ESSENER KOLLEG FÜR GESCHLECHTERFORSCHUNG

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Some links between Music and Mathematics –

algebraic aspects

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´´Mathematics and Music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind...´´
Hermann von Helmoltz, 1884

1. Introduction

The relationship between music and mathematics has been in focus of attention of a number of scientists who deal with mathematics and also biology and medicine on the one hand and are interested in some parts of art on the other hand. The numerous and deep connections between these kinds of human creative activity can be explained by the strict structures in music constructions and emotional nature of any challenging mathematical investigation. I belong to a group of mathematicians for whom music gives ever lasting inspiration for the algebraic research.

One of the brightest experiences in my professional life was the course of lectures on algebra in St. Petersburg (Leningrad) State University delivered by Professor Dmitriy Konstantinovich Faddeev, corresponding member of the Academy of Sciences of the former USSR. Our student stream turned out to be the last one to have the happiness of getting the basic algebraic knowledge from him. His lectures were never boring, always emotional and full of wonders and colours. We got a feeling from him that algebra is the nicest piece of human creations. Being never completed, it conceals lots of mysteries for everybody who is willing to open its secrets step by step. Greatly impressed by the beauty of algebra revealed to me by Professor Faddeev I have chosen it as the main subject in the university for my professional activity.

By that time I had got an almost 15 year experience of playing the piano and a Music School diploma. The fact that Professor

Faddeev was a brilliant pianist gave me a clue to discover the secret of his great Algebra presentations. He was able to show the "music harmony" of algebraic results and constructions. On the other hand the algebraic structures in music had been known for me by that time. These links, the presence of algebra in music and music in algebra, connected the two favourite activities in my mind and life forever. I am still grateful to Professor Faddeev for his fantastic (even if not intentional) opening this interplay, which completely coincides with my own perception.

Before that my strong subconscious feeling that playing the piano helps my algebraic thinking made me play intensively during the periods of taking exams in Mathematics. Music gave me emotional and even physical relief, the latter being explained by the sport ingredient of a music performance, some pieces of which took a lot of physical strength. I consider the combination of playing the piano and doing Algebra as a very agreeable union, with the parts complementing each other and not contradicting. Later I have learnt about the so-called "Mozart effect", the statement that the early musical lessons and training in childhood, especially with respect to some kinds of classical music, lead to the development of logical thinking in terms of abstract categories. Visiting Macquarie University in Sydney in 1999 I heard about some biological investigation on balancing left and right brain thinking and a very positive role of music exercises in combination with abstract thinking. In my opinion the "finger memory" is an absolutely special kind of mind activity, which never works under other circumstances. It affects a human brain like an ordering mechanism and distributes information systematically. That's why intensive playing the piano after long mathematical work brushes up mathematician's mind like a night sleep passing different phases (the organising role of those have been quite well investigated).

The present essay does not pretend to give a comprehensive review of mathematics-music links and can be considered only as an attempt to describe and summarise my own experiences related to the combination of both. This is done in connection with some facts accessible in a series of articles, books and www.presentation, see [3-5].

First, we discuss some algebraic aspects of music harmony which can be associated with the basic algebraic knowledge. Some of them are partially known, while the others have arisen for me in process of certain algebraic research and got a detailed considera-tion below.

2. Algebra in Music
Cyclic Group Z/12 Z and Tonality Circle

We start with the main idea of identifying octave with a “cyclic identity”, which was yet known to ancient Greeks. So all the piano keys factorised modulo 12 represent a cyclic structure in the sense that the notes sound in unison (we would say, coincide) if and only if the visible distance between their keys is equal to an octave that is just 12 keys. The vibration frequencies are in the same ratio for any two neighbouring (successive) keys. Any two successive keys form an interval called *semitone* in accordance with Johann Sebastian Bach’s system which is used at present. It means that octaves (characterised by the ratio 1:2) are divided into twelve equal semitones. From algebraic point of view a semitone (the so-called minor second interval) can be considered as a generator of group **I** of all intervals (in increasing order: minor second, major second, minor third, major third, perfect fourth, tritone, perfect fifth, minor sixth, major sixth, minor seventh, major seventh, octave) with respect to the commutative addition of

