DEFECT CORRECTION
FOR
CONVECTION DOMINATED FLOW

by

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Abstract. A defect correction scheme with the first order upwind preconditioner is considered. By Fourier analysis the preconditioning properties for the second order upwind scheme, the central scheme and spectral methods are examined. Since the eigenvalues of the preconditioned operator are complex the GMRES iteration is used for the iterative solution. This procedure is applied to the Boussinesq flow problem in vorticity–streamfunction formulation. Numerical results are presented for increasing Rayleigh numbers.

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1. INTRODUCTION

Here we consider convection–diffusion problems which can in its most general form be written as

$$\begin{aligned} -\epsilon\Delta u + cu_x + du_y &= f \ \text{ in } \Omega = (-1,1)^2, && (1.1)\\ u &= g \ \text{ on } \partial\Omega, && (1.2) \end{aligned}$$

where $\epsilon > 0$ denotes a constant, c, d, f are functions defined in Ω, and g is defined on $\partial\Omega$. Such problems arise after a linearization of the Navier-Stokes equations (or Boussinesq flow problem). Here ϵ correspondts to $\frac{1}{Re}$. The part $-\epsilon\Delta u$ denotes the diffusive part and $cu_x + du_y$ denotes the convective part of the above equation. Here we are mainly interested in convection dominated flows where $\epsilon << h$. Here h denotes the step size of the finite difference (FD) scheme. For the FD approximation of convection–diffusion problems one observes certain phenomenae of instability. For small ϵ standard discretizations can lead to a solution of the discrete problem which has nothing to do with the solution of the original problem. For instance, the discretization of

$$\begin{aligned} -\epsilon u'' \ - \ 2u' \ = \ 0 \ \ in \ [0,\infty)\,, \\ u(0) = 1, \ u(\infty) = 0 \end{aligned}$$

with central differences yields as discrete solution:

$$u_{\epsilon,h}(ih) \ = \ \left(\frac{\epsilon - h}{\epsilon + h}\right)^i .$$

This is an $O(h^2)$ discretization. For ih fixed and $\frac{h}{\epsilon} \ \to \ 0$ we obtain:

$$|(u_{\epsilon,h} - u)(ih)| = |\left(\frac{\epsilon - h}{\epsilon + h}\right)^i - \left(e^{-2h/\epsilon}\right)^i| \le C\left(\frac{h}{\epsilon}\right)^2$$

with C independent of i, h and ϵ. But the solution of the reduced equation is:

$$u_{0,h}(ih) = \lim_{\epsilon\to 0} u_{\epsilon,h}(ih) = (-1)^i$$

which means that for $\epsilon << h$ the solution of the FD problem has nothing to do with the exact solution

$$u(x) = e^{-2x/\epsilon}.$$

For this special problem we further observe a boundary layer in which the first derivative behaves as $O(\epsilon^{-1})$ for $\epsilon \to 0$. One possibility to avoid the phenomenon of instability is to use upstream discretization for u'. Clearly, an obvious disadvantage of this scheme lies in the fact that the method now becomes only first order accurate. Hence it makes sense to use the first order upstream scheme only as a preconditioner for a higher order scheme. We analyze the preconditioning properties of this method for the following higher order schemes:

- second order upstream scheme
- central finite difference scheme
- Chebyshev pseudospectral scheme.

In the spectral scheme (see [4]) the solution is approximated by Chebyshev polynomials of degree $\leq N$. This space is denoted by $\mathbf{P}_N$. By a Fourier analysis it can be shown that the eigenvalues of the preconditioned operator are bounded but complex. Hence one has to employ a nonsymmetric matrix iteration for the solution. Here we recommend the GMRES iteration which belongs to the residual minimization methods. Clearly, for the general convection–diffusion problem (1.1), (1.2) the first derivatives u_x and u_y have to be approximated according to the sign of the coefficients c and d, respectively. Therefore for the iterative solution we recommend flow directed schemes. Since the Chebyshev nodes are dense near the boundary it is necessary to use line Gauss–Seidel relaxation (in an alternating manner). Finally this iterative solver is applied to the Boussinesq flow problem in vorticity–streamfunction formulation. We obtained numerical results for increasing Rayleigh numbers up to $Ra = 10^5$.

