Knuth's 0-1-Principle

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May 19th, 2020

The Sorting Problem

Task: Given a list and an order on the type of elements of this list, produce a sorted list (with same content)!

Example:

12	7	9	8	4	6

$\mapsto$

4	6	7	8	9	12

Many Solutions:

- Quicksort
- Insertion Sort
- Merge Sort
- Bubble Sort
- ...

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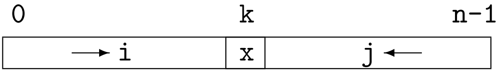

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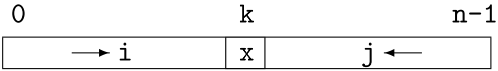

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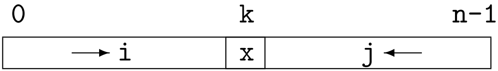

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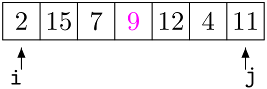

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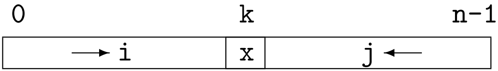

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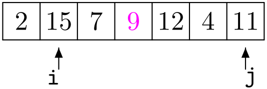

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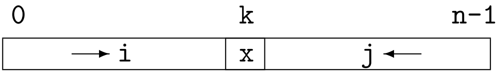

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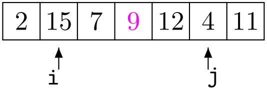

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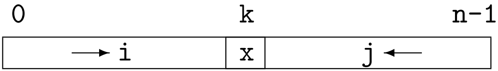

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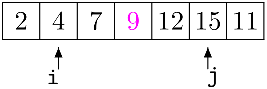

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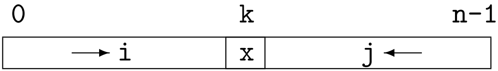

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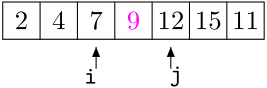

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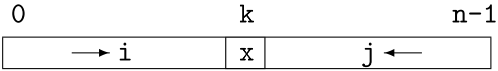

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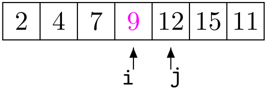

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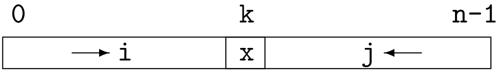

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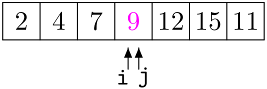

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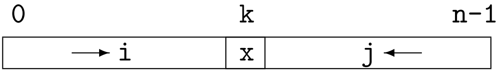

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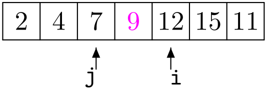

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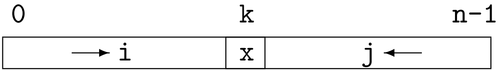

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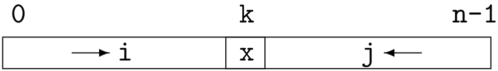

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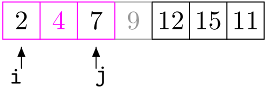

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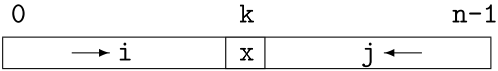

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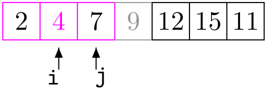

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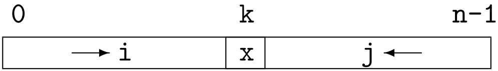

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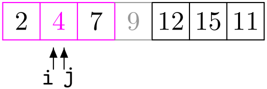

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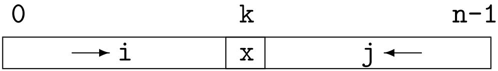

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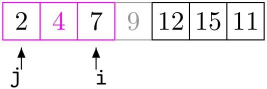

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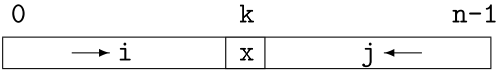

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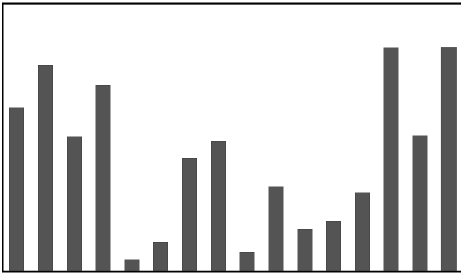

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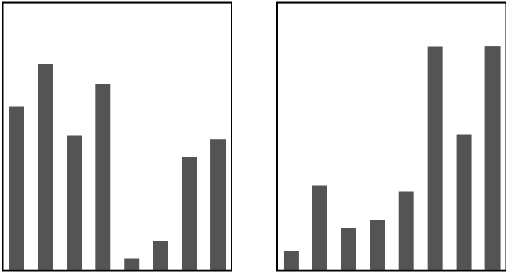

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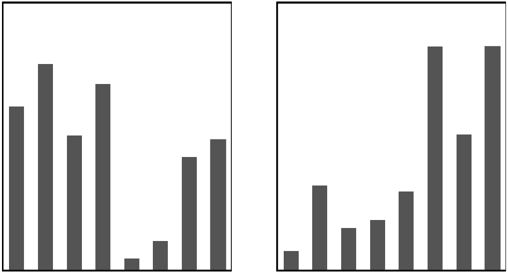

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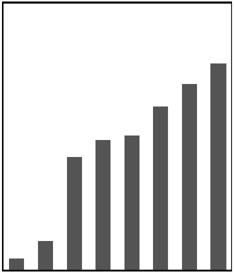

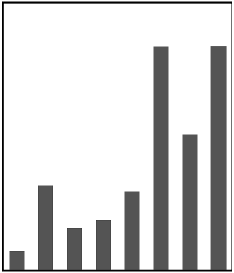

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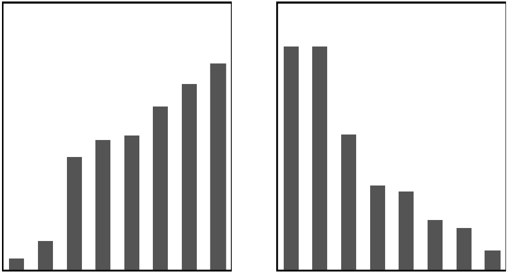

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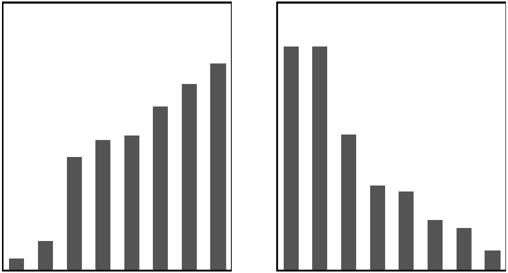