intervals in which the *highest note of the predecessor coincides with the lowest note of the follower*. (*)
For example, major third + minor third = perfect fifth. According to this definition of sum in **I**, any interval can be considered as a multiple of the minimal interval, semitone, and the algebraic structure of **I** coincides with a cyclic group of order 12.
In the theory of music the piano keyboard is traditionally used for theoretical constructions and their explanations and we use it as well. Commonly, for the seven main notes **DO, RE, MI, FA, SOL, LA, TI** we use letters **C, D, E, F, G, A, B** (Chugunov 1985) accordingly.
All the other notes can be obtained from those by adding the signs called sharp (#) and flat (ь), which make the note a semitone higher or lower. We may imagine a piano that infinitely extends in both directions, to the left and right. It will be the piano with countably many keys, which are in one-to-one correspondence with the set **Z** of integer numbers. Let's take for certain that **DO** of the primary octave corresponds to zero, then all the **DO**s correspond to a subgroup of integers, say **C**, which is **C** = 12 **Z,** furthermore, according to the above notation, **C#**=1+12 **Z, D** =2 + 12 **Z , … , B** = 11 + 12 **Z** are congruence classes modulo 12. Since each note of the primary octave can be determined by its distance (i.e. interval) from the note **DO** of the same octave, the above group **I** of all intervals is isomorphic to the group of the classes **C, C#, D, D#, E, F, F#, G, G#, A, A#, B** (Fuchs 1973) or, in other notation, **C, D ь, D, Eь E, F, G ь, G, A ь, A, B ь, B**, which form the cyclic group of order 12, traditionally denoted by **Z /12 Z,** with respect to the addition of the classes. We call the chain (Fuchs 1973) *the proper chain* of the main octave because its elements are arranged in the same order as those at the piano keyboard. The classical group theory states that the cyclic group **Z/12 Z** has

different generators **K**=k+12 **Z** for integers k relatively prime with 12, that is k=1, 5, 7, 11.

It is known that a melody performance (in particular singing a song) can start from any note which determines the key note, or, simpler, the *key* (i. e. the main note, or, in other words, tonality) that completes the melody performance in most cases similar to a full stop at the end of a sentence pronounced. From now on we use the word "key" in different sense, as tonality and also, as a note of the piano keyboard.

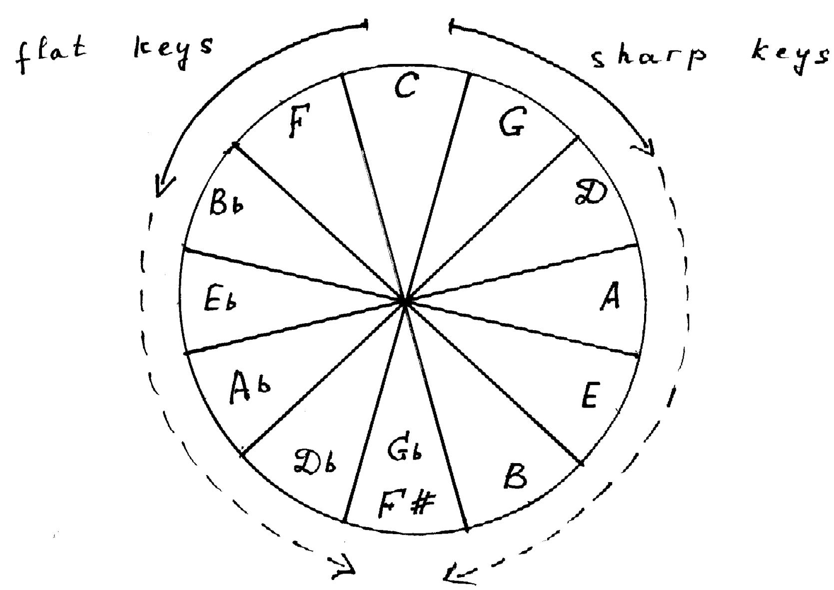
If we list the notes (Chugunov 1985) at the piano keyboard from left to right starting from any of them, not necessarily **C,** the ordinal number of each is called its position number with respect to the first one, which has the 1st position. For instance, **G** is the 5th position with respect to **C** and the 3d position with respect to **E**. The interval between notes **i** and **j** can be denoted by the ordered pair **(i**, **j)** according to their positions at the piano keyboard (**i** is supposed to be of a lower pitch than **j**). It is easy to see that **(j**, **i)=–(i**, **j)** in **I,** because the rule **(*)** implies **(j**, **i) +(i**, **j)=octave** (Beer).

