DEFECT CORRECTION
FOR THE
ADVECTION–DIFFUSION EQUATION

by

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Abstract. A defect correction method for the advection–diffusion equation is presented. The discretization is performed by $2nd$ order finite difference schemes (β-schemes) where the $2nd$ order upstream scheme is combined with the standard central scheme. For preconditioning the usual first order upstream scheme is employed. The defect correction iteration is used for relaxation inside a multigrid procedure. It is shown that the smoothing analysis yields rather pessimistic results with rates of about 0.7–0.8. In the practical computation the discretization error is reached in 2 V-cycles. Numerical results are presented which demonstrate the high efficiency of our treatment.

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1. INTRODUCTION

The advection–diffusion equations yields a good model for the numerical solution of the Navier-Stokes equations with high Reynolds numbers. One is interested in finding a stable method which is of at least $2nd$ order accuracy. Unfortunately, the standard stable schemes as the usual upstream scheme or the artificial viscosity discretization lead to only $1st$ order methods. But in many applications the $2nd$ order accuracy is necessary in order to get a realistic impression of the flow.

Here the advection–diffusion equation is discretized by the $2nd$ order β-scheme which yields a combination of the standard $2nd$ order upstream scheme and the central scheme. These schemes were already analyzed by Desideri & Hemker [**?**] and Luh [**?**]. For $\beta = 1$ we obtain the standard $2nd$ order upstream scheme and for $\beta = 0$ the central scheme. $\beta = \frac{1}{2}$ results in the *Fromm's* scheme. For $\beta = \frac{1}{3}$ we obtain the *upwind biased* scheme which is of $3rd$ order accuracy. Since an iterative solver for these higher order schemes yields bad convergence factors we propose a defect correction procedure. We already made good experience with this method for spectral discretizations (see [**?**], [**?**]). Here the higher order scheme is preconditioned by the standard $1st$ order upwind scheme. We investigate the smoothing properties of this procedure for the advection equation. Finally this method is used for relaxation in a multigrid cycle. Numerical results for the advection–diffusion equation are presented which demonstrate the improvements by using multigrid techniques.

2. DISCRETIZATION

Here we consider advection–diffusion problems which can in its most general form be written as

$$-\epsilon\Delta u + au_x + bu_y = f \quad \text{in } \Omega = (0,1)^2, \tag{2.1}$$

$$u = g \quad \text{on } \partial\Omega, \tag{2.2}$$

where $\epsilon = \frac{1}{Re}$ and Re denotes the Reynolds number. f is defined in Ω, and g is defined on $\partial\Omega$. a and b denote given constants. Such problems arise after a linearization of the Navier–Stokes equations. The part $-\epsilon\Delta u$ denotes the diffusive part and $au_x + bu_y$ denotes the advective part of the above equation. Here we are mainly interested in advection dominated flows where $\epsilon << h$. Here h denotes the step size of the finite difference (FD) scheme. In particular, we are interested in cases of large Reynolds numbers $Re = 10^4 - 10^6$. For the FD approximation of advection–diffusion problems one observes certain phenomenae of instability. For small ϵ standard discretizations can lead to a solution of the discrete problem which has nothing to do with the solution of the original problem. For instance, the discretization of

$$-\epsilon u'' - 2u' = 0 \quad in \ [0,\infty)\,,$$
$$u(0) = 1,\ u(\infty) = 0$$

with central differences yields as discrete solution:

$$u_{\epsilon,h}(ih) = \left(\frac{\epsilon - h}{\epsilon + h}\right)^i .$$

This is an $O(h^2)$ discretization. For ih fixed and $\frac{h}{\epsilon} \to 0$ we obtain:

$$|(u_{\epsilon,h} - u)(ih)| = |\left(\frac{\epsilon - h}{\epsilon + h}\right)^i - \left(e^{-2h/\epsilon}\right)^i| \leq C\left(\frac{h}{\epsilon}\right)^2$$

with C independent of i, h and ϵ. But the solution of the reduced equation is:

$$u_{0,h}(ih) = \lim_{\epsilon\to 0} u_{\epsilon,h}(ih) = (-1)^i$$

which means that for $\epsilon << h$ the solution of the FD problem has nothing to do with the exact solution

$$u(x) = e^{-2x/\epsilon}.$$

For this special problem we further observe a boundary layer in which the first derivative behaves as $O(\epsilon^{-1})$ for $\epsilon \to 0$. One possibility to avoid the phenomenon of instability is to use upstream discretization for u'. Clearly, an obvious disadvantage of this scheme lies in the fact that the method now becomes only first order accurate. Hence it makes sense to use the first order upstream scheme only as a preconditioner for a higher order scheme. We analyze the preconditioning properties of this method for the following higher order schemes:

- second order upstream scheme,
- central finite difference scheme,
- β-schemes, $\beta \in [0,1]$.