2. PRECONDITIONING BY THE UPSTREAM SCHEME

From the one–dimensional model problem it can be seen that for $\epsilon \to 0$ we first have to find a good preconditioner for the derivative operator

$$\frac{du}{dx}.$$

Here we employ the first order upstream scheme (L^1_{up}) for preconditioning, i.e.

$$\frac{du}{dx}(x_i) \;\cong\; \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i} \quad if \quad x_{i+1} < x_i$$

or

$$\frac{du}{dx}(x_i) \;\cong\; \frac{u(x_i) - u(x_{i-1})}{x_i - x_{i-1}} \quad if \quad x_{i-1} < x_i.$$

In the spectral discretization we have collocation points $x_i = \cos\theta_i,\ \theta_i = \frac{i\pi}{N}$, so that $x_{i+1} < x_i$, whereas in the finite difference case we have grid points $x_i = ih,\ h = \frac{1}{N}$, so that $x_{i-1} < x_i$ for $i = 1, \ldots, N-1$. Now we were interested in the preconditioning properties of L^1_{up} for the following three higher order methods:

- second order upstream scheme: $L^2_{up}u \;:=\; \frac{1}{h}\left(\frac{1}{2}u(x_{i-2}) - 2u(x_{i-1}) + \frac{3}{2}u(x_i)\right)$
- second order central scheme: $L_{ce}u \;:=\; \frac{1}{2h}\left(u(x_{i+1}) - u(x_{i-1})\right)$
- Chebyshev pseudospectral scheme: L_{sp} (see [4], [12], [13]).

Hence we are interested in the eigenvalues of the discrete operators:

$$\left(L^1_{up}\right)^{-1} L^2_{up},\ \left(L^1_{up}\right)^{-1} L_{ce},\ \left(L^1_{up}\right)^{-1} L_{sp}.$$

Eigenvalue bounds are obtained by a Fourier analysis. It is well known (see [4]) that the Fourier analysis also yields a good prediction for the eigenvalues in the Chebyshev case. For $\left(L^1_{up}\right)^{-1} L^2_{up}$ the absolute value of the eigenvalues are given by

$$|\lambda^{up}_p| \;=\; \sqrt{4 - 3\cos^2\frac{p\pi}{2N}}\ ,\ p = 1, \ldots, N-1.$$

For $\left(L^1_{up}\right)^{-1} L_{ce}$ they are given by:

$$|\lambda_p^{ce}| \;=\; \cos\frac{p\pi}{2N},\;\; p = 1, \ldots, N-1.$$

Since $\cos\frac{p\pi}{2N} \in [0,1]$ we obtain

$$|\lambda_p^{up}| \in [1,2] \;\;\textit{and}\;\; |\lambda_p^{ce}| \in [0,1].$$

Because zero is the lower bound for the eigenvalues of the preconditioned central scheme it is already clear that this method is not good. For the second order upstream scheme we observed that the imaginary parts are small compared to the real parts. Hence a simple Richardson iteration can be applied. By choosing one relaxation parameter we obtain a convergence factor of $\frac{1}{3}$. Clearly, the convergence speed can be accellerated by using more parameters (nonstationary Richardson relaxation [11]). As shown in [11] the convergence factor ρ_k for k relaxations is here given by

$$\rho_k \;:=\; |T_k(3)|^{-\frac{1}{k}},$$

where T_k denotes the kth Chebyshev polynomial. In table I we present ρ_k for $k = 1, \ldots, 4$. We recommend to use $k = 3$ since the improvement for increasing k is no more significant.

k	ρ_k
1	0.3333
2	0.2425
3	0.2162
4	0.2040

Table I. ρ_k for $k = 1, \ldots, 4$

For the spectral Fourier operator (see [4, 5.2.2]) the eigenvalues of the preconditioned spectral operator are

$$\lambda_p^{sp} \;=\; \frac{p\pi/N}{\sin(p\pi/N)} e^{-i\frac{p\pi}{N}}, p = -\frac{N}{2}, \ldots, \frac{N}{2}$$

and hence

$$|\lambda_p^{sp}| \;=\; \frac{p\pi/N}{\sin(p\pi/N)},\; p = -\frac{N}{2}, \ldots, \frac{N}{2}.$$