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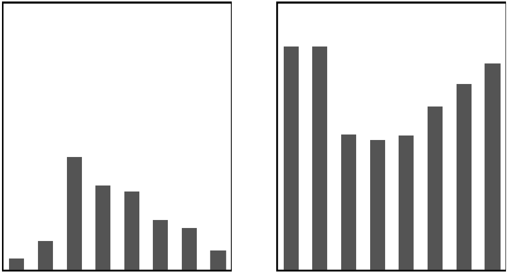

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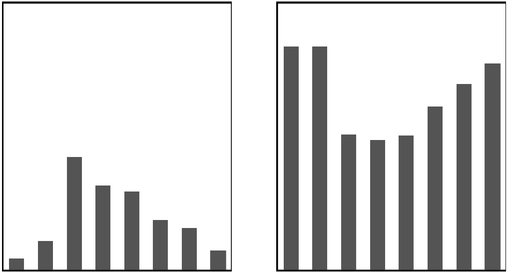

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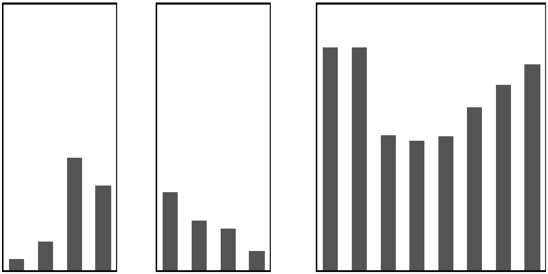

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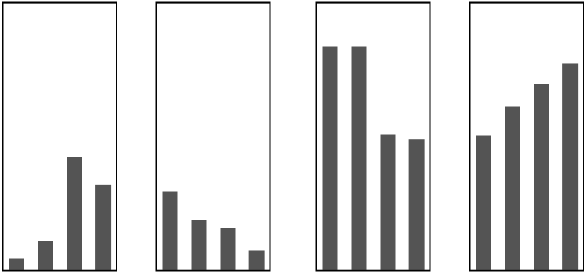

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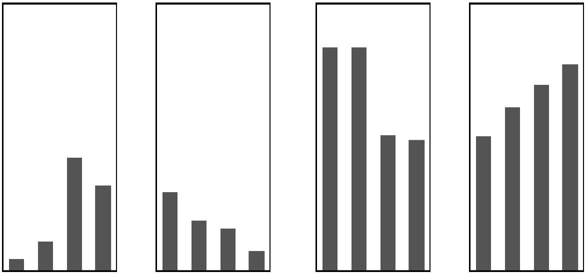

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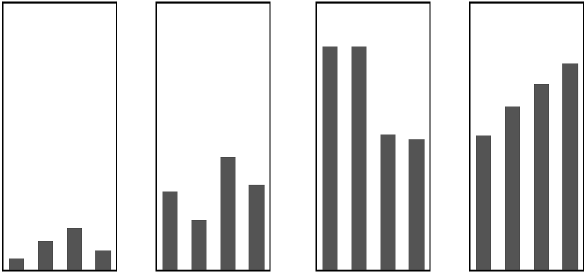

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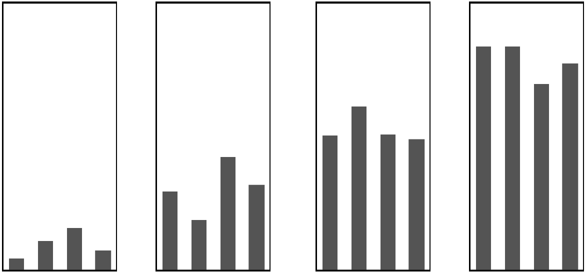

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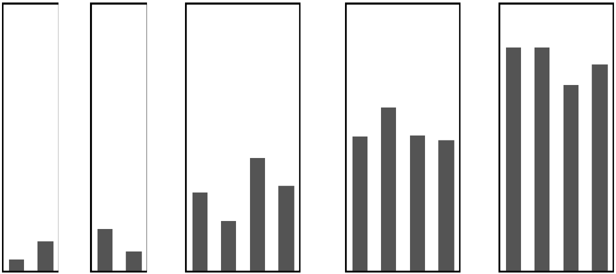

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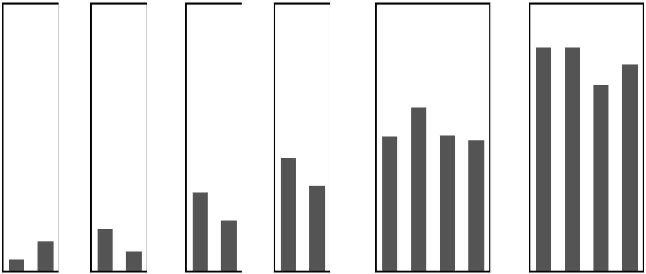

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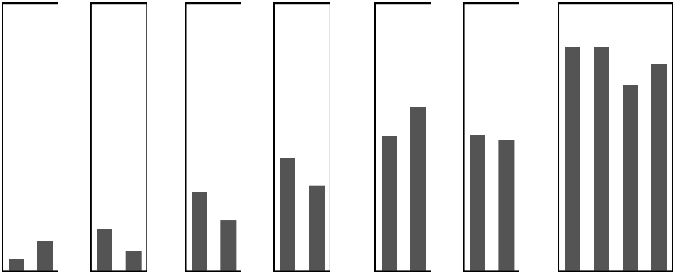

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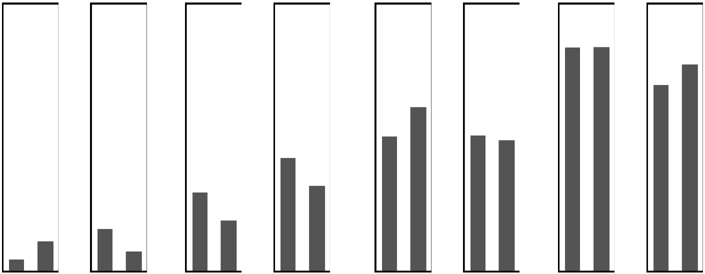

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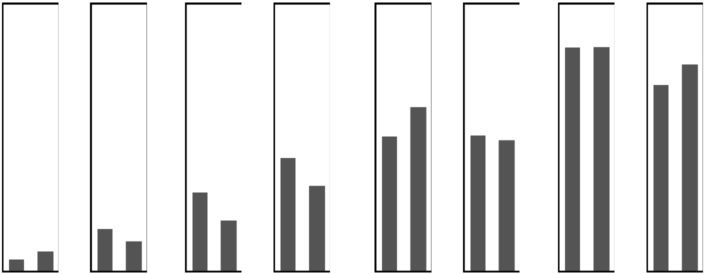

