".

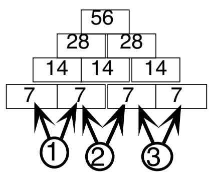
In the following episode, they explain this remark more precisely; no doubt, they point at the exercise sheet but it is impossible to identify the exact number wall they refer to:

28 So One has always-, one only has to calculate three times, because these bo-, because, why this one gives that. ... One only has to calculate three times. [*points at some numbers on the sheet while she is explaining; then Sonja is interrupted by her neighbour Julia*]

29 Ju This, this plus this (1), this plus this (2) and this plus this (3). [*in this moment So and Ju speak simultaneously, they point both at exercise sheet in front of So*]



29a Ju Because, and when one has calculated these, these plu-, ehm, this plus this(1), gives this (2). And then one needs no longer to calculate these.[*meanwhile she points again at different places on the exercise sheet*.]



30 So And here the same [*points at different the exercise sheet*.]

Later, Julia and Sonja report other striking observations: They have recognised that the calculated numbers in the fourth wall belong to the multiplication row of nine; moreover, the numbers double from one row to the next one above: "That's‘ always double, the double".

On the basis of their calculations Julia and Sonja construct her "abbreviation" and then they observe striking facts in these number walls. But they do not discover the "little trick" of an immediate calculation of the top number. The children's considerations remain on the level of a direct calculation and of visible numbers; for instance, no relations between the changes of the base stones and the resulting changes in the top stone are used (as has been done in the lesson before in some way). The teacher simply notes these explanations, partly she confirms them, but she remarks, that only one calculation task is necessary.

Summarising, one can conclude that the children in this episode do not construct a really new knowledge relation but they rather re-construct the interpretation of mathematical signs on the basis of already known solid knowledge and then they are able to abbreviate the calculation process – because of the observed repetitions; also the mentioning of the "ninth-row" and that there is "always the double" are forms of a re-construction of arithmetically consistent knowledge. The children reconstruct the uniquely fixed and pre-fabricated signifieds for the mathematical signs; for this purpose they use the mathematical sign–tools in the familiar manner.

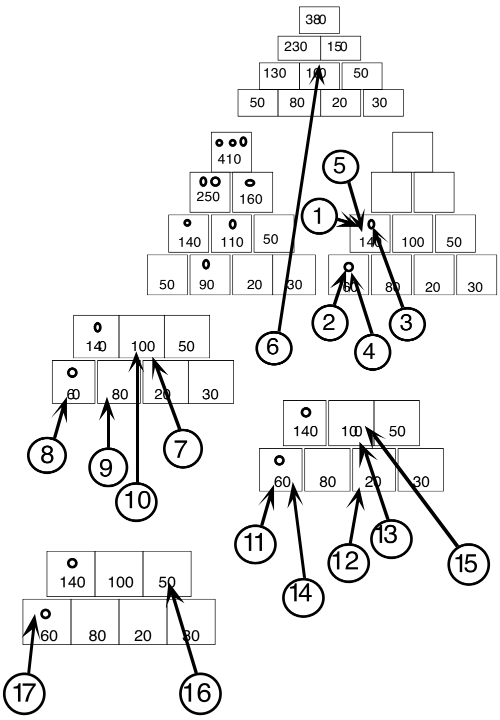
4.2 Matthi Explains the Increase of the First Stone in the Second Row

This episode is part of the same teaching unit on number walls. Now, in the course of reflective discussion, an explanation is expected what is the change of the top stone dependent on the change by "10" of one of the base stones. Just before, the effect of the increase of one of the middle base stones has been discussed; now the relation between the left border stone and the top stone is in question. The student Matthi is asked to explain why the left stone in the second row of the wall has increased by "10", dependent on the change of the left base stone. Matthi goes to the blackboard and, first, he wants to calculate the next number row; but the teacher interrupts and asks for an explaining argumentation.