The first order upstream scheme is explicitly given by the upstream operator L_h^1 (here in 1D):

$$L_h^1 \;\cong\; \frac{1}{h}\left[-1\ \underline{1}\ 0\right].$$

The second order hybrid scheme is the β-scheme L_h^β which is a combination of the standard $2nd$ order upstream scheme $L_h^{su,2}$ and the $2nd$ order central scheme $L_h^{ce,2}$:

$$L_h^\beta \;=\; \beta L_h^{su,2} \;+\; (1-\beta)L_h^{ce,2}, \quad \beta \in [0,1]. \tag{2.3}$$

For all β we obtain at least $2nd$ order discretizations. Clearly, for $\beta = 0$ the method becomes unstable. Especially, for $\beta = \frac{1}{3}$ we obtain a $3rd$ order scheme. The following special cases of the hybrid schemes are of importance (here in 1D):

$\beta = 1$	standard upstream, $2nd$ order	$\frac{1}{h}\left[\frac{1}{2}\ -2\ \underline{\frac{3}{2}}\ 0\right]$,
$\beta = \frac{1}{2}$	Fromm's scheme, $2nd$ order	$\frac{1}{h}\left[\frac{1}{4}\ -\frac{5}{4}\ \underline{\frac{3}{4}}\ \frac{1}{4}\right]$,
$\beta = \frac{1}{3}$	upwind biased scheme, $3rd$ order	$\frac{1}{h}\left[\frac{1}{6}\ -1\ \underline{\frac{1}{2}}\ \frac{1}{3}\right]$,
$\beta = 0$	central scheme, $2nd$ order, unstable	$\frac{1}{h}\left[0\ -\frac{1}{2}\ \underline{0}\ \frac{1}{2}\right]$.

3. FOURIER ANALYSIS

Here we study the preconditioning properties of L_h^1 for the 2D advection operator

$$au_x \; + \; bu_y .$$

The defect correction iteration is defined by the operator

$$M_h \;=\; I_h \;-\; \omega \left(L_h^1\right)^{-1} L_h^\beta, \quad \beta \in [0,1].$$

Here ω denotes a relaxation parameter which should accellerate the convergence speed. By a Fourier analysis for the Fourier components $\underline{\theta}$ ($\underline{\theta} = \theta_1$ in $1D$, $\underline{\theta} = (\theta_1, \theta_2)$ in $2D$) the above operator M_h leads to the *amplification factor*

$$\mu(\underline{\theta}) \;=\; 1 - \omega \frac{\lambda^\beta(\underline{\theta})}{\lambda^1(\underline{\theta})},$$

where $\lambda^\beta(\underline{\theta})$ and $\lambda^1(\underline{\theta})$ denote the factors of the Fourier analysis for the operators L_h^β and L_h^1. Now the convergence factor ρ of the defect correction procedure is defined as the supremum of $\mu(\underline{\theta})$ taken over all frequencies $\underline{\theta} \in (-\pi, \pi]$ in 1D and $\underline{\theta} \in (-\pi,\pi]^2$ in 2D:

$$\rho(M_h) \;:=\; \sup_{\underline{\theta} \in (-\pi,\pi]^2} |\mu(\underline{\theta})|.$$

Furthermore we were interested in defect correction as a smoother in a multigrid procedure. The efficiency of a smoother can be measured by means of the *smoothing rate* μ_β. This rate can be obtained by taking the above supremum only for the high frequencies $|\underline{\theta}| := \max(|\theta_1|, |\theta_2|) \in (\frac{\pi}{2}, \pi]$:

$$\mu_\beta \;:=\; \sup_{\underline{\theta} \in (\frac{\pi}{2},\pi]} |\mu(\underline{\theta})|.$$

From the Fourier analysis of Yvonne Luh [**?**] it became clear that for all $\beta \in [0,1]$, ω and independent of the *alignment* $\frac{b}{a}$ the prediction

$$\rho(M_h) \;=\; 1$$

holds.