This implies that

$$|\lambda_p^{sp}| \in [1, \frac{\pi}{2}] \cong [1, 1.57].$$

Therefore in all three cases the eigenvalues are bounded but complex. The iterative solver must be able to handle complex eigenvalues. Here we recommend the GMRES iteration (see [19], [20] and [21]) which belongs to the residual minimization methods. Consider the general linear system

$$Bv \;=\; g,$$

where B is a large nonsymmetric matrix. If v_0 is an initial approximation to the solution and $r_0 = g - Bv_0$, we define the mth Krylov subspace

$$K_m \;:=\; span\{r_0, Br_0, B^2 r_0, \ldots, B^{m-1} r_0\}.$$

Then the GMRES approximation v_m with

$$v_m \;\in\; v_0 + K_m \tag{2.1}$$

is determined such that the mth residual r_m fulfills

$$||r_m||_2 \;=\; \text{minimum}. \tag{2.2}$$

An equivalent statement is the orthogonality condition

$$r_m \;\perp\; BK_m. \tag{2.3}$$

The GMRES iteration is a robust implementation of (2.1)–(2.3) by means of an Arnoldi construction of an orthonormal basis for the Krylov space, which leads to an $(m+1) \times m$ Hessenberg least-squares solution [21].

3. STABILIZATION TECHNIQUES AND ITERATIVE SOLVER

Here we consider convection-diffusion problems which are in its most general form given by (1.1), (1.2). It is well known that these problems lead to instabilities for $\epsilon << h$ in the FD case (see [7], [8], [10]) and for $\epsilon << N^{-2}$ in the spectral case (see [2], [15]). Canuto [2] has shown that the spectral approximations are affected by spurious oscillations which deteriorate the spectral accuracy. For instance, for the one-dimensional model problem

$$\begin{aligned} -\epsilon u'' + u' &= 0 \text{ in } (-1,1), && (3.1)\\ u(-1) &= 0,\ u(1) = 1 && (3.2) \end{aligned}$$

the exact solution is given by

$$u_\epsilon(x) = \frac{e^{\frac{x+1}{\epsilon}} - 1}{e^{\frac{2}{\epsilon}} - 1}. \tag{3.3}$$

Hence the boundary layer exhibited near $x = 1$ when $\epsilon \to 0$ has a width of order $O(\epsilon)$.

The pseudo spectral approximation $u_N \in \mathbf{P}_N$ of (3.1), (3.2) is now defined by

$$\begin{aligned} -\epsilon u_N''(x_i) + u_N'(x_i) &= 0 \text{ for } i = 1, \ldots, N-1, && (3.4)\\ u_N(-1) = 0,\ u_N(1) &= 1, && (3.5) \end{aligned}$$

where $x_i = \cos\frac{i\pi}{N}$ denote the Chebyshev collocation points. In [2] the spectral approximation is explicitly calculated and finally one obtains for

<u>odd N</u>: $u_N \cong \frac{1}{2} + \frac{1}{2}T_N$,

<u>even N</u>: $u_N \cong \hat{u}_0 + \hat{u}_N T_N,\ |\hat{u}_0| \cong \hat{u}_N \cong O(\epsilon N^2)^{-1}$

as $\epsilon \to 0,\ \epsilon << N^{-2}$. Here T_N denotes the Nth Chebyshev polynomial. Therefore in both cases u_N is strongly oscillating but for a given ϵ the oscillations created by the boundary layer are less pronounced if N is chosen to be odd. This shows that attention should be paid to the parity of the degree of polynomials to be used in a spectral approximation of boundary layer problems. This example further demonstrates the instability of the convection dominated problem. In particular, for even N we can read from the coefficients $\hat{u}_0$ and $\hat{u}_N$ that the approximation error is perturbed by an instability rate of $O(\epsilon N^2)^{-1}$.

The problem of instability is also well known for finite difference (or finite element) discretizations if central finite differences are used (see, e.g., [9]). Here also spurious oscillations are introduced by the discretization scheme. This problem is avoided by applying an upstream scheme where the first derivative is approximated by a one-sided finite difference star. Another possibility is to add an artificial diffusion (or viscosity) term of the form $-N^{-1}\Delta u$ to the convection-diffusion equation. Now these methods are stable and produce non-oscillating solutions but are only first-order accurate being based on solving a modified problem. Clearly such techniques are of no interest for spectral approximations since the high spectral accuracy is completely lost.

