Least-Squares Spectral Collocation
for the Stokes Problem

by

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Abstract

A least-squares spectral collocation formulation for the Stokes problem is presented. By this new approach the well known *Babuška – Brezzi* condition can be avoided. Here we are able to employ polynomials of the same degree both for the velocity components and for the pressure. The collocation conditions and the boundary conditions lead to a overdetermined system which can be efficiently solved by least-squares. The solution technique will only involve symmetric positive definite linear systems. The numerical simulations confirm the usual exponential rate of convergence for the spectral scheme.

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1 Introduction

Spectral methods (see, e.g., Canuto et al. [3] or Gottlieb and Orszag [7], [14]) employ global polynomials for the discretization of the Stokes problems. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. For analytical data exponential convergence can be achieved. For the Stokes problem the velocity and the pressure cannot be approximated independently due to the well known $Babuška - Brezzi$ condition. If the velocity and the pressure are approximated by polynomials of the same degree eight spurious modes are introduced which lead to an instable system (see Bernardi, Canuto and Maday [1]). A well-known compatible approximating velocity-pressure pair is the so-called $\mathbb{P}_N \times \mathbb{P}_{N-2}$ formulation of Bernardi, Maday [2] and Rønquist [16]. Heinrichs [8] employed this technique for the splitting of the Stokes equations. Here the velocity components are approximated by polynomials in $\mathbb{P}_N$ and the pressure by two degrees lower order polynomials in $\mathbb{P}_{N-2}$. The resulting discrete system constitutes a saddle point problem which is difficult to solve numerically. Least-squares techniques offer theoretical and numerical advantages over the classical methods. In particular, the least-squares formulation of the Stokes [4, 11, 12] and Navier-Stokes equations [10, 13] leads to symmetric and positive definite algebraic systems which circumvent the LBB stability condition. Furthermore it has the following advantages:

- equal order interpolation polynomials can be employed
- iterative solvers for positive definite systems (e.g. conjugate gradient methods) are available
- efficient preconditioners can be derived, see [9] for elliptic problems
- improved stability properties for convection-diffusion problems (see [5]) and the Navier-Stokes equations
- implementation and parallelization is straightforward.

Least-squares spectral element methods were already investigated by Proot and Gerritsma [15]. Here we consider a least-squares collocation (or pseudospectral) scheme. We employ an approximation with Chebyshev polynomials and Chebyshev Gauss-Lobatto nodes. This allows the efficient implementation with Fast Fourier Transforms (FFTs). First the Stokes problem is rewritten in a first-order velocity-vorticity-pressure form. All functions are approximated by equal-order Chebyshev polynomials of degree N in each variable. Collocation is performed in all collocation nodes including the boundary. Hence we obtain $4(N+1)^2$ conditons of collocation for the same number of unknowns. By further requiring boundary conditions on the velocity components the system becomes overdetermined. After a suitable scaling of the boundary conditions we numerically solve the corresponding least-squares problem by a conjugate gradient method. By numerical experiments we observe the high order of the spectral scheme. For analytical data exponential convergence can be achieved.

The paper is organized as follows. In Section 2, the velocity-vorticity-pressure form of the Stokes problem is derived. In the subsequent section the least-squares spectral collocation scheme is introduced. The results of the numerical simulation are discussed in Section 4. Finally a conclusion is presented.

2 The First-Order Formulation

In order to apply least-squares the Stokes problem is transformed into an equivalent first-order system of partial differential equations. This is accomplished by introducing the vorticity ω as an auxiliary variable. By using the identity

$$\nabla \times \nabla \times \mathbf{u} = -\Delta \mathbf{u} + \nabla(\nabla \cdot \mathbf{u})$$

and by using the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$ we obtain

$$\begin{aligned} \nabla p + \nu \nabla \times \omega &= \mathbf{f} \;\; in\ \Omega &(1)\\ \nabla \cdot \mathbf{u} &= 0 \;\; in\ \Omega &(2)\\ \omega - \nabla \times \mathbf{u} &= 0 \;\; in\ \Omega &(3) \end{aligned}$$

where $\mathbf{u}^T = [u_1, u_2]$ denotes the velocity vector, p the pressure, $\mathbf{f}^T = [f_1, f_2]$ the forcing term and ν the kinematic viscosity. Here it is assumed that the density equals unity. In two dimensions the above system (1)-(3) contains four equations for four unknown functions. It can explicitly be written as

$$\begin{bmatrix} 0 & 0 & \nu\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \\ 0 & 0 & -\nu\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 1 & 0 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \omega \\ p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix} \quad in\ \Omega. \tag{4}$$