In general, the choice of the key note is a principal step in a music performance and determines the tonality in which a music piece sounds. Normally we consider the note position with respect to the key note, which can be supplied with flat or sharp.

The traditional circle of the flat and sharp keys, the so-called *circle of fifths*, is the nicest ingredient of the classical music theory, which makes dealing with it simple, logical and pleasant for musicians and composers. However, the key circle construction has a strict group theory explanation based on the group isomorphism between **I** and **Z/12 Z**. In fact, the *key circle* determines a special order in which all the keys considered as elements of **Z/12 Z,** are listed and each of them is taken only once. If we take any generator **K**=k+12 **Z** of **Z/12 Z** then the list of all its multiples **sK** with **s**=0, 1, ...11, coincides with the group

Z/12 Z in the set theory sense. Denote by **e** the generator of **Z/12 Z** defined by k=1, which corresponds to the **minor second,** i. e. **semitone,** in **I**. Then 7**e** and 5**e** will be also generators of **Z/12 Z**. If we take the generator **e**, then the chain {**se:s**=0, 1,...11} coincides with the above *proper chain* **(2)**. Let's take the generator 7**e,** then the set {**7se:s**=0, 1,...11} lists all the elements of **Z/12 Z** in a different order, which is accepted in music as the *key chain.* If we allow **s**=0,1,...11 and also **s**=12 we get the *key circle* because the beginning and the end of the chain coincide with **C** and correspond to the note **DO** (as 12 = 0 mod 12). So, we have the following key circle in the clockwise succession starting from **C**.

picture 1.



It is called the circle of *fifths* because the distance of 7**e** corresponds to the interval between the 1st and 5th positions and is equal to the **perfect fifth**. Furthermore, the equality 7**e**+5**e**=12**e** implies that **–** 7**e**=5**e** in **Z/12 Z**. Then the passing of the key circle in the opposite direction, i. e. counter-clockwise begun from **C**, according to (Beer), corresponds to the choice of the generator that it equal to **–** 7**e**, or 5**e** in a different form. It is called the circle of *fourths* because the distance of 5**e** corresponds to the interval between the 1^{st} and 4^{th} positions and is equal to the **perfect fourth**. Note that on the 6^{th} step from the beginning **C** these clockwise and counter-clockwise circles meet at the same key **F#=G ь** because (7**e**) 6=42**e,** (5**e**) 6=30 **e** and 42–30=12=0 mod 12. Traditionally, the list of seven successive keys starting with **C** of the clockwise circle, form the chain of the sharp keys, **C, G, D, A, E, B, F#,** and, analogously, the seven keys of the counter-clockwise circle form the chain of the flat keys, **C, F, B ь, E ь, A ь, D ь, G ь**. These chains have intersections only in the first and last elements and together give the list of all 14 – 2 = 12 major keys. The minor keys can be considered in a parallel connection with the major ones and are situated **minor third** lower than the corresponding major keys. So, all the above can be extended to the circles of the minor keys which start with the note **A**. We see that the elegant tonality construction which is automatically used by composers and musicians as a technical tool has a strict group theory structure.

Music pieces as elements of an algebraic structure

A music phrase can be considered as a sequence of sounds characterized by their relative pitches and durations. Any music piece has its own time duration. In this sense the time characteristic of a sound is very important. The music theory is based on the well-known rhythm arithmetic. According to the latter, the time of the performance is divided into equal **bars**, which contain a certain number of beats of a certain duration. In music notation it is determined by a fraction, for instance, $\frac{6}{8}$ means that there are six bits of $\frac{1}{8}$ duration of a standard time interval in each bar, $\frac{4}{4}$ orresponds to the existence of four quarter beats in a bar, i.e. the whole measure denoted by **C**.