Therefore we thought of other techniques of stabilization which maintain the high accuracy of spectral methods. Together with Eisen [8] we obtained a stable scheme by adding one additional equation of collocation to the original system. Hence we obtain an overdetermined system of equations and the instability caused by the highest mode is avoided. This is a certain kind of penalty method. We proved stability independent of ϵ and present the numerically calculated condition numbers for several types of collocation points (Chebyshev Gauss or Gauss-Lobatto nodes). A drawback of this approach is that the method is not flow directed and therefore in our model problem for $\epsilon \to 0,\ \epsilon << N^{-2}$ the spectral solution approximates the straight line $u_s(x) = 0.5(x+1)$ (see [8, Figure 2]) instead of the boundary layer solution (3.3). Another drawback is that for overdetermined systems no efficient iterative solvers like multigrid methods are available. For these reasons this method is probably not the best way to go.

A flow directed method is the *streamline diffusion method* which was introduced by Hughes and Brooks [17], [18] for a finite element discretization. This method is stable and no accuracy is lost. It is a Petrov-Galerkin modification of the standard Galerkin method where artificial diffusion in the streamline direction is introduced by modifying the test functions from v to

$$v \ + \ \delta_x c v_x + \delta_y d v_y,$$

where $\delta_x = \delta_y = O(h)$ and h denotes the step size of the finite element scheme. Clearly, for the spectral method with Chebyshev Gauss-Lobatto points one has to choose a point dependent viscosity given by $\delta_x = \delta_x(x),\ \delta_y = \delta_y(y)$, where

$$\begin{aligned}\delta_x(x_i) &= C_x \sin(\frac{\pi}{N}) \sin(i\frac{\pi}{N}), \quad i = 1,\dots,N-1,\\ \delta_y(y_j) &= C_y \sin(\frac{\pi}{N}) \sin(j\frac{\pi}{N}), \quad j = 1,\dots,N-1\end{aligned}$$

with suitable constants C_x, C_y. These formulas result from the finite difference discretization with central differences (see [11]). The constants C_x, C_y can be chosen such that the resulting finite difference matrix yields an M-matrix (see [9]), i.e., its inverse has only nonnegative entries. For the practically more efficient pseudo spectral method the stabilization is achieved by adding the viscosity term to the right hand side f. Here f is replaced by

$$f + \delta_x c f_x + \delta_y d f_y$$

and the corresponding differential operator is modified such that the new problem is equivalent to the original system (1.1), (1.2). In [15] we investigated this method in connection with a multi domain approach for the above boundary layer problem. Stability is shown and suitable multigrid components for the efficient solution of the stabilized problem are presented. Clearly, a drawback of this approach is that high order derivatives have to be computed which makes it quite expensive.

For preconditioning of the original spectral system we recommend an upwind FD method, for which each of the first derivatives u_x and u_y is differenced according to the sign of the coefficients c and d, respectively. The Laplace operator is discretized by standard central finite differences and the first derivatives u_x and u_y in $(x_i, y_j),\ i,j = 1,\dots,N-1$ are approximated as follows:

$$\begin{aligned}
c(x_i,y_j) \geq 0: \quad u_x(x_i,y_j) &\cong \frac{u(x_{i+1},y_j) - u(x_i,y_j)}{x_{i+1} - x_i},\\
c(x_i,y_j) < 0: \quad u_x(x_i,y_j) &\cong \frac{u(x_i,y_j) - u(x_{i-1},y_j)}{x_i - x_{i-1}},\\
d(x_i,y_j) \geq 0: \quad u_y(x_i,y_j) &\cong \frac{u(x_i,y_{j+1}) - u(x_i,y_j)}{y_{j+1} - y_j},\\
d(x_i,y_j) < 0: \quad u_y(x_i,y_j) &\cong \frac{u(x_i,y_j) - u(x_i,y_{j-1})}{y_j - y_{j-1}}.
\end{aligned}$$

For the iterative solution we recommend flow directed schemes. Since the Chebyshev nodes are dense near the boundary it is necessary to use line Gauss-Seidel relaxation. For smoothing it is recommended to use alternate iterations of FDHI (Flow Directed Horizontal Iterations) and FDVI (Flow Directed Vertical Iterations). In the literature this combination is called FDHVI (see [7], [10]). The iterative scheme FDHI is a variant of line Gauss-Seidel relaxation. Let P_i denote the mesh points on the vertical line $x = x_i$. We divide P_i into two subsets:

$$\begin{aligned} P_{i,E} &:= \{(i,j) : c(x_i, y_j) \geq 0\}, \\ P_{i,W} &:= \{(i,j) : c(x_i, y_j) < 0\}. \end{aligned}$$

