NONCONFORMING SPECTRAL COLLOCATION

by

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Abstract

Domain decompositions based on patching techniques for nonconforming spectral collocation schemes are considered. On the interface lines continuity of normal derivatives in suitable collocation nodes is required. It is shown that the spectral accuracy is maintained. Global preconditioning with finite differences yields dramatically improved condition numbers. The methods are applied to elliptic problems with cross points and nonconforming interfaces. A domain decomposition technique for the iterative solution of a nonconforming patching is investigated. Numerical results are presented which demonstrate the high accuracy of our spectral collocation schemes.

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Figures: 1

Tables: 6

1 Introduction

Spectral methods (see, e.g., Canuto et al. [?] or Gottlieb and Orszag [?], [?]) employ global polynomials for the discretization of elliptic boundary value problems. Hence they give very accurate approximations for smooth solutions with relatively few degrees of freedom. In order to make these methods applicable for complex geometries or adaptive refinements one has to use domain decomposition methods. Most versions of spectral domain decomposition methods use patching techniques originally proposed by Orszag [?]. Here the original problem is decomposed into subproblems where continuity of the normal derivatives on the interface lines is required. Special problems arise at cross points where more than two spectral elements are matching. Here it is precribed that the sum of the outward normal derivatives over all elements vanish. On each subdomain the solution is approximated by a collocation (or pseudospectral) scheme. Here we employ an approximation with Chebyshev polynomials and Chebyshev Gauss-Lobatto nodes. This allows the efficient implementation with Fast Fourier Transforms (FFTs). The collocation conditions are required in the interior nodes and on the interface lines. For nonconforming spectral elements where one elements matches two refined elements the continuity of normal derivatves can only be enforced when the polynomial degree of the refined elements is reduced to the half of the original elements. In general, we enforce continuity on the interface lines and continuity of the normal derivatives only in the collocation points of the not-refined elements. Here spectral interpolation techniques on the refined elements are necessary. If we would require collocation conditions also on the refined interface points the complete system would be overdetermined. Hence C^1 - continuity cannot be enforced. For different degrees of polynomials the spectral approximation properties of the nonconforming elements are studied. It is shown that the high order of the spectral schemes is maintained. The somewhat reduced smootheness on the interface does not disturb the high spectral accuracy. Similar results were obtained by mortar spectral elements introduced by Maday et al. [?], [?]. Here the interface conditions are given by suitable L^2 - projections which are more appropriate for spectral finite elements introduced by Patera [?].

Since the spectral operators have bad condition numbers increasing as $O(N^4)$ preconditioning becomes necessary. Here we employ global preconditioning based on finite difference schemes. On the interface lines interpolation is required. Numerical results for cross points and nonconforming elements are presented which show the highly improved condition numbers. For the efficient solution we recommend spectral multigrid techniques which were already introduced in [?], [?] and by Zang et al. [?], [?]. For cross point problems spectral multigrid techniques were already successfully tested in [?]. Finally we investigate domain decomposition techniques for the iterative solution of nonconforming patchings. Here we employ a Dirichlet-Neumann interface relaxation (see [?]) which is iterated until continuity of normal derivatives at the collocation nodes is achieved. By numerical results the efficiency of this treatment is demonstrated. With a fixed relaxation parameter convergence is always achieved within a few numbers of iterations.

The paper is organized as follows. First we consider cross point problems. The spectral discretization with Chebyshev collocation is presented. It is shown by numerical experiments that spectral accuracy can be achieved. An efficient finite difference preconditioner is developed. In section 3 these techniques are extended to nonconforming spectral collocation schemes. Global preconditiong with finite differences yields strongly improved condition numbers. Finally in section 4 a domain decomposition technique based on a Dirichlet-Neumann iteration is studied.

2 Cross point problem

We consider second-order elliptic problems given by

$$\begin{aligned} -\Delta u &= f \ in \ \Omega^c = (-1,1)^2, \qquad (1)\\ u &= g \ on \ \Gamma^c = \partial\Omega^c. \qquad (2) \end{aligned}$$

Here f, g denote given functions defined on Ω^c, Γ^c. An adaptive refinement strategy makes it necessary to decompose the original domain Ω^c into four subdomains (see figure I)

$$\begin{aligned} \Omega_1^c &= (0,1)\times(0,1), \quad \Omega_2^c = (0,1)\times(-1,0),\\ \Omega_3^c &= (-1,0)\times(-1,0), \ \Omega_4^c = (-1,0)\times(0,1). \end{aligned}$$

The corresponding interface lines are given by

$$\begin{aligned} \Gamma_{1,2}^c &= (0,1)\times\{0\}, \ \ \Gamma_{2,3}^c = \{0\}\times(-1,0),\\ \Gamma_{3,4}^c &= (-1,0)\times\{0\}, \ \ \Gamma_{1,4}^c = \{0\}\times(0,1). \end{aligned}$$