198 T # No. Matthi, please wait a moment? Did you notice something? #

199 Ma # Ehm, here (1) also is 10 more (2, chip)

20 Ma .... because also here is 10 more. [places *a chip* (3)] Because here, too, because it's 10 more here, (4, chip), here is 10 more (5) than there 10 more (6). Here then is the same (7) [*and then several times alternately* (8), (9)], because one cannot this here somehow plus that (10). One does not get with this here (*alternately* (11), (12)) then there (13), (14), or so. That one gets this. (15). And here (16) it's also the same, because this is on the outside (17).



Obviously, the student Matthi comes with the intention to the blackboard to calculate the numbers in the next row of the wall with the consistent rules. But the teacher's demand forces him to develop spontaneously an explanation for the increase of the according stone. Then Matthi constructs a new mathematical sign; he uses neither concrete numbers nor more general descriptions for the positions of numbers, but he always points at certain places in the number wall: This way of pointing is his personal way of expressing the mathematical signs he is constructing; he mainly bases his communication on signs produced in a deictic manner. He argues indirectly: If the sixth stone should increase then the first base stone must contribute to the corresponding addition; but this increased stone cannot be a term in the sum with the second nor with the third base stone, for producing the middle stone of the second row.

Matthi constructs a new mathematical sign that is not yet known or familiar. In the course of common interaction the students actively have to interpret this sign. Further, the construction is not simply only his very personal, individual construction; especially the discussion episode, just before, on the increase of numbers in the second row induced by an increase of a middle stone obviously had important influences on the way of how Matthi constructed his argument.

5 The Genesis of New Mathematical Knowledge: Individual or Social Learning?

In an exemplary way, the discussed episodes make visible two forms of interaction, on the one hand, the orientation on the mediation of consistent factual knowledge, on the other hand, the potential construction of new conceptual relations. In the context of consistent arithmetical rules, Julia and Sonja reduce the number of calculation steps to three. No really new knowledge is constructed, but while observing repetitions of calculations, the procedure is abbreviated. The knowledge here is derived from existing properties immediately.

Behind Matthi's indirect justification there is a construction of really new knowledge relations: The increased left stone in the base row is never linked with any other base number and therefore it is impossible to increase the stones except the left one in the second row. This argument cannot be deduced in this moment from existing factual knowledge. The question »What is a sufficient reason for looking at these – impossible – arithmetical relations?« requires to be aware of an "idea behind the logical structure".

At first sight, the constructions of knowledge always seems to be the result of personal insights of individual students. Of course, during processes of communication always some participant introduces his or her contribution and, reversal, the communicative process depends on these personal contributions. But this is not sufficient to deduce that the construction of new knowledge is exclusively an individual process. The obligation for constructing new mathematical signs and relations in the end is an external cause "imposed" to the individual by a contradiction, a conflict or a contrast in the course of social interaction – or also in a personal struggle with a contradiction in the existing knowledge structure.

The proposal made by Julia and Sonja rather shows a linear, step-by-step, algorithmic knowledge construction in the context of existing factual knowledge. The episode where Matthi develops his argument, exemplifies the interactive manner of a construction of new mathematical knowledge. The requirement to give a justification forces Matthi to exceed his own knowledge and to make “experiences relevant for learning”.

In the course of every day, traditional mathematics teaching the requirement that students develop justifications and arguments plays a minor role; when students are more strongly asked to give reasons for mathematical insights – as it was the case in the teaching units observed and analysed in the mentioned research project – then one can recognise that especially phases with joint reflection and discussion offer opportunities to interactively construct really new mathematical knowledge. Social, collective communication is a particular basis for young students in primary grades to question their actual available, individual knowledge archive and to further develop their knowledge substantially. In authentic mathematical communication and argumentation the meaning of mathematical signs must not be predetermined and fixed, but the possibility of an intentionally open meaning of signs is the inevitable fundament for a collective learning process in which only mathematical meaning for these signs of new knowledge can develop interactively.

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