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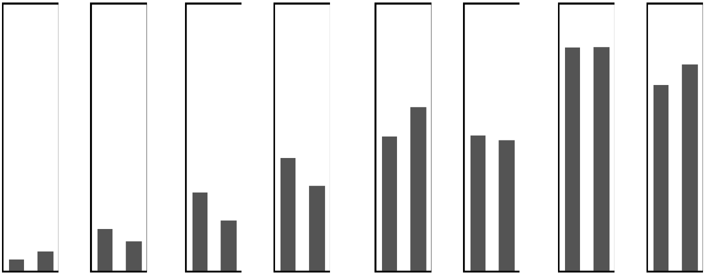

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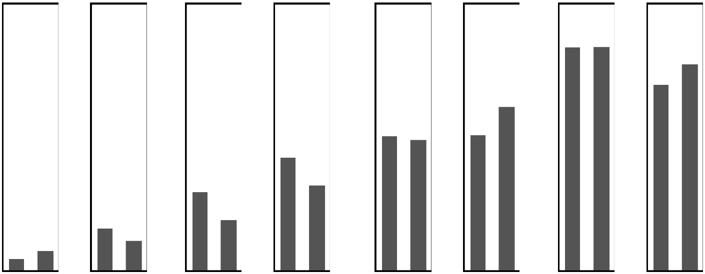

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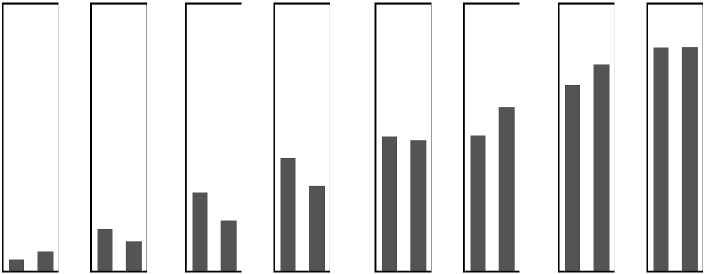

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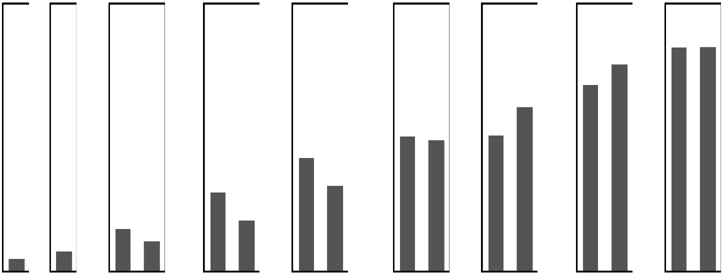

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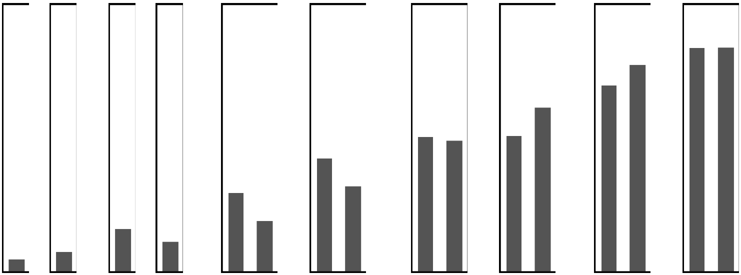

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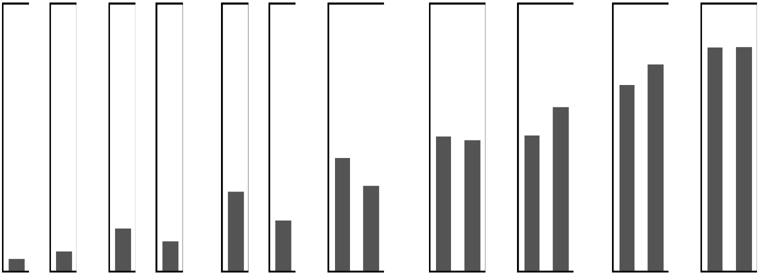

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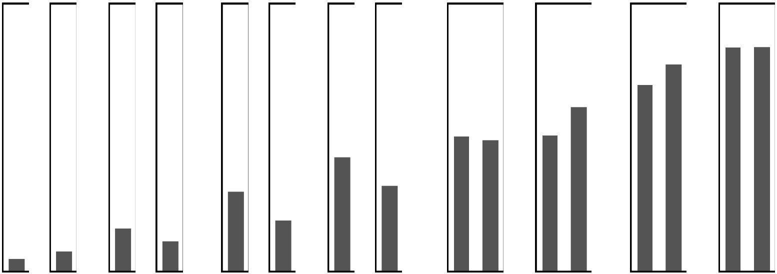

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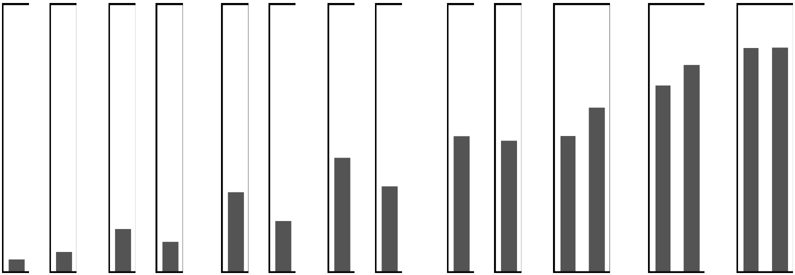

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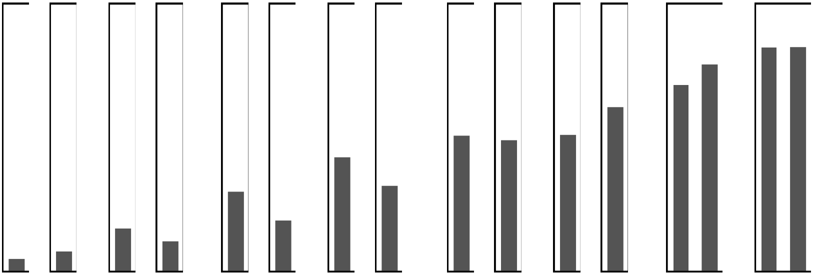

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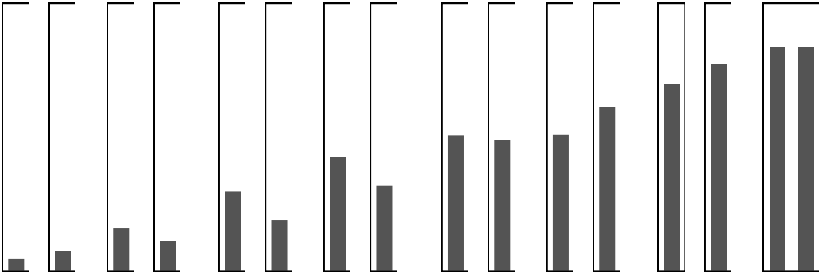