If we fix **d**, the smallest note length (i. e. sound duration) that takes place in music pieces considered, we may form an additive semi-group **T** generated by **d**, of time intervals which are multiples of **d**. The additive operation of summing intervals is commutative and will be denoted by (+).

Let **N** be the set of all notes of the piano keyboard. Assume that zero-note **0,** which means no note, is also an element of **N** and plays the same role as empty set considered as a subset of any set. We may define the addition (+) for a finite number of notes assuming that the sum **m+n** means that the notes **m** and **n** sound simultaneously and generalizing this to finitely many notes. So, the note sum is a chord and we see that the note addition is defined as an associative and commutative operation, i. e. it does not depend on the order of summands. Let **M** be a semi-group of all chords, then **M** includes all elements of **N** since a separate note is a particular case of chords. More precisely, **M** is generated by the set **N**.

Furthermore, for any **a**$\in$**M** and **t**$\in$**T** let's denote by **a+t** the "elementary piece" which coincides with the sound of the chord **a** lasting for the time **t**. If **a=0** then **t+a** means a pause of duration **t**. Construct a semi-group of elementary pieces as the direct sum **M**$\oplus$**T** which is also a semi-group with respect to the component-wise addition.

For any two elements **x** and **y** from **M**$\oplus$**T** determine their product **x*** **y** as the sequence **(x, y)** and extend this associative but not commutative operation to any finitely many elements of **M**$\oplus$**T**. The essence of this second operation (*) is very simple and means that the elementary piece **y** sounds just after **x**. Let **P** be the set of all possible pieces, i. e. finite consequences. Clearly, **P** contains **M**$\oplus$**T** because the elementary pieces coincide with one element sequences. Extend the addition from **M**$\oplus$**T** to all elements of **P** setting the component-wise addition of sequences in accordance with the ordinal numbers of the components, elements of **M**$\oplus$**T** (we keep in mind that the summands are not necessarily of the same length as sequences).

We see that **P** is closed under the two associative operations, addition (**+**) and multiplication ($*$), with the addition being also commutative, and **P** can be considered as a set with two algebraic operations. Then any piece of music is represented as an element of **P**. The most important kind of sets with two operations is called a **ring** in the classical Algebra. The introduced set **P** does not satisfy the ring axioms because **P** cannot be a group with respect to the addition as it is necessary for a ring. This fact can be easily explained by the following. If we have a note sounding for some time, there is no other note pressed together with the first one on the piano keyboard so that the impression of the sound of both would be like nothing sounds. It means that for an element **p** of **P** we can not find an element **–p** with the property **p+(–p)=O** if **O** is naturally supposed to be a symbolic silence. However, **P** is a semi-group with respect to each of the operation (**+**) and ($*$). In this connection we mention that the set of all positive integers Z_+ is also a semi-group with respect to the regular addition and multiplication of numbers.

We discussed the algebraic structure of a music piece and did not take care of its harmony. So, masterpieces on the one hand and very disharmonic pieces on the other hand can be represented by the elements of **P**. In this sense some elements of **P** sound nice, while the others cannot be called musical pieces if we suppose that music should not be a chaos of sounds. This leads us to emotional perception of elements of algebraic structures, and this view is accessible for people with very outstanding abilities. It is known that for some of them who are very good at mental arithmetic, the integer numbers, elements of Z_+**,** are differently coloured. It makes the arithmetic operations with the numbers visible by their inner sight. Such people demonstrate wonderful abilities operating with very big numbers. Our construction, perhaps, reveals the possibility of not seeing but hearing the "melody of algebra", which sometimes sounds in a soul of a mathematician even if this connection is not quite clear for him/her. I still remember Professor Faddeev's words that if an algebraic hypothesis looks elegant, there are many chances that it is true and can be proved. May be in some cases we should say that "it sounds nice" instead of "it looks elegant". Anyway, the search for beauty in Algebra quite often coincides with the search for the truth.

