Least-Squares Spectral Collocation
for Discontinuous and Singular Perturbation Problems

by

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Abstract

A least-squares spectral collocation scheme for discontinuous problems is proposed. For the first derivative operator the domain is decomposed in subintervals where the jumps are imposed at the discontinuities. Equal order polynomials are used on all subdomains. For the discretization spectral collocation with Chebyshev polynomials is employed. Fast Fourier Transforms (FFTs) are now available. The collocation conditions and the interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. The solution technique will only involve symmetric positive definite linear systems. This approach is further extended to singular perturbation problems where least-squares are used for stabilization. By a suitable decomposition of the domain the boundary layer is well resolved. Numerical simulations confirm the high accuracy of our spectral least-squares scheme.

AMS(MOS) classification: 65N35

Keywords: Least-squares, spectral collocation, discontinuous approximation, singular perturbation problems.

1 Introduction

Spectral methods (see, e.g., Canuto et al. [1] or Gottlieb and Orszag [5], [14]) employ global polynomials for the numerical solution of differential equations. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. For analytical data exponential convergence can be achieved. For problems with discontinuities the usual (global) continuous spectral approach yields very poor approximation results. In general, only a first-order method can be obtained. For this purpose the original domain has to be decomposed into several subdomains where the jumps are explicitly enforced at the corresponding interface nodes. This technique is introduced for a one-dimensional problem with the first derivatives. Least-squares spectral element methods were already investigated by Gerritsma and Proot [4]. Here we prefer a spectral collocation scheme where the differential equation is collocated at the usual Chebyshev Gauss-Lobatto nodes including the boundary nodes. Hence on each subdomain we require $N+1$ conditions of collocation. Equal order Chebyshev polynomials of degree N are employed which allow the efficient implementation with Fast Fourier Transforms (FFTs). The collocation conditions together with the interface conditions lead to an overdetermined system which can be efficiently solved directly or by least-squares. The least-squares scheme yields a large condition number so that the truncation error spoils the high spectral accuracy for increasing N.

Furthermore our approach is extended to singular perturbation problems where the least-squares schemes lead to a stabilization. This was already observed by Eisen and Heinrichs in an earlier paper [2]. After a suitable decomposition of the domain we are able to resolve the boundary layer with the usual high spectral accuracy (exponential convergence). Summarizing our approach has the following advantages:

- equal order interpolation polynomials can be employed
- iterative solvers for positive definite systems (e.g. conjugate gradient methods) are available
- efficient preconditioners can be derived, see [8] for elliptic problems
- improved stability properties for convection-diffusion problems (see [2])

and the Navier-Stokes equations (see [3], [9] and Jiang et al. [10, 11, 12, 13])

- implementation and parallelization is straightforward.

The paper is organized as follows. In Section 2, the the least-squares spectral collocation scheme for a sample one-dimensional problem is derived. The approach is extended in section 3 to singular perturbation problems. By numerical simulations the stability and high accuracy of our spectral scheme is demonstrated. Finally a conclusion is presented.

2 The discontinuous problem

As a sample problem we consider the differential equation

$$\begin{aligned} u' &= f \quad in\ \Omega = (0, L), && (1) \\ u(0) &= c_0, && (2) \end{aligned}$$

where f denotes a given function and $c_0 \in \mathbb{R}$ an initial value. The domain Ω is decomposed into K subdomains $\Omega_k = (x_{k-1}, x_k),\ k = 1, \ldots, K$ with

$$0 = x_0 < x_1 < x_2 < \ldots < x_{K-1} < x_K = L.$$

We assume that the solution is smooth in between the discontinuities and jumps should only occur at interface points x_k. Now problem (1),(2) can be decomposed as follows: Find functions $u_k, k = 1, \ldots, K$ defined on $\overline{\Omega}_k$ such that

$$\begin{aligned} u_k' &= f \quad in\ \Omega_k,\ k = 1, \ldots, K, && (3) \\ u_{k+1}(x_k) - u_k(x_k) &= c_k,\ k = 1, \ldots, K-1, && (4) \\ u_1(0) &= c_0. && (5) \end{aligned}$$

Here the coefficients c_k denote the jumps at the interface nodes. The classical solution can only be retrieved if we set the $K-1$ coefficients $c_k = 0$. In this case the solution has no jumps.