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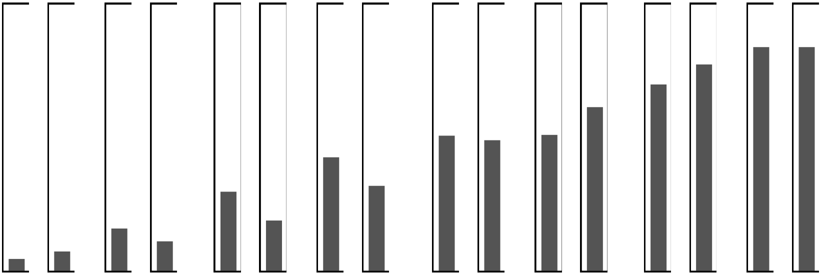

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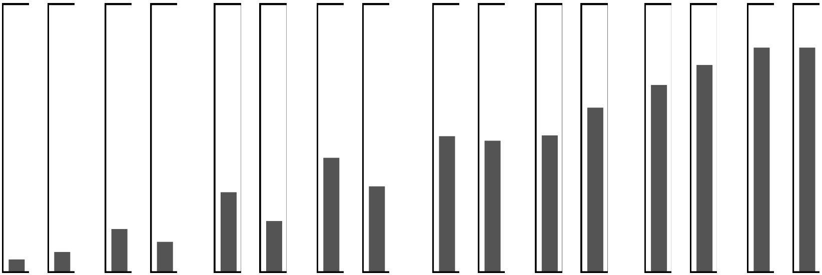

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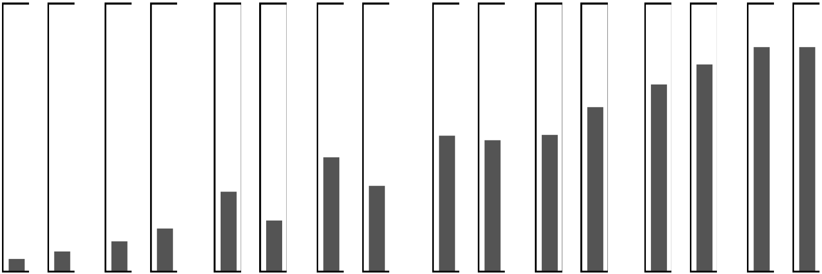

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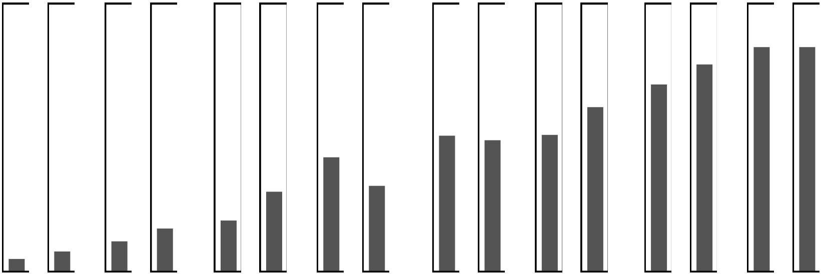

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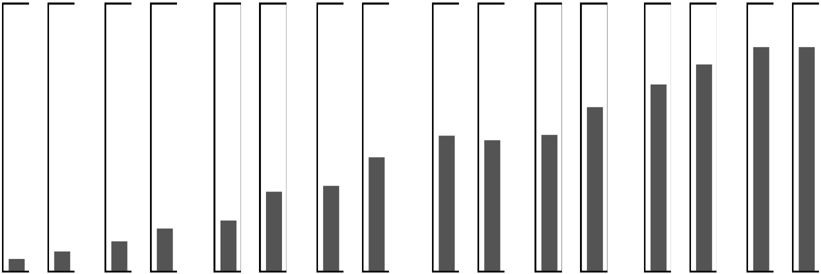

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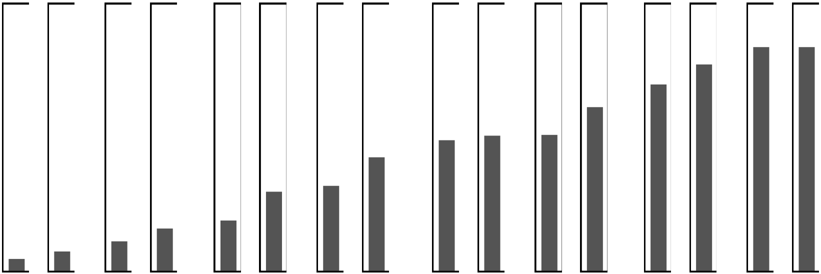

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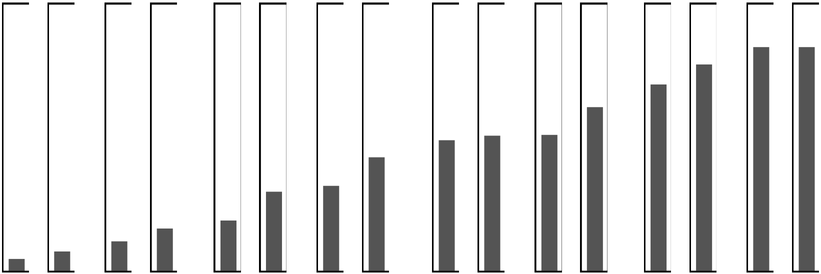

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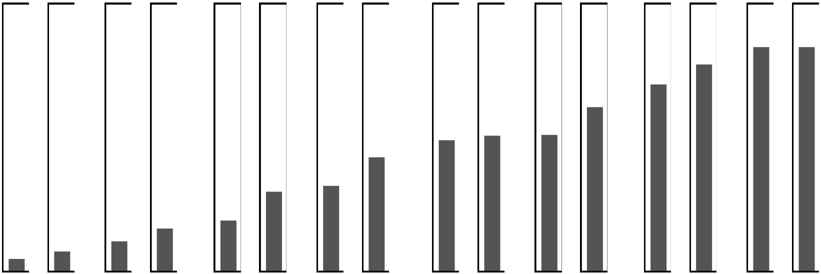

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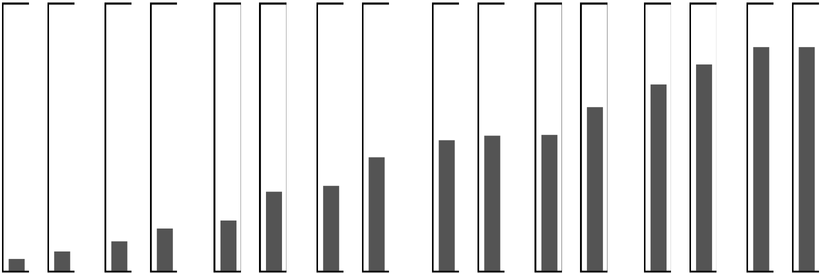