3 The Spectral Collocation Scheme

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N \quad = \quad \{Polynomials\ of\ degree\ \leq N\ in\ both\ variables\ x_1, x_2\}.$$

Now all unknown functions are approximated by polynomials of the same degree N, i.e. u_1, u_2, ω, p are approximated by interpolating polynomials $u_{1,N}, u_{2,N}, \omega_N, p_N \in \mathbb{P}_N$. Furthermore we have to introduce the standard Chebyshev Gauss-Lobatto collocation nodes. They are explicitly given by

$$(x_i, y_j) = \left(\cos\frac{i\pi}{N}, \cos\frac{j\pi}{N}\right), \ \ i, j = 0, \ldots, N.$$

We further introduce the discrete set

$$\overline{\Omega}_N = \{(x_i, y_j) : \ i, j = 0, \ldots, N\}.$$

Now the collocation (or pseudo spectral) scheme determines the polynomial functions $u_{1,N}, u_{2,N}, \omega_N, p_N \in \mathbb{P}_N$ such that (4) holds in $\overline{\Omega}_N$. These are $4(N+1)^2$ conditions of collocation for the same number of unknowns. Furthermore we have to enforce discrete boundary conditions for the velocity components. For each component we have $4N$ boundary conditions resulting in $8N$ conditions. Hence the complete system is overdetermined and least-squares techniques become necessary. In the following we write the spectral system in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T \quad = \quad \left(\cos(k\frac{i\pi}{N})\right), \;\; i,k = 0,\ldots,N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitly given by $\hat{D} = (\hat{d}_{i,j}) \in \mathrm{IR}^{N+1,N+1}$ with

$$\hat{d}_{i,j} \quad = \quad \begin{cases} \frac{2j}{c_i}, \; j = i+1, i+3, \ldots, N \\ 0\,, \qquad\qquad \text{else} \end{cases}$$

and

$$c_i \quad = \quad \begin{cases} 2, \; i = 0, \\ 1, \;\; \text{else.} \end{cases}$$

Now we are able to write the spectral derivative matrix D for the first derivative. It is explicitly given by

$$D = T\hat{D}T^{-1} \in \mathrm{IR}^{N+1,N+1}.$$

The spectral operator can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations. We further introduce the identity matrix $I \in \mathrm{IR}^{N+1,N+1}$. By tensor product representation

$$A \otimes B = (Ab_{i,j})_{i,j}$$

we are now able to write the spectral partial derivatives:

$$\frac{\partial}{\partial x_1} \cong D_1 = D \otimes I, \;\; \frac{\partial}{\partial x_2} \cong D_2 = I \otimes D.$$

The 2D spectral operators can be efficiently evaluated by FFTs in $O(N^2 \log N)$ arithmetic operations. Furthermore we have to introduce the boundary operator $B = (b_{i,j}) \in \mathrm{IR}^{4N,(N+1)^2}$ which represents the boundary conditions. The first index i numbers the boundary nodes where a lexicographic numbering beginning from the left lower corner to the right upper corner is used. The second index j runs through all nodes of $\overline{\Omega}_N$ in the same manner. The entries of B are given by

$$b_{i,j} = \begin{cases} 1, & \textit{when } (i,j) \textit{ belongs to a boundary point} \\ 0, & \textit{else.} \end{cases}$$

For instance, for $N = 2$ we obtain the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{IR}^{8,9}.$$