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N = \{Polynomials\ of\ degree\ \leq N\}.$$

Now all unknown functions u_k are approximated by polynomials $u_{k,N}$ of the same degree N, i.e., $u_{k,N} \in \mathbb{P}_N, k = 1,\ldots,K$. Furthermore we have to introduce the standard Chebyshev Gauss-Lobatto collocation nodes. They are explicitly given by

$$\xi_i = -\cos\frac{i\pi}{N},\ i = 0,\ldots,N.$$

In order to define the nodes on Ω_k we have to introduce the mapping from the interval $(-1,1)$ to Ω_k. The nodes $x_{k,i}, i = 0,\ldots,N, k = 1,\ldots,K$ are explicitly given by

$$x_{k,i} = \frac{1}{2}\left[(x_k - x_{k-1})\xi_i + x_{k-1} + x_k\right].$$

Hence we retrieve the interval left and right bounds by $x_{k,0} = x_{k-1}, x_{k,N} = x_k$. Now the collocation (or pseudo spectral) scheme determines the polynomial functions $u_{k,N} \in \mathbb{P}_N$ such that

$$\begin{aligned} u'_{k,N}(x_{k,i}) &= f(x_{k,i}),\ i = 0,\ldots,N,\ k = 1,\ldots,K, && (6)\\ u_{k+1,N}(x_k) - u_{k,N}(x_k) &= c_k,\ k = 1,\ldots,K-1, && (7)\\ u_{1,N}(0) &= c_0. && (8) \end{aligned}$$

These are $K(N+1)$ conditions of collocation for the same number of unknowns. The K initial and interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. In the following we write the spectral system in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T = \left(\cos(k\frac{(N-i)\pi}{N})\right),\ i,k = 0,\ldots,N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\hat{D} = (\hat{d}_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$\hat{d}_{i,j} = \begin{cases} \frac{2j}{c_i},\ j = i+1, i+3,\ldots,N \\ 0, \qquad \text{else} \end{cases}$$

and

$$c_i \;=\; \begin{cases} 2, \; i = 0, \\ 1, \;\; \text{else.} \end{cases}$$

Now we are able to write the spectral derivative matrix D for the first derivative. It is explicitly given by

$$D = T\hat{D}T^{-1} \in \mathrm{IR}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations. Because of the above transform the derivative operator on Ω_k is given by

$$D_k = d_k D, \quad d_k = 2/(x_k - x_{k-1}), \;\; k = 1, \ldots, K.$$

Furthermore we have to introduce the operator which represents the inital and interface conditions. We define vectors $B^-, B^+ \in \mathrm{IR}^{1,N+1}$ as follows

$$B^- = (0, \ldots, 0, 1), \;\; B^+ = (1, 0, \ldots, 0).$$

Now it is an easy task to write the spectral system. For instance, for $K = 4$ it reads as follows

$$\begin{bmatrix} D_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 \\ 0 & 0 & D_3 & 0 \\ 0 & 0 & 0 & D_4 \\ B^+ & 0 & 0 & 0 \\ -B^- & B^+ & 0 & 0 \\ 0 & -B^- & B^+ & 0 \\ 0 & 0 & -B^- & B^+ \end{bmatrix} \begin{bmatrix} u_{1,N} \\ u_{2,N} \\ u_{3,N} \\ u_{4,N} \end{bmatrix} = \begin{bmatrix} f_{1,N} \\ f_{2,N} \\ f_{3,N} \\ f_{4,N} \\ c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{9}$$