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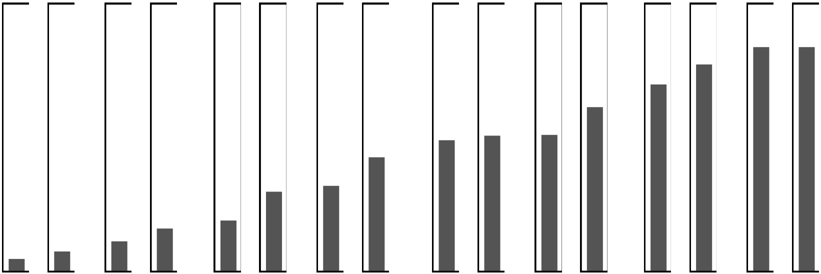

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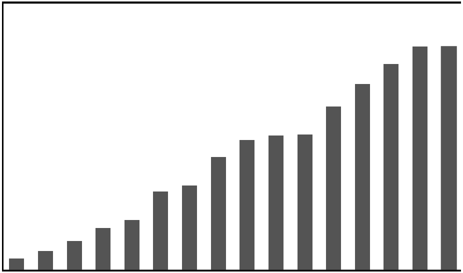

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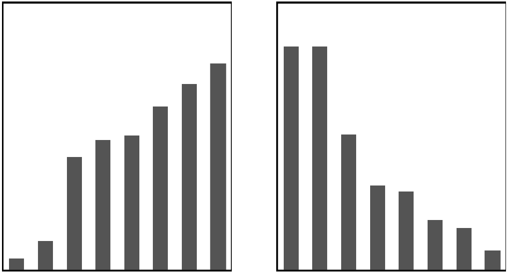

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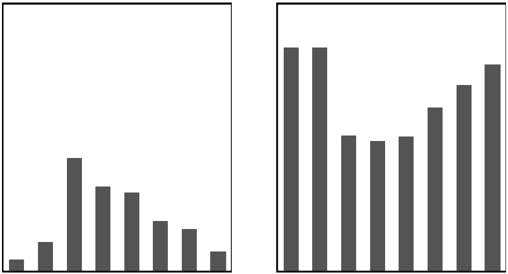

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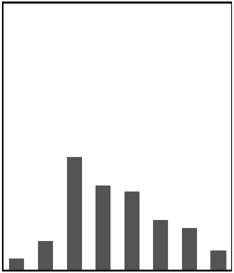

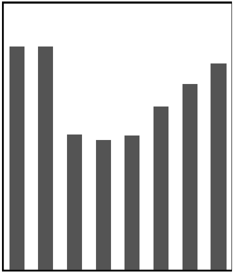

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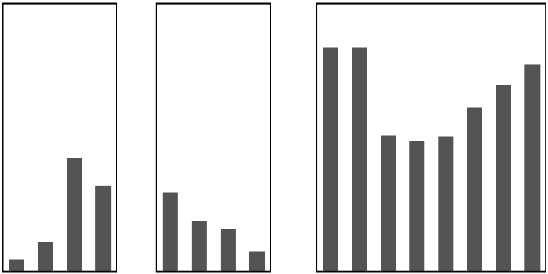

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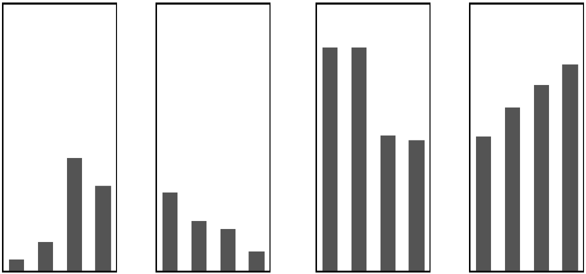

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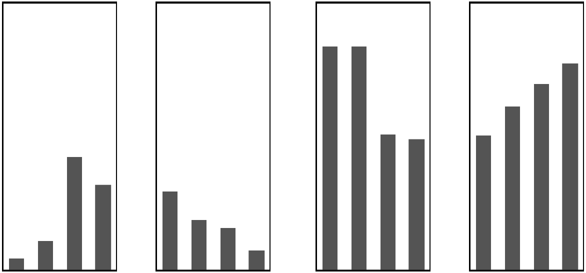

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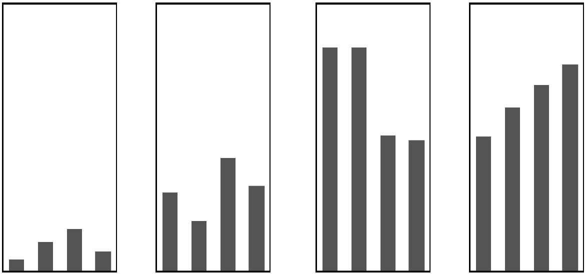

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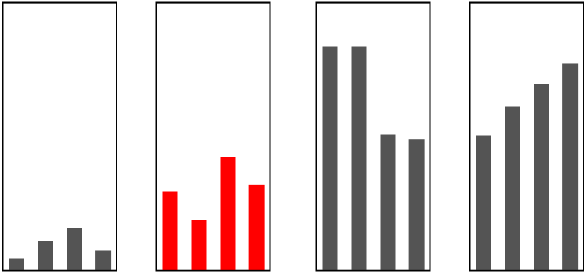

Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

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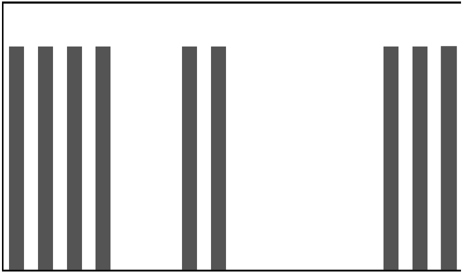

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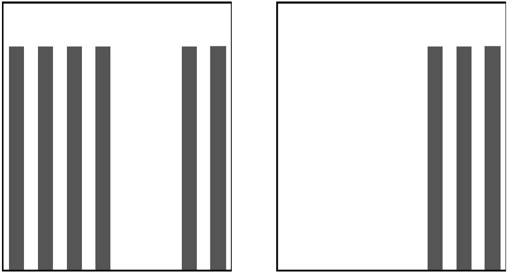

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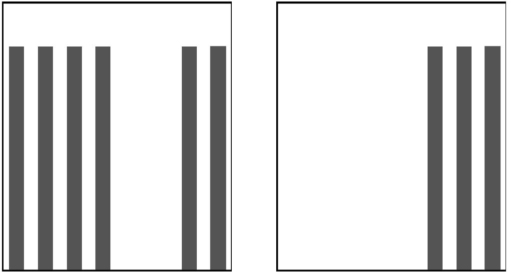