The boundary conditions lead to an overdetermined system. Since we want to consider different scalings of boundary conditions against collocation conditions we further introduce a scaling factor α. In the numerical simulation we present suitable choices. Now it is an easy task to write the spectral system. It is given by

$$\begin{bmatrix} 0 & 0 & \nu D_2 & D_1 \\ 0 & 0 & -\nu D_1 & D_2 \\ D_2 & -D_1 & I \otimes I & 0 \\ D_1 & D_2 & 0 & 0 \\ \alpha B & 0 & 0 & 0 \\ 0 & \alpha B & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,N} \\ u_{2,N} \\ \omega_N \\ p_N \end{bmatrix} = \begin{bmatrix} f_{1,N} \\ f_{2,N} \\ 0 \\ 0 \\ \alpha b_{1,N} \\ \alpha b_{2,N} \end{bmatrix}. \tag{5}$$

Here $f_{1,N}, f_{2,N} \in \mathrm{IR}^{(N+1)^2}$ denote the discrete forces evaluated in $\overline{\Omega}_N$ and $b_{1,N}, b_{2,N} \in \mathrm{IR}^{4N}$ denote the discrete boundary values of the velocity components u_1, u_2.

4 Numerical simulation

The convergence rates of the least-squares spectral collocation scheme is demonstrated by means of a model problem introduced by Gerritsma-Phillips

[6] with $\nu = 1$. The exact velocity components and the pressure are defined on the square $\Omega = (-1,1) \times (-1,1)$ by

$$u_1(x,y) = -\sin(\pi x)\cos(\pi y) \tag{6}$$
$$u_2(x,y) = \cos(\pi x)\sin(\pi y) \tag{7}$$

and

$$p(x,y) = \sin(\pi x)\sin(\pi y). \tag{8}$$

This exact solution satisfies the Stokes equations if the following forcing term is used

$$\mathbf{f} = \begin{pmatrix} \pi\cos(\pi x)\sin(\pi y) - 2\pi^2\sin(\pi x)\cos(\pi y) \\ \pi\sin(\pi x)\cos(\pi y) + 2\pi^2\cos(\pi x)\sin(\pi y) \end{pmatrix}.$$

We numerically calculated the discrete H^1-error norms of the velocity components and the H^0-error norm of the pressure. Before introducing the scaling factor α let us denote by $A_{st} = (a_{i,j})$ the spectral Stokes operator given in (5). The scaling factor should quantify a suitable weighting of the boundary conditions versus the collocation conditions. Here different matrix norms of the matrix belonging to the collocation conditions could be chosen. We define α by means of the L^1-norm, i.e.,

$$\alpha = \max\{\sum_{j=1}^{4(N+1)^2} |a_{i,j}| : \ i = 1, \ldots, 4(N+1)^2\}.$$

The corresponding numerical results are presented in table I. The results for the u_2-velocity component resp. vorticity are not shown since they yield a similar convergence rate as for the u_1-component resp. the pressure. The results for the pressure are about one order less accurate than for the velocity. We observe an exponential rate of convergence for the spectral collocation scheme.

Conclusion

A highly accurate least-squares spectral collocation scheme for the Stokes equations is presented. For this purpose the Stokes problem is transformed into a first-order system. This is done by introducing the vorticity as an auxiliary variable. The spectral collocation conditions and the boundary conditions lead to an overdetermined system which is efficiently solved by least-squares. The solution technique only involves symmetric positive definite algebraic systems which circumvent the $\inf - \sup$ stability condition and subsequently allow the use of equal order interpolation polynomials for all variables. The usual problems with spurious modes are avoided. Robust iterative solvers such as the conjugate gradient method can be employed. The numerical simulations confirm the exponential rate of convergence for the spectral least-squares scheme.

TABLE

N	$\|\|u_1 - u_{1,N}\|\|_{H^1}$	$\|\|p - p_N\|\|_{H^0}$
4	$3.51 \cdot 10^{-1}$	$1.15 \cdot 10^{0}$
8	$1.03 \cdot 10^{-3}$	$1.45 \cdot 10^{-2}$
12	$3.83 \cdot 10^{-7}$	$4.94 \cdot 10^{-6}$
16	$4.12 \cdot 10^{-11}$	$3.65 \cdot 10^{-10}$
20	$4.04 \cdot 10^{-13}$	$8.27 \cdot 10^{-11}$

Table I. Numerical results for example (6)-(8).

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