Here $f_{k,N} \in \mathrm{IR}^{N+1}$ denote the discrete forces evaluated in $\overline{\Omega}_k$ for $k = 1, \ldots, 4$. Clearly, the above system is overdetermined with a spectral matrix ranging in $\mathrm{IR}^{K(N+2),K(N+1)}$. Two different techniques of solution are considered. In the direct approach we first solve the problem $D_1 u_{1,N} = f_{1,N}$ with the singular matrix D_1; it is only determined up to an arbitrary constant which is provided by the inflow condition $B^+ u_{1,N} = c_0$. Then we solve the equation $D_2 u_{2,N} = f_{2,N}$ where the constant is now provided by the interface condition

$B^+u_{2,N} = c_1 + B^+u_{1,N}$. This procedure is continued until the Kth element is reached. The resulting scheme is called the 'upstream' scheme. We compare these results to the 'downstream' scheme where we start with the initial value in $x = L$ and successively solve the above equations from Ω_K down to Ω_1. An alternative to the above approaches results from least-squares. The linear system becomes symmetric positive definite which allows the use of efficient iterative solvers like the conjugate gradient method.

The convergence rates of the different spectral collocation schemes is demonstrated by means of a model problem introduced by Gerritsma and Proot [4]. Here the right hand side f is given by

$$f(x) = \exp(x)\cos(x^2) - 2x\exp(x)\sin(x^2) \tag{10}$$

with the exact solution

$$u(x) = \exp(x)\cos(x^2) - 1. \tag{11}$$

We numerically calculated the discrete L^2-error norms for $K = 4$ spectral elements with interface points $x_k = k, k = 1, 2, 3, 4$. Different polynomial degrees $N = 4, 8, 12, 16, 20, 24$ are considered. A jump at $x_2 = 2$ has been imposed to $c_2 = 25$ (see figure I). The numerical results for the 'upstream' and 'downstream' schemes are presented in the tables I, II. The results for the 'upstream' scheme are also plotted in figures II. For the 'upstream' scheme we observe exponential convergence which is only slightly disturbed by rounding errors. For the 'downstream' scheme the accuracy is somewhat reduced due to the fact that we first solve on the elements where the solution is less smooth. The loss of regularity is transported over the interface conditions onto all elements. From the numerical results it becomes clear that 'downstream' resp. 'upstream' information strongly influences 'upstream' resp. 'downstream' solution. The results for the least squares approach with 'upstream' discretization are given in the tables III, IV. Here different scalings of the interface conditions are compared. For the scaling factor $sc = 1$ the interface conditions are fulfilled up to an accuracy of about 10^{-10} whereas for the factor $sc = 100$ we obtain an accuracy of about $10^{-12} - 10^{-13}$. Furthermore the discrete L^2-errors are somewhat reduced for increasing scaling factors. The discontinuous least-squares method captures the jump very well; something that would not be possible with the continuous least-squares method.

Due to the increasing oscillation of u the errors become larger for increasing k. The condition number which is already high for spectral methods becomes prohibitively high for least-squares, since this method roughly squares the condition number. Since the condition number for the first spectral derivative behaves as $O(N^2)$ now a behavior of $O(N^4)$ can be expected. This is the reason why for increasing N the results are much worse compared to the 'upstream' solver. However, for moderate N we observe an exponential rate of convergence for the spectral collocation scheme.

3 Singular perturbation problem

Here we consider the singular perturbation problem

$$\begin{aligned} -\epsilon u'' + u' &= 0 \quad in\ \Omega = (0,1), && (12)\\ u(0) &= 0, && (13)\\ u(1) &= 1. && (14) \end{aligned}$$

The exact solution is given by

$$u(x) = \frac{e^{(x-1)/\epsilon} - e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}. \qquad (15)$$