The FDHI partitioning and ordering of the unknowns consists of the subsets $P_{i,E}$ arranged in order of increasing i, followed by the subsets $P_{i,W}$, arranged in order of decreasing i. The difference equations on each of the subsets $P_{i,E}$ or $P_{i,W}$ are a collection of tridiagonal systems. By considering the mesh points P_j on a horizontal line $y = y_j$ and dividing P_j into subsets $P_{j,N}$, $P_{j,S}$ we may construct the iterative scheme FDVI. Finally one alternates between FDHI and FDVI, resulting in FDHVI. For a more detailed description including numerical results we refer to [7] and [10]. Han et al. [10] describe a procedure based on directed graphs to partition and order the unknowns of the Gauss-Seidel process. This is performed by inspection of the coefficient matrix. Nevertheless, this algorithm is expensive for non-linear problems, like those coming from the Navier-Stokes or Boussinesq equations, when the coefficients are solution dependent and require the reconstruction of the directed graph several times. The penalty for such a choice is proportional to the number of mesh points. In Section 4 the FDHVI scheme is applied to the Boussinesq flow problem.

4. APPLICATION TO THE BOUSSINESQ FLOW PROBLEM

The problem specifically considered here is that of the two-dimensional flow of a Boussinesq fluid of Prandtl number $Pr = 0.71$ (i.e., air) in an upright square cavity (see [1], [6]). The walls are non-slip and impermeable. The horizontal walls are adiabatic and the vertical sides are at fixed temperatures. In addition to the Navier-Stokes equations we have one further equation for the temperature T. By Ra we denote the Rayleigh number. The Boussinesq flow problem in vorticity-streamfunction formulation reads as follows:

$$\begin{aligned} 4\Delta\psi + \omega &= 0 \ \text{ in } \Omega = (-1,1)^2, && (4.1)\\ -2Pr\Delta\omega + \frac{\partial}{\partial x}(v_1\omega) + \frac{\partial}{\partial y}(v_2\omega) &= RaPr\frac{\partial T}{\partial x} \ \text{ in } \Omega, && (4.2)\\ -2\Delta T + \frac{\partial}{\partial x}(v_1 T) + \frac{\partial}{\partial y}(v_2 T) &= 0 \ \text{ in } \Omega. && (4.3) \end{aligned}$$

As usual $(v_1, v_2)^t$ denotes the velocity. The scalar factors 2 in equations (4.2),
(4.3) and 4 in equation (4.1) are due to the fact that we here define the problem in $(-1,1)^2$ instead of the original square cavity $(0,1)^2$. ψ fulfills homogeneous Dirichlet boundary conditions, i.e.,

$$\psi = \frac{\partial\psi}{\partial\nu} = 0 \ \text{ on } \partial\Omega$$

and T fulfills mixed Dirichlet/Neumann boundary conditions

$$\begin{aligned} &T(-1,y) = 1,\ T(1,y) = 0 \ \text{ for } y \in (-1,1),\\ &\frac{\partial T}{\partial y}(x,-1) = \frac{\partial T}{\partial y}(x,1) = 0 \ \text{ for } x \in [-1,1]. \end{aligned}$$

The homogenous Neumann boundary conditions correspond to the fact that the horizontal walls are adiabatic.

Now the equations (4.1)–(4.3) are linearized by a Quasi-Newton method (see [16]), where the velocity from the previous iteration is employed. Hence we have to solve the linear problem:

$$4\Delta\psi^{n+1} + \omega^{n+1} = 0 \ \text{ in } \Omega, \qquad (4.4)$$

$$
\begin{aligned}
-2Pr\Delta\omega^{n+1} + \frac{\partial}{\partial x}(v_1^n\omega^{n+1}) + \frac{\partial}{\partial y}(v_2^n\omega^{n+1}) &= RaPr\frac{\partial T^{n+1}}{\partial x} \ \text{ in } \Omega, &(4.5)\\
-2\Delta T^{n+1} + \frac{\partial}{\partial x}(v_1^n T^{n+1}) + \frac{\partial}{\partial y}(v_2^n T^{n+1}) &= 0 \ \text{ in } \Omega, &(4.6)
\end{aligned}
$$

where the index n denotes the n-th iterate.
Clearly, the velocity is related to the streamfunction by

$$
v_1^n = \frac{\partial \psi^n}{\partial y}, \quad v_2^n = -\frac{\partial \psi^n}{\partial x}.
$$