However, in the multigrid procedure we are more interested in the damping only of the high frequencies. Here we consider the special cases $\beta = 1$ and $\beta = \frac{1}{2}$. From the analysis in [**?**] the following was seen:

- $\beta = 1$, $\omega_{opt} = 0.68$: $\mu_1 \leq 0.77$
- $\beta = \frac{1}{2}$, $\omega_{opt} = 1.00$: $\mu_{\frac{1}{2}} \leq 0.72$.

For $\beta = 1$ the optimal parameter ω is about 0.68 for all alignments $\frac{b}{a}$ (see figure 1). The corresponding smoothing rate μ_1 is decreasing for increasing alignment. The maximum is attained at $\frac{b}{a} = 0.1$ where $\mu_1 = 0.77$. For $\beta = \frac{1}{2}$ the value $\omega_{opt} = 1$ yields a quite good choice for all alignments $\frac{b}{a}$ (see figure 2). In figure 3 we present the optimal parameters ω and corresponding smoothing rates for all $\beta \in [0, 1]$ and the special alignment $\frac{b}{a} = 0.4142$ (i.e. 22,5 degrees). For $\beta = 0$ the smoothing rate μ_0 is always equal to 1 (independent of the parameter choice of ω). From these considerations it becomes clear that the smoothing rates are rather bad compared to the usual rates for symmetric problems (e.g. the Poisson problem).

4. NUMERICAL RESULTS

Here we apply a standard multigrid method. The transfer operators are given by *full weighting* restriction and *bilinear* interpolation. For relaxation we choose the already mentioned *Richardson* iteration with *defect correction.* It is explicitly defined as follows:

$$u_h^{j+1} \;=\; u_h^j \;-\; \omega \left(L_h^1\right)^{-1} \left(L_h^\beta u_h^j - f_h\right)$$

for $j = 0, 1, 2, \ldots$. u_h^0 denotes a start approximation which is chosen to be identically zero. ω denotes the relaxation parameter. Optimal choices are given by a smoothing analysis (see Section 3). Instead of solving the first order problem relative to L_h^1 exactly we employ a *lexikographic* Gauß-Seidel step which nearly yields an exact solver if $a,\ b > 0$. If a and b have different sign then after a renumbering of the grid points the same effect can be achieved. In case of variable coefficients a and b which change sign one has to use the *flow directed point relaxation.*

Here we consider a multigrid method with 7 grids and corresponding step sizes $h_\nu \;=\; \frac{1}{2^\nu},\ \nu = 1, \ldots, 7$. In general, we employ a *V–cycle.* Other cycle structures as *W–cycle, F–cycle* or the *full multigrid* technique could not improve the convergence speed significantly. We employ two relaxations before and one after the coarse grid correction. The results are compared with the pure Richardson iteration without using multigrid. The absolute errors between the exact solution and the $IT.-$ iterate are measured in the discrete L^1 and L^2 norms: $L1,\ L2$. By $Q(L1),\ Q(L2)$ we denote the quotient of the errors for two successive iterates. Numerical results are provided for the example where the exact solution is given by

$$u(x,y) \;=\; \sin\left(8\pi(y - \frac{b}{a}x)\right),$$

where $a = 4,\ b = 1$ and $\epsilon = 10^{-6}$ (see Y. Luh [**?**]). The right hand side f and the Dirichlet boundary conditions are determined by u. We present results for $\beta \in \{\frac{1}{3}, \frac{1}{2}, 1\}$. The corresponding numerical results are given in the tables I–VI.

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$2.98 \cdot 10^{-2}$	$4.07 \cdot 10^{-2}$	0.047	0.058
2	$5.56 \cdot 10^{-3}$	$8.58 \cdot 10^{-3}$	0.186	0.211
3	$7.99 \cdot 10^{-3}$	$1.01 \cdot 10^{-2}$	1.437	1.176
4	$7.31 \cdot 10^{-3}$	$9.40 \cdot 10^{-3}$	0.915	0.931
5	$7.42 \cdot 10^{-3}$	$9.51 \cdot 10^{-3}$	1.016	1.012
6	$7.39 \cdot 10^{-3}$	$9.48 \cdot 10^{-3}$	0.996	0.997
7	$7.40 \cdot 10^{-3}$	$9.49 \cdot 10^{-3}$	1.001	1.001

Table I. $\beta = \frac{1}{3}$, relaxations

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$6.12 \cdot 10^{-3}$	$1.19 \cdot 10^{-2}$	0.009	0.016
2	$7.56 \cdot 10^{-3}$	$9.76 \cdot 10^{-3}$	1.236	0.818
3	$7.44 \cdot 10^{-3}$	$9.53 \cdot 10^{-3}$	0.984	0.976
4	$7.40 \cdot 10^{-3}$	$9.49 \cdot 10^{-3}$	0.994	0.996