Here $0 < \epsilon << 1$ denotes a small perturbation parameter. Figure II displays the solutions for $\epsilon = 0.1, 0.05, 0.02, 0.01$. In $x = 1$ we observe a boundary layer of the width $O(\epsilon)$. For small ϵ the solution behaves as

$$u(x) \cong e^{(x-1)/\epsilon}.$$

For a resolution of the layer the domain Ω has to be decomposed into K subdomains $\Omega_k = (x_{k-1}, x_k),\ k = 1, \ldots, K$ with

$$0 = x_0 < x_1 < x_2 < \ldots < x_{K-1} < x_K = 1.$$

Since we consider a second-order problem we require C^1-continuity at the interface points x_k. Now problem (12)-(14) can be decomposed as follows:

Find functions $u_k, k = 1, \ldots, K$ defined on $\overline{\Omega}_k$ such that

$$\begin{aligned}
-\epsilon u_k'' + u_k' &= 0 \;\; in\ \Omega_k,\ k = 1, \ldots, K, && (16)\\
u_{k+1}(x_k) - u_k(x_k) &= 0,\ k = 1, \ldots, K-1, && (17)\\
u_{k+1}'(x_k) - u_k'(x_k) &= 0,\ k = 1, \ldots, K-1, && (18)\\
u_1(0) &= 0,\ u_K(1) = 1. && (19)
\end{aligned}$$

Now the collocation (or pseudo spectral) scheme determines the polynomial functions $u_{k,N} \in \mathbb{P}_N$ such that

$$\begin{aligned}
-\epsilon u_{k,N}''(x_{k,i}) + u_{k,N}'(x_{k,i}) &= 0,\ i = 0, \ldots, N,\ k = 1, \ldots, K, && (20)\\
u_{k+1,N}(x_k) - u_{k,N}(x_k) &= 0,\ k = 1, \ldots, K-1, && (21)\\
u_{k+1,N}'(x_k) - u_{k,N}'(x_k) &= 0,\ k = 1, \ldots, K-1, && (22)\\
u_{1,N}(0) &= 0,\ u_{K,N}(1) = 1. && (23)
\end{aligned}$$

These are $K(N+1)$ conditions of collocation for the same number of unknowns. The $2K$ boundary and interface conditions lead to an overdetermined system which can be efficiently solved by least-squares. The stabilizing effect of such an approach was already observed by Eisen and Heinrichs [2]. In the following we write the spectral system in matrix notation. The singularly perturbed differential operator on Ω_k is now given by

$$D_{\epsilon,k} = -\epsilon D_k^2 + D_k,\ k = 1, \ldots, K.$$

Furthermore we have to introduce the operators which represents the boundary and interface conditions. The vectors $D_k^-, D_{k+1}^+ \in \mathbb{R}^{1,N+1}$ represent the spectral derivatives of u_k, u_{k+1} in x_k. Hence D_k^- corresponds to the last row of D_k and D_{k+1}^+ to the first row of D_{k+1}. Now it is an easy task to write the spectral system. For instance, for $K = 4$ it reads as follows

$$\begin{bmatrix} D_{\epsilon,1} & 0 & 0 & 0 \\ 0 & D_{\epsilon,2} & 0 & 0 \\ 0 & 0 & D_{\epsilon,3} & 0 \\ 0 & 0 & 0 & D_{\epsilon,4} \\ B^+ & 0 & 0 & 0 \\ -B^- & B^+ & 0 & 0 \\ -D_1^- & D_2^+ & 0 & 0 \\ 0 & -B^- & B^+ & 0 \\ 0 & -D_2^- & D_3^+ & 0 \\ 0 & 0 & -B^- & B^+ \\ 0 & 0 & -D_3^- & D_4^+ \\ 0 & 0 & 0 & B^- \end{bmatrix} \begin{bmatrix} u_{1,N} \\ u_{2,N} \\ u_{3,N} \\ u_{4,N} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{24}$$

Clearly, the above system is overdetermined with a spectral matrix ranging in $\mathbb{R}^{K(N+3),K(N+1)}$. It can efficiently be solved by least-squares which leads to a symmetric and positive definite system.