The linear system (4.4)–(4.6) is now approximately solved by a spectral multigrid (SMG) method (see [12], [13]). We first describe the pseudo spectral discretization of the system (4.4)–(4.6). The functions ψ^{n+1} and ω^{n+1} are spectrally approximated by polynomials $u_{N+2} \in \mathbf{P}_{N+2}$, $v_N \in \mathbf{P}_N$. T^{n+1} is approximated by a polynomial $w_N \in \mathbf{P}_N$. Here the index $n+1$ is omitted. Hence v_1^n, v_2^n are approximated by the polynomials $v_{1,N}$, $v_{2,N} \in \mathbf{P}_{N+2}$, where

$$
v_{1,N} = \frac{\partial \overline{u}_{N+2}}{\partial y}, \ v_{2,N} = -\frac{\partial \overline{u}_{N+2}}{\partial x},
$$

and $\overline{u}_{N+2} \in \mathbf{P}_{N+2}$ corresponds to the polynomial u_{N+2} from the previous iteration. Now the pseudo spectral problem related to (4.4)–(4.6) reads as follows:
Find $u_{N+2} \in \mathbf{P}_{N+2}$, $v_N \in \mathbf{P}_N$, $w_N \in \mathbf{P}_N$ such that

$$
[4\Delta u_{N+2} + v_N](x_i, y_j) = 0 \tag{4.7}
$$

for $i, j = 0, \ldots, N$ and

$$
[-2Pr\Delta v_N + \frac{\partial}{\partial x}(v_{1,N}v_N) + \frac{\partial}{\partial y}(v_{2,N}v_N)](x_i, y_j) = RaPr\frac{\partial w_N}{\partial x}(x_i, y_j), \tag{4.8}
$$

$$
[-2\Delta w_N + \frac{\partial}{\partial x}(v_{1,N}w_N) + \frac{\partial}{\partial y}(v_{2,N}w_N)](x_i, y_j) = 0 \tag{4.9}
$$

for $i, j = 1, \ldots, N-1$. Since u has to fulfill two types of boundary conditions we choose $u_{N+2} \in \mathbf{P}_{N+2}$ fulfilling the homogeneous Dirichlet boundary conditions. For v_N there are no boundary conditions. The pseudo spectral boundary conditions for $w_N \in \mathbf{P}_N$ are given by

$$w_N(-1, y_j) = 1,\ w_N(1, y_j) = 0 \ \text{ for } j = 1, \ldots, N-1, \tag{4.10}$$

$$\frac{\partial w_N}{\partial y}(x_i, -1) = \frac{\partial w_N}{\partial y}(x_i, 1) = 0 \ \text{ for } i = 0, \ldots, N. \tag{4.11}$$

The system (4.7)–(4.11) completely determines the spectral approximations $u_{N+2},\ v_N,\ \ w_N$. It is clear that the polynomials $u_{N+2},\ v_N$ are uniquely determined by the equations (4.7), (4.8). Furthermore the equation (4.9) together with the boundary conditions (4.10), (4.11) uniquely determine the polynomial w_N. Hence in the linearized version the systems (4.7), (4.8) for determining $u_{N+2},\ v_N$ and (4.9), (4.10), (4.11) for determining w_N can be handled separately. First, one solves the system (4.9), (4.10), (4.11) for w_N by a SMG method, then one calculates $\frac{\partial w_N}{\partial x}(x_i, y_j),\ i, j = 1, \ldots, N-1$, and finally one solves the system (4.7), (4.8) by the SMG method introduced in [14]. Here we employed 6 V-cycles of SMG in order to get a nearly exact solution of the linear systems.