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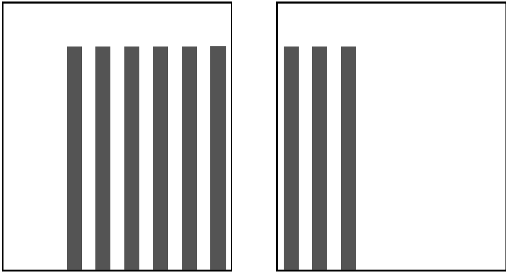

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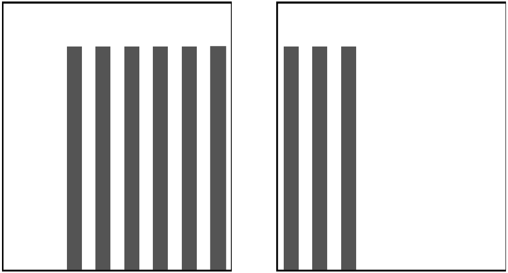

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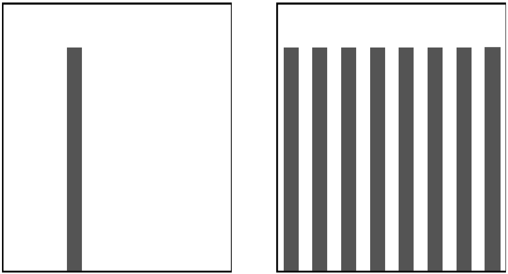

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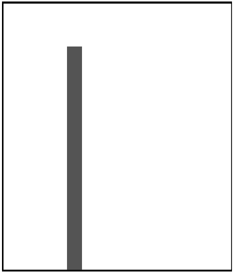

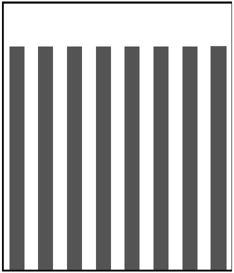

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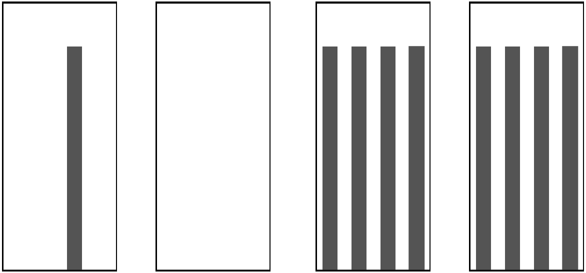

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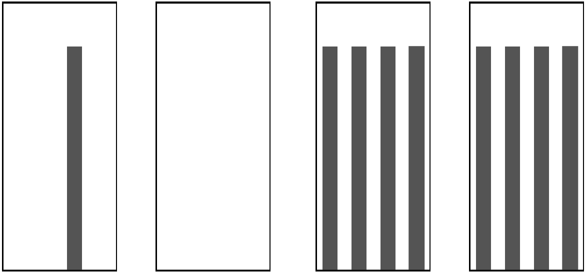

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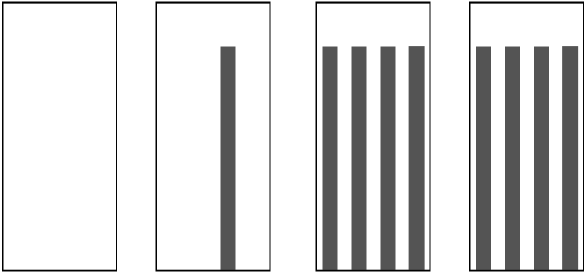

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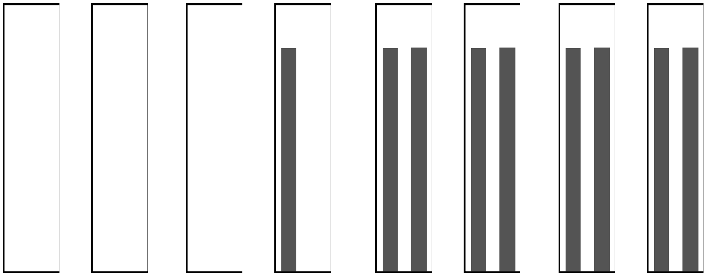

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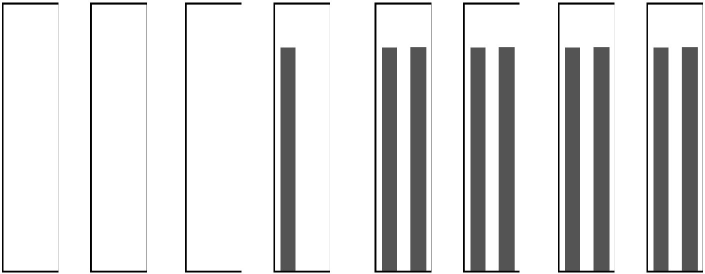

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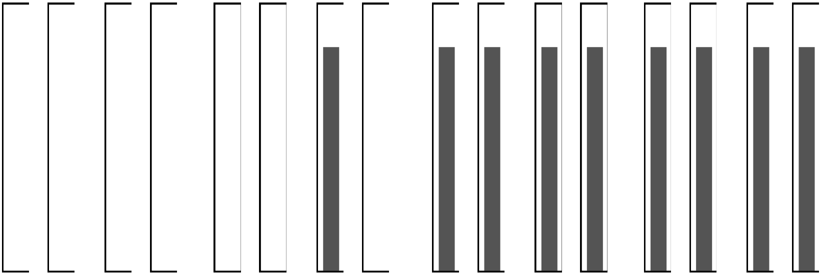

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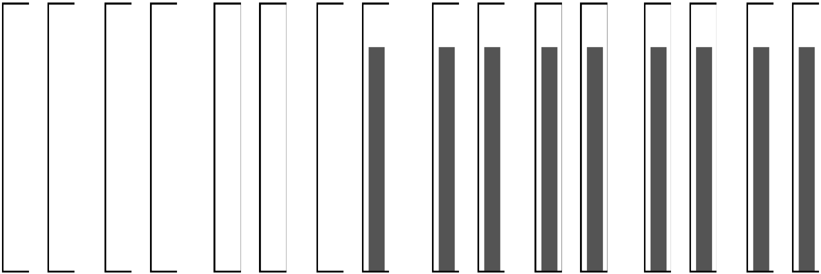

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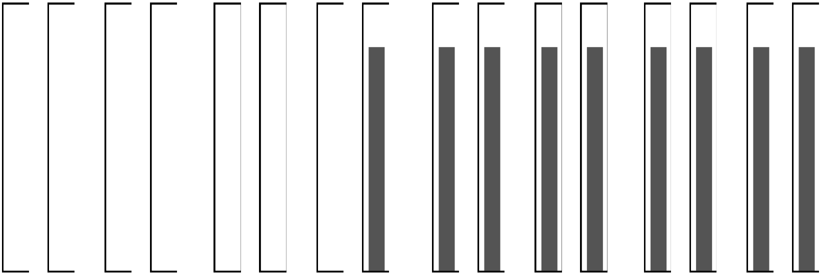