An 'upstream' solution technique as for the first order problem cannot be employed since we consider elliptic problems. The convergence rates of the least-squares spectral collocation scheme are demonstrated by a numerical simulation. In order reduce the condition number of the above system we recommend scaling $D_{\epsilon,k}$ by $1/d_k$ and the C^1-interface operators D_k^-, D_{k+1}^+ by $1/d_{k+1}$ We numerically calculated the discrete L^2-error norms for $K = 2$ spectral elements with the interface point $x_1 = 1 - \beta\epsilon$. From numerical experiments of Heinrichs in [6], [7] we observe that $\beta = N$ yields a good choice. Different polynomial degrees $N = 4, 8, 12, 16, 20, 24$ are considered. We fix $\epsilon = 10^{-6}$. The corresponding numerical results are presented in table V and figure III. Since the lines in figure III are nearly straight we observe a nearly exponential convergence. Clearly, the strong boundary layer in the right element disturbs the spectral convergence somewhat. But we think that for such kind of singularity we still obtain a very good convergence behavior. Obviously, in the smooth part of the solution (element 1) the constant approximation $u_{1,0} \equiv 0$ yields a better approximation (except at the interface point) but this leads to a much worse approximation in the boundary layer region (element 2). The decomposition of the domains is chosen such that the error on both elements is well-balanced. A different weighting of the jump conditions leads to worse approximation results in the boundary layer region. Here some refinement (either h or N) becomes necessary. This was confirmed by many numerical experiments (see [6, 7]). The influence of scaling the interface conditions is further studied for a moderate perturbation parameter of $\epsilon = 0.02$. We use an decomposition into two uniform elements. Different scaling factors $sc = 100, 1, 0.01$ are considered. In table VI we present the discrete L^2-errors e_1, e_2 in both elements. Table VII contains the values of $u_{1,N}$ and $u_{2,N}$ in $x = 0.5$ and hence shows the jumps. Obviously, for $sc = 100$ the solution seems to be continuous whereas for the other scaling factors we observe jumps at the interface. This becomes more obvious by our plots (Fig. IV - VIII for $N = 4, 8, 12, 16, 20$). For small scaling factors the error in the smooth part of the solution (also for small N) is minimized at the expense that significant jumps are introduced. The accuracy in the second element is somewhat disturbed which is a clear indicator that h-refinement is necessary. On the other hand, for large scaling factors we observe a fast global convergence for increasing N (see Fig. IX). A good compromise between both approaches is reached for the scaling factor $sc = 1$. These results are also confirmed after one step of h-refinement where the second element is subdivided into two uniform elements. Hence in $x = 0.75$ a new interface point is introduced. From table VIII and Fig. X,XI,XII for $N = 4, 8, 12$ we once more observe jumps at the interfaces for $sc = 1$ whereas for $sc = 100$ the approximation looks continuous. Once more the smooth approximation yields a faster convergence for increasing N. On the other hand, for small N scaling with $sc = 1$ yields a better approximation of the smooth part of the solution. From our numerical results we recommend a scaling factor of $sc = 1$.

Conclusion

A highly accurate least-squares spectral collocation scheme for discontinuous and singular perturbation problems is presented. The original domain is decomposed into several subdomains. The spectral collocation conditions and the interface conditions lead to an overdetermined system which is efficiently solved by least-squares. The solution technique only involves symmetric positive definite algebraic systems which allow the use of equal order interpolation polynomials in all subdomains. Robust iterative solvers such as the conjugate gradient method can be employed. The discontinuous solution is approximated with the usual high spectral order. For singular perturbation problems a stabilized least-squares scheme is obtained. The boundary layer is well resolved.

Acknowledgement

The author wishes to acknowledge helpful comments of the referee.