Table II. $\beta = \frac{1}{3}$, V–cycle

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$1.07 \cdot 10^{-1}$	$1.30 \cdot 10^{-1}$	0.169	0.184
2	$1.24 \cdot 10^{-2}$	$1.61 \cdot 10^{-2}$	0.115	0.124
3	$1.03 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	0.828	0.776
4	$1.00 \cdot 10^{-2}$	$1.23 \cdot 10^{-2}$	0.978	0.985
5	$1.01 \cdot 10^{-2}$	$1.23 \cdot 10^{-2}$	1.003	1.001

Table III. $\beta = \frac{1}{2}$, relaxations

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$8.81 \cdot 10^{-3}$	$1.22 \cdot 10^{-2}$	0.014	0.017
2	$1.03 \cdot 10^{-2}$	$1.25 \cdot 10^{-2}$	1.164	1.032
3	$1.01 \cdot 10^{-2}$	$1.23 \cdot 10^{-2}$	0.982	0.981

Table IV. $\beta = \frac{1}{2}$, V–cycle

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$2.92 \cdot 10^{-1}$	$3.31 \cdot 10^{-1}$	0.459	0.468
2	$9.09 \cdot 10^{-2}$	$1.08 \cdot 10^{-1}$	0.311	0.327
3	$3.53 \cdot 10^{-2}$	$4.35 \cdot 10^{-2}$	0.388	0.402
4	$2.57 \cdot 10^{-2}$	$3.19 \cdot 10^{-2}$	0.727	0.734
5	$2.45 \cdot 10^{-2}$	$3.04 \cdot 10^{-2}$	0.952	0.950
6	$2.43 \cdot 10^{-2}$	$3.01 \cdot 10^{-2}$	0.992	0.992
7	$2.42 \cdot 10^{-2}$	$3.01 \cdot 10^{-2}$	0.998	0.998

Table V. $\beta = 1$, relaxations

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$4.28 \cdot 10^{-2}$	$5.42 \cdot 10^{-2}$	0.067	0.076
2	$2.44 \cdot 10^{-2}$	$3.04 \cdot 10^{-2}$	0.569	0.560
3	$2.42 \cdot 10^{-2}$	$3.01 \cdot 10^{-2}$	0.994	0.989

Table VI. $\beta = 1$, V–cycle

IT	$L1$	$L2$	$Q(L1)$	$Q(L2)$
1	$3.81 \cdot 10^{-2}$	$4.82 \cdot 10^{-2}$	0.060	0.068
2	$2.43 \cdot 10^{-2}$	$3.02 \cdot 10^{-2}$	0.637	0.627
3	$2.42 \cdot 10^{-2}$	$3.01 \cdot 10^{-2}$	0.997	0.994

Table VII. $\beta = 1$, W–cycle

β	Relaxation	V–cycle
$\frac{1}{3}$	0.44	0.24
$\frac{1}{2}$	0.31	0.14
1	0.11	0.032

Table VIII. Asymptotic rates for different β

The numerical results show the highly improved efficiency of the V-cycle compared to the scheme with two relaxations. There is nearly no improvement by using other cycle structures (see W–cycle, table VII). In particular, the first rate of the multigrid scheme is very small and yields already an error which is nearly equal to the discretization error. In one case (see table IV) it happened to be better than the discretization error. Hence in general it can be seen that 2 or 3 V-cycles are enough to reach the truncation error. So the smoothing analysis in Section 3 gives rather pessimistic results. In practice, the discretization error is reached very fast. By comparing the results for different step sizes we also confirmed the at least second order accuracy. For $\beta = \frac{1}{3}$ we obtain a $3rd$ order method. Here we obtain the most precise results (see tables I, II). For increasing β the results become less accurate. In table VIII we present the asymptotic convergence rates in the $L1$ norm. Here we study the convergence for $u \equiv 0$. It becomes obvious that the asymptotic rates become smaller for increasing β. This result is really different from the first rates $Q(L1)$ where $Q(L1)$ becomes smaller for decreasing β. For one V-cycle ($IT = 1$) we obtain $Q(L1) = 0.009,\ 0.014,\ 0.067$ for $\beta = \frac{1}{3},\ \frac{1}{2},\ 1$ (see tables II, IV, VI). Asymptotically the more accurate schemes yield a slower convergence. Hence we found at least second order methods and also an efficient iterative solver with defect correction. Once more the use of multigrid techniques is recommended.

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