Now we turn to a more precise description of the SMG method. We use the same components as already introduced in Section 3. A somewhat different treatment results from the fact that the diffusive part is now perturbed by the first order derivatives $\frac{\partial}{\partial x},\ \frac{\partial}{\partial y}$. For an increasing Rayleigh number the convective part becomes dominant. Hence in the defect correction step one has to use a FD approximation which remains stable also for an increasing Rayleigh number. Furthermore the FD problem has to be solved approximately by a suitable iterative method which also works for convection dominated flows. Here we employed the FDHVI iteration for preconditioning of the spectral system resulting from the equations (4.8), (4.9). In order to handle the complex eigenvalues of the preconditioned spectral operator we employ nonsymmetric matrix iterations. Here we choose the GMRES iteration. For a more detailed description of these components we refer to Section 3.

By using these components we numerically calculated for various Rayleigh numbers and mesh sizes the following quantities:

$|\psi|_{mid}$: absolute value of the streamfunction at the midpoint of the cavity,

$|\psi|_{max}$: maximum absolute value of the streamfunction,
$v_{1,max}$: maximum horizontal velocity on the vertical mid-plane of the cavity,
$v_{2,max}$: maximum horizontal velocity on the horizontal mid-plane of the cavity.

The local heat flux in a horizontal direction at any point in the cavity is given by

$$Q \;:=\; v_1 T \;-\; 2\frac{\partial T}{\partial x}.$$

Let us further introduce the following Nusselt numbers:

$\overline{Nu} \;:=\; \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} Q(x,y)dxdy$: average Nusselt number throughout the cavity,
$Nu_{\frac{1}{2}} \;:=\; \frac{1}{2}\int_{-1}^{1} Q(0,y)dy$: average Nusselt number on the vertical mid-plane,
$Nu_0 \;:=\; \frac{1}{2}\int_{-1}^{1} Q(-1,y)dy$: average Nusselt number on the vertical boundary,
$Nu_{max} \;:=\; \max\{|Q(-1,y)| : \; y \in [-1,1]\}$: maximum value of the Nusselt number,
$Nu_{min} \;:=\; \min\{|Q(-1,y)| : \; y \in [-1,1]\}$: minimum value of the Nusselt number.

The above integrals in the definition of $\overline{Nu}$, $Nu_{\frac{1}{2}}$ and Nu_0 are evaluated by the Clenshaw-Curtis quadrature (see [5, p. 68]). In the tables II - IV we present the numerical results for different Rayleigh numbers and $N = 8, 16, 24$. The numerical results are in good accordance with the results obtained in [6]. However, for a larger Rayleigh number or increasing N the above SMG method is no more convergent. The reason is that upstream preconditioning is not good enough. Here one has to find some better ways of preconditioning. At the moment we try to find improved preconditioners where the finite difference discretization is performed on staggered grids.

N	$\|\psi\|_{mid}$	$\|\psi\|_{max}$	$v_{1,max}$	$v_{2,max}$	$\overline{Nu}$	$Nu_{\frac{1}{2}}$	Nu_0	Nu_{max}	Nu_{min}
8	1.1747	1.1747	3.5250	3.6122	1.1178	1.1173	1.1174	1.5048	0.6902
16	1.1746	1.1746	3.5690	3.6148	1.1178	1.1178	1.1178	1.5060	0.6913
24	1.1746	1.1746	3.6441	3.6759	1.1178	1.1178	1.1178	1.5060	0.6913

Table II. Results for $Ra = 10^3$.

N	$\|\psi\|_{mid}$	$\|\psi\|_{max}$	$v_{1,max}$	$v_{2,max}$	$\overline{Nu}$	$Nu_{\frac{1}{2}}$	Nu_0	Nu_{max}	Nu_{min}
8	5.0713	5.0713	15.8009	18.9305	2.2474	2.1946	2.1870	3.6170	0.5067
16	5.0736	5.0736	15.8352	19.0474	2.2448	2.1946	2.1870	3.5314	0.5853
24	5.0981	5.0980	16.1800	19.5000	2.2340	2.2350	2.2420	3.5450	0.5920

Table III. Results for $Ra = 10^4$.

N	$\|\psi\|_{mid}$	$\|\psi\|_{max}$	$v_{1,max}$	$v_{2,max}$	$\overline{Nu}$	$Nu_{\frac{1}{2}}$	Nu_0	Nu_{max}	Nu_{min}
8	14.3409	18.8519	37.8844	40.2643	4.4140	4.7345	4.7590	10.4740	0.3438
16	11.3720	12.3330	36.3420	61.3420	4.5030	4.5061	4.5313	7.9010	0.7551
24	9.1600	9.6530	34.6320	67.9120	4.5100	4.5120	4.5231	7.7700	0.7361

Table IV. Results for $Ra = 10^5$.