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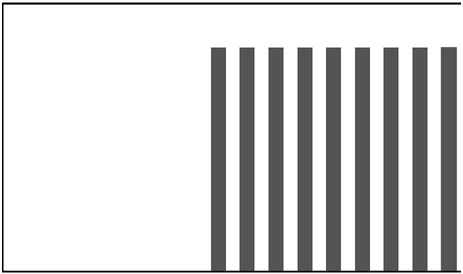

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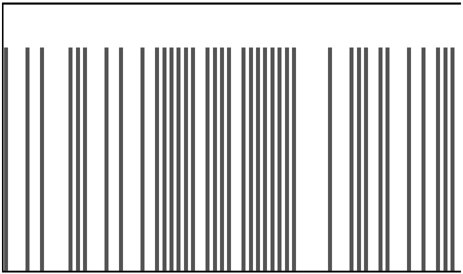

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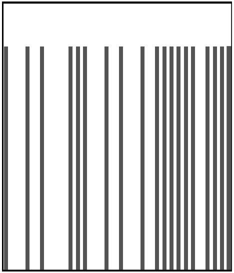

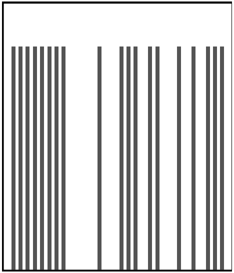

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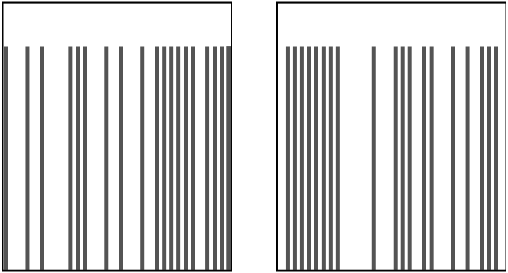

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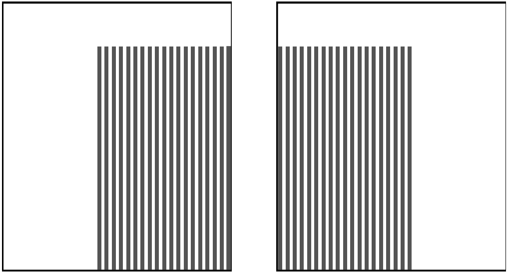

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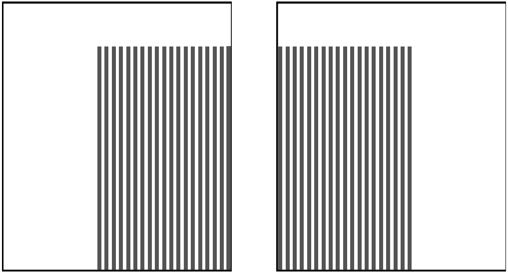

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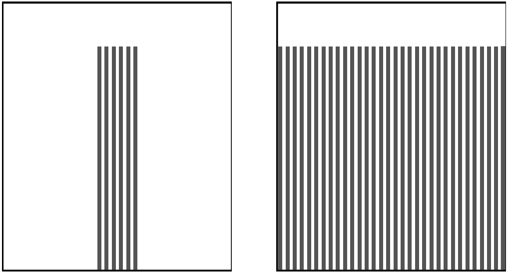

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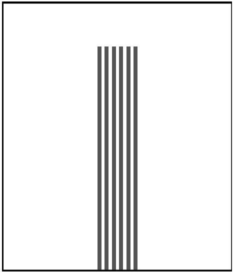

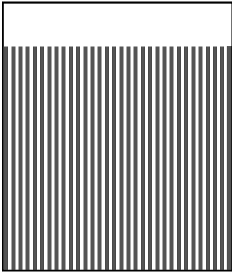

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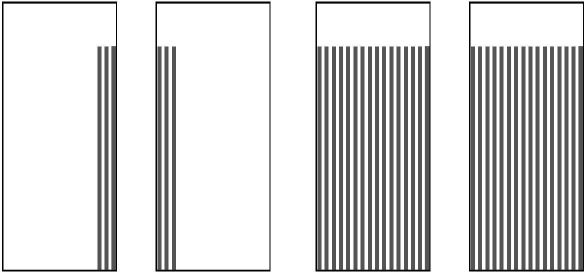

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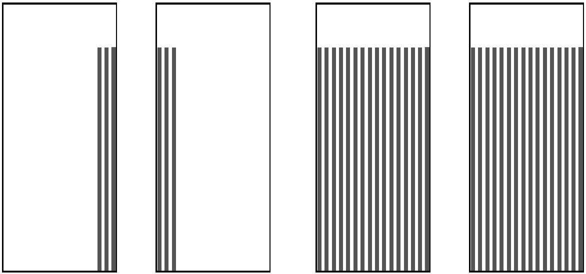

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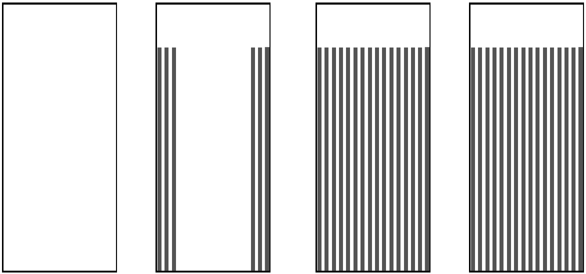

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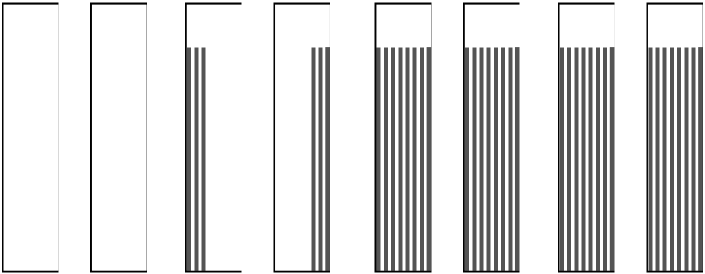

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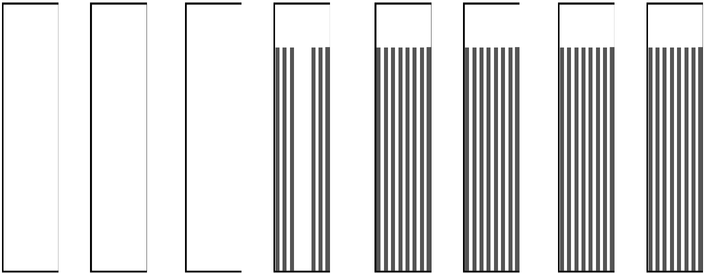