Melody and accompaniment in algebraic connection

Now we turn to harmonious music, i. e. music itself, and try to analyse some connections between the melody and its accompaniment. The music line is based on the row of the harmonic chords. A very simple chord row is an alternation of the classical chords constructed on the first, fourth and fifth positions with respect to the first one, that is the key note (tonality). In general the music row consists of numerous chords and their modifications, sometimes they are very special and fit a particular genre, for instance, jazz (Chugunov 1985). Recall that the final chord in most cases is the chord of the first position if the music piece (phrase) has a complete narrative intonation. So, a certain chord dominates in each small part of the whole music piece. Being an accompaniment, this chord, in fact, determines the melody line, whereas the melody, as it may be explained to beginners, like a queen from a fairy tale, dictates 'who' (or which chord, i. e. group of notes-servants) should accompany 'her' during a certain time period. The change of a dominating chord (harmony move) is associated with the change of the queen suite, who serve the queen-melody and at the same time protect her as bodyguards, so that she cannot make a step away from them. Then the queen is able to step only in a certain direction and her moves are restricted by her suite.

Let's return to the group of all note classes with respect to the addition, see (Fuchs 1970). It was shown that this group is isomorphic to the group **Z/12 Z**. Denote the additive semi-group of all possible chords, consisting of these 12 elements, by **L**. It is the semi-group generated by the set (Fuchs 1970), and the sum of its elements means the simultaneous sound of any notes from the corresponding classes.

A dominating chord determines a sub-semi-group, say **K**, generated by its notes. Then the melody notes, say **v,** can be considered as elements of the quotient **L/K** in the sense that they sound together with elements of **K** in a harmonic alliance. Since the simultaneous sound means the sum, we get **v + K**, a congruence class modulo **K**, and one more additional note **w** from **K** does not change the harmony, which corresponds to the equality **v + K=v+w+K** of the elements in **L/ K** if **w**$\in$**K**. The factor-semi-group remains until another chord comes up to replace the previous dominating chord. Summarizing this we suggest an algebraic model of music constructions in which a sequence of harmonies is a sequence of sub-semi-groups of **L** and melody elements belong to the corresponding factor-semi-group in proper time. The model also reflects the fact that a melody transposition to any octave is an admissible trick, which makes the music performance more subtle but does not really change it.

We logically complete this Section quoting H. E. Huntley: "The syntax and the grammar of the language of music are not capricious; they are dictated by the texture and organization of the deep levels of mind, so with mathematics".

Music in Algebra (one example)

In comparison with the previous section this one is connected with a very special area of Abelian Group Theory and can be difficult and, perhaps, not interesting for a reader who has never dealt with Algebra of such kind. The links between Music and direct decompositions of some groups I am describing now became evident for me in the process of solution of two problems stated by Professor Laszlo Fuchs, one of the brightest mathematicians among our contemporaries. I feel that the music interpretation of direct decompositions of *almost completely decomposable groups* described here helped me to cope with the complicated combinatorial construction which led to the successful solution of the above problems (Fuchs 1970: Problems 67, 68).

For short and to be kind to readers, I am demonstrating the presence of music in the direct decomposition theory by consideration of only one example. Let A be a completely decomposable group of rank 10 which is a direct sum of 5 homogeneous components of pair-wise incomparable types, each of rank 2, that is

> $A = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5$ where group $A_i = \langle a_i \rangle_* \oplus \langle b_i \rangle_*$ is a direct sum of its pure rank-one subgroups $\langle a_i \rangle_*$ and $\langle b_i \rangle_*$ of the same type, i =1, 2, 3, 4, 5.

Consider a group X generated by A and certain additional elements b_1, b_2, b_3, b_4 so that

$$p_1 b_1 \in A_1 \oplus A_2, \quad p_2 b_2 \in A_2 \oplus A_4, \quad p_3 b_3 \in A_3 \oplus A_5, \quad p_4 b_4 \in A_2 \oplus A_3$$

for some distinct integers p_1, p_2, p_3, p_4 which don't divide any of the elements $\{a_i, b_i : i$ =1, 2, 3, 4, 5$\}$ in A.
According to the Abelian Group Theory, group X belongs to the class of almost completely decomposable groups and admit non-