TABLES

N	$\|\|u_1 - u_{1,N}\|\|_{L^2}$	$\|\|u_2 - u_{2,N}\|\|_{L^2}$	$\|\|u_3 - u_{3,N}\|\|_{L^2}$	$\|\|u_4 - u_{4,N}\|\|_{L^2}$
4	$1.89{\cdot}10^{-3}$	$5.02{\cdot}10^{-2}$	$8.68{\cdot}10^{-1}$	$9.84{\cdot}10^{0}$
8	$4.19{\cdot}10^{-7}$	$3.69{\cdot}10^{-5}$	$9.82{\cdot}10^{-4}$	$2.76{\cdot}10^{-2}$
12	$5.91{\cdot}10^{-11}$	$8.71{\cdot}10^{-9}$	$5.39{\cdot}10^{-7}$	$1.19{\cdot}10^{-5}$
16	$4.82{\cdot}10^{-15}$	$8.30{\cdot}10^{-13}$	$5.73{\cdot}10^{-11}$	$1.08{\cdot}10^{-8}$
20	$2.65{\cdot}10^{-16}$	$1.85{\cdot}10^{-15}$	$1.69{\cdot}10^{-14}$	$1.08{\cdot}10^{-12}$
24	$2.12{\cdot}10^{-16}$	$1.16{\cdot}10^{-15}$	$9.90{\cdot}10^{-15}$	$5.23{\cdot}10^{-14}$

Table I. Numerical results for example (11), 'upstream' scheme.

N	$\|\|u_1 - u_{1,N}\|\|_{L^2}$	$\|\|u_2 - u_{2,N}\|\|_{L^2}$	$\|\|u_3 - u_{3,N}\|\|_{L^2}$	$\|\|u_4 - u_{4,N}\|\|_{L^2}$
4	$1.53{\cdot}10^{1}$	$1.37{\cdot}10^{1}$	$1.29{\cdot}10^{1}$	$9.20{\cdot}10^{0}$
8	$1.72{\cdot}10^{-2}$	$1.62{\cdot}10^{-2}$	$1.62{\cdot}10^{-2}$	$2.69{\cdot}10^{-2}$
12	$1.61{\cdot}10^{-5}$	$1.55{\cdot}10^{-5}$	$1.52{\cdot}10^{-5}$	$1.18{\cdot}10^{-5}$
16	$3.76{\cdot}10^{-9}$	$3.65{\cdot}10^{-9}$	$3.70{\cdot}10^{-9}$	$1.08{\cdot}10^{-8}$
20	$1.50{\cdot}10^{-12}$	$1.46{\cdot}10^{-12}$	$1.47{\cdot}10^{-12}$	$1.11{\cdot}10^{-12}$
24	$5.02{\cdot}10^{-14}$	$4.94{\cdot}10^{-14}$	$4.83{\cdot}10^{-14}$	$4.21{\cdot}10^{-14}$

Table II. Numerical results for example (11), 'downstream' scheme.

N	$\|\|u_1 - u_{1,N}\|\|_{L^2}$	$\|\|u_2 - u_{2,N}\|\|_{L^2}$	$\|\|u_3 - u_{3,N}\|\|_{L^2}$	$\|\|u_4 - u_{4,N}\|\|_{L^2}$
4	$2.51{\cdot}10^{-3}$	$5.19{\cdot}10^{-2}$	$1.29{\cdot}10^{0}$	$1.30{\cdot}10^{1}$
8	$5.22{\cdot}10^{-7}$	$4.22{\cdot}10^{-5}$	$9.87{\cdot}10^{-4}$	$3.09{\cdot}10^{-2}$
12	$3.96{\cdot}10^{-10}$	$1.12{\cdot}10^{-8}$	$6.43{\cdot}10^{-7}$	$1.64{\cdot}10^{-5}$
16	$5.56{\cdot}10^{-10}$	$1.10{\cdot}10^{-9}$	$1.55{\cdot}10^{-9}$	$1.18{\cdot}10^{-8}$
20	$2.34{\cdot}10^{-10}$	$4.76{\cdot}10^{-10}$	$6.01{\cdot}10^{-10}$	$1.22{\cdot}10^{-9}$
24	$9.16{\cdot}10^{-10}$	$1.81{\cdot}10^{-9}$	$2.77{\cdot}10^{-9}$	$2.48{\cdot}10^{-9}$

Table III. Numerical results for example (11) with least squares, sc = 1.