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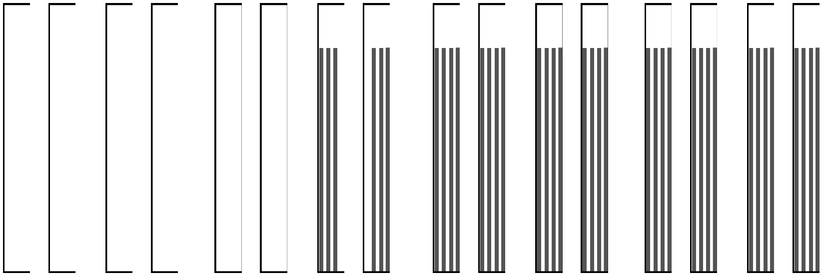

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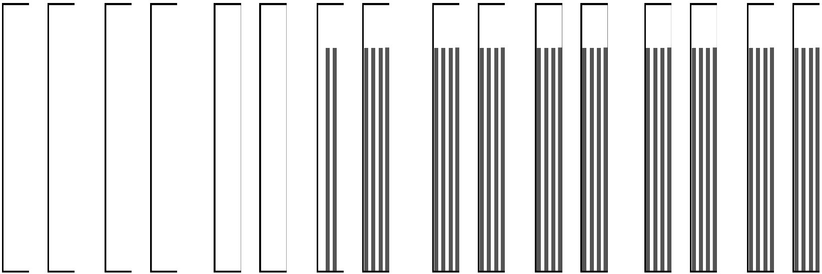

Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

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$$f :: (\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$$
$$f\ (x, y) = \mathit{if}\ x > y\ \mathit{then}\ (y, x)\ \mathit{else}\ (x, y)$$

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If for every xs :: [Bool], `sort` g xs gives the correct result, then for every xs :: [Int], `sort` f xs gives the correct result.

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Using a Free Theorems Generator

Input: `sort::((a,a)->(a,a))->[a]->[a]`

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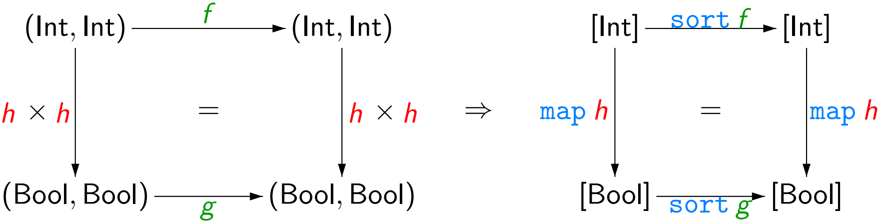
Output:

```
forall t1,t2 in TYPES, h::t1->t2.
 forall f::(t1,t1)->(t1,t1).
  forall g::(t2,t2)->(t2,t2).
   (forall (x,y) in lift_{(,)}(h,h).
     (f x,g y) in lift_{(,)}(h,h))
   ==> (forall xs::[t1].
         map h (sort f xs) = sort g (map h xs))

lift_{(,)}(h,h)
 = {((x1,x2),(y1,y2)) | (h x1 = y1)
                        && (h x2 = y2)}
```

More Specific (and Intuitive)

For every `sort` :: $((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$,
f :: (Int, Int) $\to$ (Int, Int), g :: (Bool, Bool) $\to$ (Bool, Bool), and
h :: Int $\to$ Bool:



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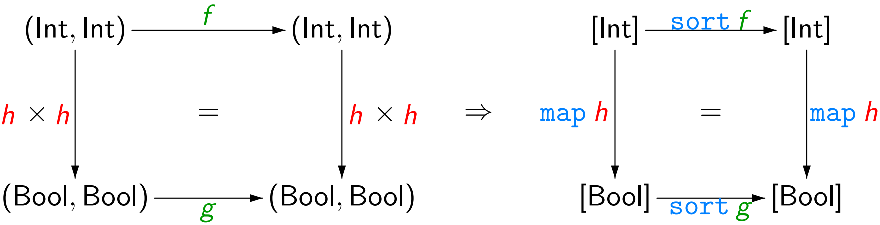

If f and g are as defined before, then the precondition is fulfilled for any h of the form $h\ x = n < x$ for some n :: Int.

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Proof of "P on [Bool] implies P on [Int]"

Recall: $P(xs, ys) :=$ *xs* and *ys* contain the same elements in the same multiplicity

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$$xs = \texttt{map}\ h\ us$$
$$ys = \texttt{sort}\ g\ (\texttt{map}\ h\ us)$$

Proof of "P on [Bool] implies P on [Int]"

Recall: $P(xs, ys) :=$ xs and ys contain the same elements in the same multiplicity

Given: $\forall xs :: [\text{Bool}], ys = \texttt{sort}\ g\ xs.\ P(xs, ys)$

To prove: $\forall xs :: [\text{Int}], ys = \texttt{sort}\ f\ xs.\ P(xs, ys)$

Assume there exist $us :: [\text{Int}]$ and $vs = \texttt{sort}\ f\ us$ with $\neg P(us, vs)$. Then there is a smallest integer n such that the multiplicities of n in us and vs are not the same. Then for $h\ x = n < x$ the multiplicities of False in ($\texttt{map}\ h\ us$) and ($\texttt{map}\ h\ vs$) are different. But this is in contradiction to the precondition with:

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Then there are $n < m$ such that an m occurs in vs before an n.

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Knuth's 0-1-Principle [Knuth 1973]

Informally: If a comparison-swap algorithm sorts Booleans correctly, it sorts integers correctly as well.

Formally: Use Haskell. Let

$$\texttt{sort} :: ((\alpha, \alpha) \to (\alpha, \alpha)) \to [\alpha] \to [\alpha]$$

$$f :: (\mathsf{Int}, \mathsf{Int}) \to (\mathsf{Int}, \mathsf{Int})$$
$$f\ (x, y) = \mathit{if}\ x > y\ \mathit{then}\ (y, x)\ \mathit{else}\ (x, y)$$

$$g :: (\mathsf{Bool}, \mathsf{Bool}) \to (\mathsf{Bool}, \mathsf{Bool})$$
$$g\ (x, y) = (x \mathbin{\&\&} y, x \mathbin{||} y)$$

If for every xs :: [Bool], $\texttt{sort}\ g\ xs$ gives the correct result, then for every xs :: [Int], $\texttt{sort}\ f\ xs$ gives the correct result.

And Beyond?

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- Free theorems allow for a particularly elegant proof of this principle. (This was not my idea: [Day et al. 1999]!)
- Can we do something similar for other algorithm classes?
- Good candidates: algorithms parametrised over some operation, like *cswap* $:: (\alpha, \alpha) \rightarrow (\alpha, \alpha)$ in the case of sorting.

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