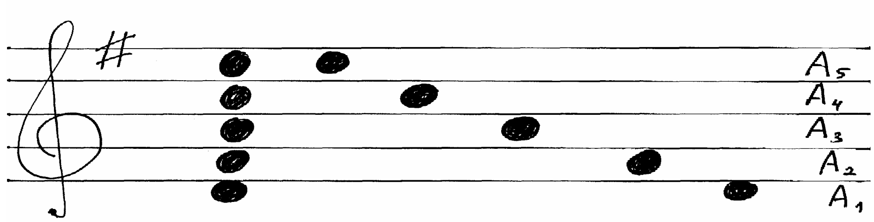
isomorphic direct decompositions. The combinatorial approach developed shows, in particular, that X has different decompositions into indecomposable summands, whose ranks give the following decompositions of the rank of group X,

10 = 5+1+1+1+1+1 = 3+3+1+1+1+1 = 2+3+2+1+1+1.

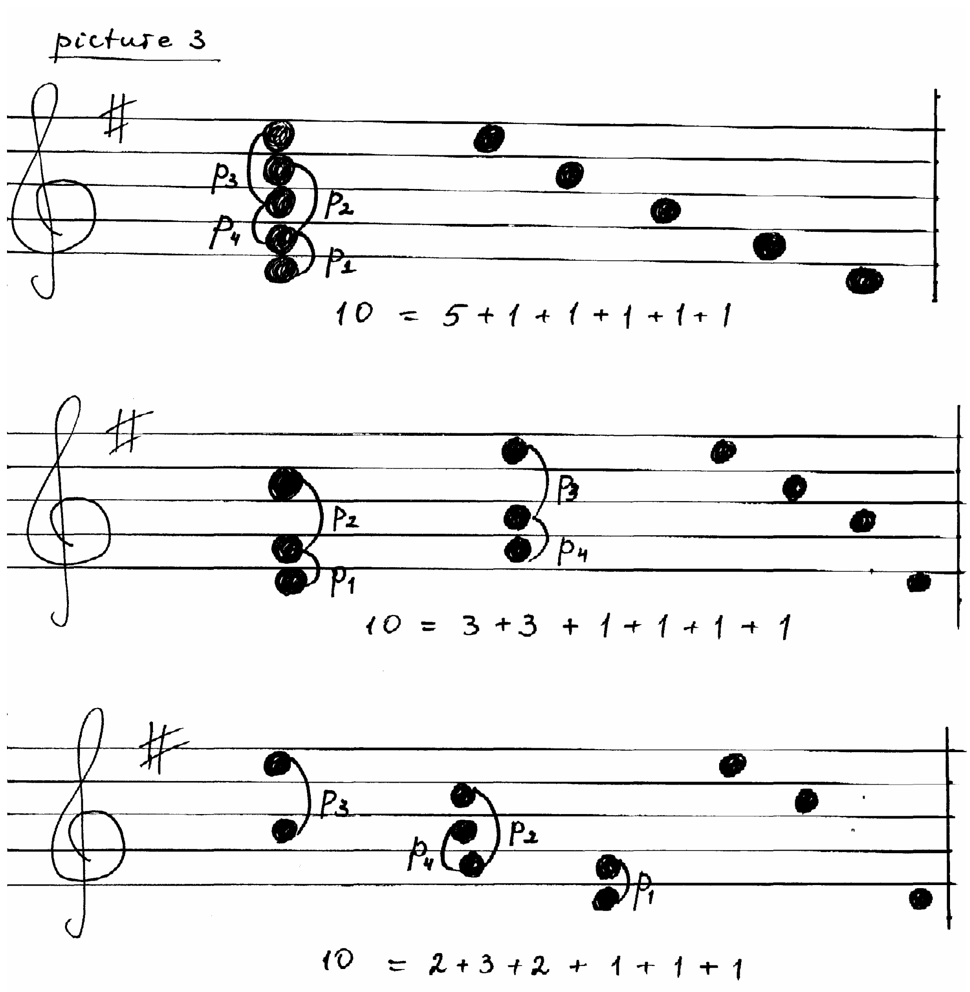
The first decomposition of the number 10 corresponds to the so-called *main decomposition of* X that contains exactly one indecomposable summand of the maximum possible rank and the others of rank 1.