N	$\|\|u_1 - u_{1,N}\|\|_{L^2}$	$\|\|u_2 - u_{2,N}\|\|_{L^2}$	$\|\|u_3 - u_{3,N}\|\|_{L^2}$	$\|\|u_4 - u_{4,N}\|\|_{L^2}$
4	$2.51{\cdot}10^{-3}$	$5.19{\cdot}10^{-2}$	$1.29{\cdot}10^{0}$	$1.30{\cdot}10^{1}$
8	$5.22{\cdot}10^{-7}$	$4.22{\cdot}10^{-5}$	$9.87{\cdot}10^{-4}$	$3.09{\cdot}10^{-2}$
12	$8.34{\cdot}10^{-11}$	$1.07{\cdot}10^{-8}$	$6.42{\cdot}10^{-7}$	$1.64{\cdot}10^{-5}$
16	$2.74{\cdot}10^{-11}$	$6.25{\cdot}10^{-11}$	$4.67{\cdot}10^{-11}$	$1.12{\cdot}10^{-8}$
20	$9.23{\cdot}10^{-12}$	$2.22{\cdot}10^{-11}$	$4.28{\cdot}10^{-11}$	$7.54{\cdot}10^{-11}$
24	$3.30{\cdot}10^{-11}$	$7.67{\cdot}10^{-11}$	$9.52{\cdot}10^{-11}$	$7.98{\cdot}10^{-11}$

Table IV. Numerical results for example (11) with least squares, sc = 100.

N	$\|\|u_1 - u_{1,N}\|\|_{L^2}$	$\|\|u_2 - u_{2,N}\|\|_{L^2}$
4	$8.87{\cdot}10^{-2}$	$1.26{\cdot}10^{-1}$
8	$2.81{\cdot}10^{-4}$	$2.43{\cdot}10^{-4}$
12	$1.69{\cdot}10^{-6}$	$8.47{\cdot}10^{-6}$
16	$2.77{\cdot}10^{-8}$	$2.04{\cdot}10^{-7}$
20	$4.72{\cdot}10^{-10}$	$5.17{\cdot}10^{-9}$
24	$4.38{\cdot}10^{-10}$	$1.19{\cdot}10^{-9}$

Table V. Numerical results for example (15), $\epsilon = 10^{-6}$.

	$sc = 100.$		$sc = 1.$		$sc = 0.01$	
N	e_1	e_2	e_1	e_2	e_1	e_2
4	$3.70{\cdot}10^{-1}$	$7.14{\cdot}10^{-1}$	$4.76{\cdot}10^{-2}$	$9.55{\cdot}10^{-1}$	$5.47{\cdot}10^{-6}$	$9.94{\cdot}10^{-1}$
8	$2.63{\cdot}10^{-1}$	$5.34{\cdot}10^{-1}$	$2.39{\cdot}10^{-2}$	$9.17{\cdot}10^{-1}$	$3.25{\cdot}10^{-6}$	$9.57{\cdot}10^{-1}$
12	$2.27{\cdot}10^{-3}$	$4.86{\cdot}10^{-3}$	$2.02{\cdot}10^{-3}$	$1.11{\cdot}10^{-1}$	$9.32{\cdot}10^{-5}$	$9.45{\cdot}10^{-1}$
16	$2.14{\cdot}10^{-6}$	$1.25{\cdot}10^{-5}$	$2.15{\cdot}10^{-6}$	$7.87{\cdot}10^{-5}$	$5.93{\cdot}10^{-5}$	$4.09{\cdot}10^{-1}$
20	$8.31{\cdot}10^{-9}$	$7.61{\cdot}10^{-8}$	$8.34{\cdot}10^{-9}$	$8.02{\cdot}10^{-8}$	$2.13{\cdot}10^{-8}$	$9.38{\cdot}10^{-5}$
24	$9.93{\cdot}10^{-11}$	$7.00{\cdot}10^{-11}$	$1.18{\cdot}10^{-10}$	$1.11{\cdot}10^{-8}$	$1.44{\cdot}10^{-8}$	$1.05{\cdot}10^{-4}$

Table VI. Numerical results for example (15), L^2-errors, 2 elements, $\epsilon = 0.02$.