Bibliography

[1] Behnia, M., Wolfstein, M., and De Vahl Davis, G.: A stable fast marching scheme for computational fluid mechanics. Int. J. for Num. Meth. in Fluids **10**, 607-621(1990)

[2] Canuto, C.: Spectral methods and a maximum principle. Math. Comp. **51**, 615-629(1988)

[3] Canuto, C.: Parallelism in spectral methods. I.A.N.-C.N.R. Report No. 609, Pavia, 1988 (unpublished)

[4] Canuto, C., Hussaini, M.Y., Quarteroni, A., and Zang, T.A.: Spectral Methods in Fluid Dynamics, Springer Series in Computational Physics, Springer-Verlag, Berlin-Heidelberg-New York, 1989

[5] Davis, P. and Rabinowitz,P.: Methods of Numerical Integration. Academic Press, Orlando, 1984

[6] De Vahl Davis, G.: Natural convection of air in a square cavity: a bench mark numerical solution. Int. J. for Numer. Meth. in Fluids **3**, 249-264(1983)

[7] Deville, M.D. and Mund, E.H.: Finite element preconditioning of collocation schemes for advection-diffusion equations. Proceedings of the IMACS International Symposium on Iterative Methods in the Linear Algebra, edited by R. Beauwens and P. de Groen, Brüssel, 181-189, 1992

[8] Eisen, H. and Heinrichs, W.: A new method of stabilization for singular perturbation problems with spectral methods. SIAM J. Numer. Anal. **29**, 107-122(1992)

[9] Hackbusch, W.: Theorie und Numerik elliptischer Differentialgleichungen. Teubner Studienbücher, B.G. Teubner, Stuttgart, 1986

[10] Han, H., Il'in, V.P., and Kellogg, R.B.: Flow directed iterations for convection dominated flow, *in* BAIL V, Proceedings of the Fifth Int. Conf. on Boundary and Interior Layers - Computational and Asymptotic Methods, edited by G. Ben-yu, J.J.H. Miller and S. Zhong-ci, Shanghai, China, 1988

[11] Heinrichs, W.: Kollokationsverfahren und Mehrgittermethoden bei elliptischen Randwertaufgaben. GMD-Bericht Nr. 168, Oldenbourg-Verlag, München-Wien, 1987

[12] Heinrichs, W.: Line relaxation for spectral multigrid methods. J. Comp. Phys. **77**, 166-182(1988)

[13] Heinrichs, W.: Multigrid methods for combined finite difference and Fourier problems. J. Comp. Phys. **78**, 424-436(1988)

[14] Heinrichs, W.: Spectral multigrid techniques for the Stokes problem in streamfunction formulation, J. Comput. Phys. **102**, 310-318(1992)

[15] Heinrichs, W.: A stabilized multi domain approach for singular perturbation problems. J. Scient. Comput. **7**, No. 2, 95-125(1992)

[16] Heinrichs, W.: Spectral projective Newton methods for quasilinear elliptic boundary value problems. To appear in Calcolo, 1993

[17] Hughes, T.J.R. and Brooks, A.: A multidimensional upwind scheme with no crosswind diffusion, *in* T.J.R. Hughes, ed., Finite Element Methods for Convection Dominated Flows, AMD Vol. 34 (ASME, New York, 1979)

[18] Hughes, T.J.R. and Brooks, A.: A theoretical framework for Petrov-Galerkin methods with discontinuous weighting functions: Application to the streamline-upwind procedure, to appear in R.H. Gallagher, ed., Finite Elements in Fluids, Vol. 4 (Wiley, New York)

[19] Nachtigal, N.M., Reddy, S.C., and Trefethen, L.N.: How fast are nonsymmetric matrix iterations? Numerical Analysis Report 90-2, Department of Mathematics, Massachusetts Institute of Technology, Cambridge, 1990 (unpublished)

[20] Saad, Y. and Schultz, M.H.: Conjugate gradient-like algorithms for solving nonsymmetric linear systems. Math. Comp. **44**, 417-424(1985)

[21] Saad, Y. and Schultz, M.H.: GMRES: A generalized minimum residual algorithm for solving nonsymmetric linear systems. SIAM J. Sci. Stat. Comp. **7**, 856-869(1986)