Now we are ready to construct a link between decompositions of X and some music pieces. Any direct decompositions of the homogeneous group A_i are isomorphic and have the form $\langle a_i \rangle_* \oplus \langle b_i \rangle_*$ but the elements a_i and b_i can't be determined uniquely. Different special choices of such elements lead to different decompositions of group X into indecomposable summands. We denote any pair of elements a_i and b_i from a decomposition of A_i by a pair of notes of the same pitch, while the elements of different homogeneous components A_i and A_j with $i \neq j$ are represented by notes of different pitches. In this way we get representations of all groups A_i on the note staff.

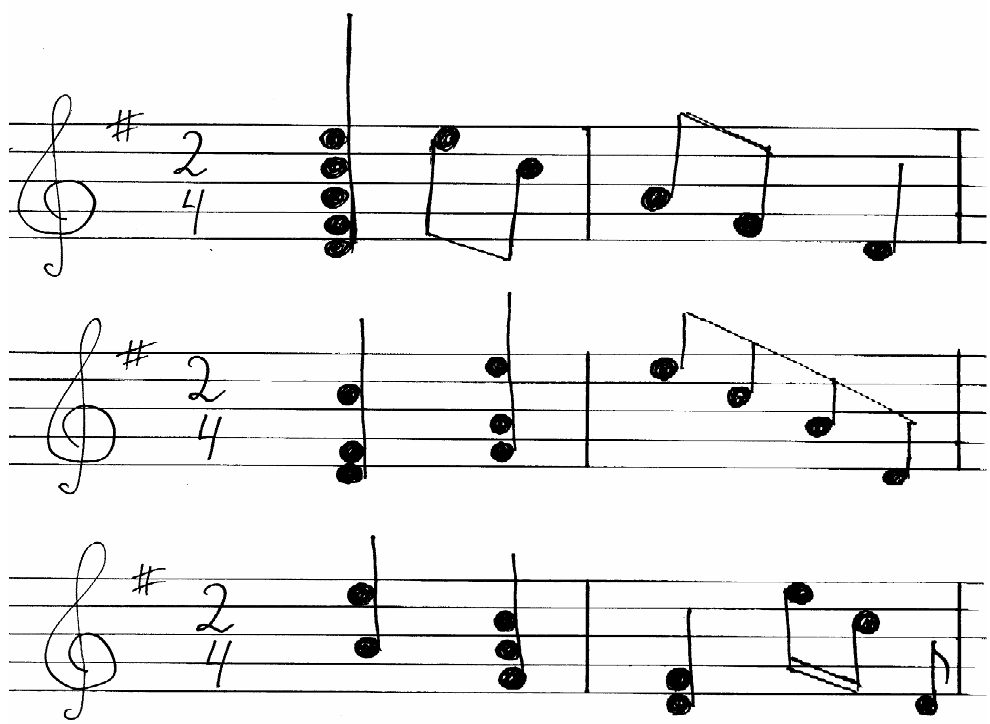
picture 2



If there exists an element from $\langle a_i \rangle_* \oplus \langle a_j \rangle_*$, $i \neq j$, which is divisible by one of the primes p_1, p_2, p_3, p_4 in X but not in A, we make the notes corresponding to a_i and a_j sound simultaneously as a chord. In our construction a chord represents indecomposable summand of the rank equal to the number of its notes. In particular, a separate note corresponds to a rank-one summand. It was proved that different choices of generating systems in groups A_i, $i=1, 2, 3, 4, 5$ lead to the following interpretations of X,



Then the above decompositions of group X "can be heard" as follows, picture 4



I am finishing the consideration of some facts which prove the profound music-mathematics connections quoting Igor Stravinsky: “Musical form is close to mathematics – not perhaps to mathematics itself, but certainly to something like mathematical thinking and relationship”. It is not possible to discuss all the links between music and algebra in detail in one article. Some of them are left out of this consideration. I believe that this subject needs a comprehensive study and the more steps have been made in this direction, the more interesting it seems to be. I am greatly thankful to Professor Doris Janshen who initiated this research and encouraged me to start this interesting work. I always felt great help from all women of her group, especially, I should thank Dr. Gudrun Schäfer for her constant warm attention.

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