	$sc = 100.$		$sc = 1.$		$sc = 0.01$	
N	u_1	u_2	u_1	u_2	u_1	u_2
4	$5.01\cdot10^{-1}$	$5.01\cdot10^{-1}$	$6.45\cdot10^{-2}$	$9.36\cdot10^{-1}$	$7.40\cdot10^{-6}$	$1.00\cdot10^{0}$
8	$3.83\cdot10^{-1}$	$3.84\cdot10^{-1}$	$3.51\cdot10^{-2}$	$9.44\cdot10^{-1}$	$6.06\cdot10^{-6}$	$1.00\cdot10^{0}$
12	$3.62\cdot10^{-3}$	$3.64\cdot10^{-3}$	$3.26\cdot10^{-3}$	$1.16\cdot10^{-1}$	$2.66\cdot10^{-4}$	$9.99\cdot10^{-1}$
16	$5.57\cdot10^{-6}$	$5.58\cdot10^{-6}$	$5.60\cdot10^{-6}$	$8.28\cdot10^{-5}$	$1.77\cdot10^{-4}$	$4.36\cdot10^{-1}$
20	$2.56\cdot10^{-8}$	$2.56\cdot10^{-8}$	$2.57\cdot10^{-8}$	$3.80\cdot10^{-8}$	$6.57\cdot10^{-8}$	$1.00\cdot10^{-4}$
24	$6.99\cdot10^{-11}$	$7.00\cdot10^{-11}$	$2.56\cdot10^{-10}$	$1.18\cdot10^{-8}$	$4.51\cdot10^{-8}$	$1.12\cdot10^{-4}$

Table VII. Numerical results for example (15), jumps, 2 elements, $\epsilon = 0.02$.

	$sc = 100.$			$sc = 1.$		
N	e_1	e_2	e_3	e_1	e_2	e_3
4	$4.96\cdot10^{-1}$	$8.49\cdot10^{-1}$	$8.61\cdot10^{-1}$	$2.59\cdot10^{-2}$	$5.74\cdot10^{-1}$	$9.57\cdot10^{-1}$
8	$2.32\cdot10^{-2}$	$4.10\cdot10^{-2}$	$4.33\cdot10^{-2}$	$8.37\cdot10^{-3}$	$3.54\cdot10^{-1}$	$6.06\cdot10^{-1}$
12	$1.22\cdot10^{-6}$	$3.51\cdot10^{-6}$	$8.94\cdot10^{-6}$	$1.23\cdot10^{-6}$	$7.78\cdot10^{-5}$	$1.37\cdot10^{-4}$
16	$8.07\cdot10^{-11}$	$2.21\cdot10^{-9}$	$9.41\cdot10^{-9}$	$2.46\cdot10^{-11}$	$2.10\cdot10^{-9}$	$8.13\cdot10^{-9}$
20	$2.81\cdot10^{-10}$	$4.98\cdot10^{-10}$	$5.20\cdot10^{-10}$	$7.00\cdot10^{-11}$	$7.22\cdot10^{-9}$	$1.26\cdot10^{-8}$
24	$5.32\cdot10^{-10}$	$9.44\cdot10^{-10}$	$9.64\cdot10^{-10}$	$6.67\cdot10^{-10}$	$8.22\cdot10^{-8}$	$1.44\cdot10^{-7}$

Table VIII. Numerical results for example (15), L^2-errors, 3 elements, $\epsilon = 0.02$.

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