The Poisson problem (1),(2) can now be decomposed as follows:

$$-\Delta u_k = f \; in \; \Omega_k^c, \tag{3}$$
$$u_k = g \; on \; \partial\Omega_k^c \cap \Gamma^c \; for \; k = 1, \dots, 4. \tag{4}$$

Here u_k defines the restriction of u to $\overline{\Omega_k^c}$. For a smooth decomposition we further require C^1 - interface conditions:

$$\begin{aligned}
u_1 - u_2 = 0, &\quad \frac{\partial}{\partial y}(u_1 - u_2) = 0 \; on \; \Gamma_{1,2}^c, \\
u_2 - u_3 = 0, &\quad \frac{\partial}{\partial x}(u_2 - u_3) = 0 \; on \; \Gamma_{2,3}^c, \\
u_4 - u_3 = 0, &\quad \frac{\partial}{\partial y}(u_4 - u_3) = 0 \; on \; \Gamma_{3,4}^c, \\
u_1 - u_4 = 0, &\quad \frac{\partial}{\partial x}(u_1 - u_4) = 0 \; on \; \Gamma_{1,4}^c.
\end{aligned}$$

Further, one has to define an additional condition in the cross point where the four adjacent subdomains meet. The usual condition in the cross point (see, e.g., [?]) results from summing up the normal derivatives and setting to zero. Hence, we obtain the condition

$$\frac{\partial}{\partial y}(u_1 - u_2) + \frac{\partial}{\partial x}(u_2 - u_3) + \frac{\partial}{\partial y}(u_4 - u_3) + \frac{\partial}{\partial x}(u_1 - u_4) = 0 \; in \; (0,0). \tag{5}$$

2.1 Spectral collocation

For the spectral approximation we introduce the polynomial subspace

$$\mathbb{P}_N = \{Polynomials \; of \; degree \; \leq N \; in \; both \; x \; and \; y\}.$$

The functions $u_k, k = 1, \dots, 4$ are approximated by polynomials $u_{k,N} \in \mathbb{P}_N$. Furthermore we have to introduce the standard Gauss-Lobatto-Chebyshev collocation nodes. They are given by

$$(x_i, y_j) = \left(\cos\frac{i\pi}{N}, \cos\frac{j\pi}{N}\right), \; i, j = 0, \dots, N,$$

The collocation nodes in Ω_k^c are denoted by $(x_i^k, y_j^k), i, j = 0, \ldots, N$ for $k = 1, \ldots, 4$. By using the Gauss-Lobatto nodes they can explicitely be written as

$$\begin{aligned} x_i^1 = x_i^2 = \frac{1}{2}\left(x_i + 1\right), \ y_j^1 = y_j^4 = \frac{1}{2}\left(y_j + 1\right), \ i, j = 0, \ldots, N, \\ x_i^3 = x_i^4 = \frac{1}{2}\left(x_i - 1\right), \ y_j^2 = y_j^3 = \frac{1}{2}\left(y_j - 1\right), \ i, j = 0, \ldots, N. \end{aligned}$$

The collocation (or pseudo spectral) scheme for (3-4) now reads as follows: Find functions $u_{k,N} \in \mathbb{P}_N, k = 1, \ldots, 4$ such that

$$\begin{aligned} -\Delta u_{k,N}(x_i^k, y_j^k) &= f(x_i^k, y_j^k) \ for \ i, j = 1, \ldots, N-1, & (6) \\ u_{k,N}(x_i^k, y_j^k) &= g(x_i^k, y_j^k) \ for \ (x_i^k, y_j^k) \in \partial\Omega_k^c \cap \Gamma^c, \ k = 1, \ldots, 4. & (7) \end{aligned}$$

The interface conditions now read as follows:

$$\begin{aligned} &(u_{1,N} - u_{2,N})(x_i^1, 0) = 0, \quad \frac{\partial}{\partial y}(u_{1,N} - u_{2,N})(x_i^1, 0) = 0 \ for \ i = 1, \ldots, N-1, \\ &(u_{2,N} - u_{3,N})(0, y_j^2) = 0, \quad \frac{\partial}{\partial x}(u_{2,N} - u_{3,N})(0, y_j^2) = 0 \ for \ j = 1, \ldots, N-1, \\ &(u_{3,N} - u_{4,N})(x_i^3, 0) = 0, \quad \frac{\partial}{\partial y}(u_{3,N} - u_{4,N})(x_i^3, 0) = 0 \ for \ i = 1, \ldots, N-1, \\ &(u_{4,N} - u_{1,N})(0, y_j^4) = 0, \quad \frac{\partial}{\partial x}(u_{4,N} - u_{1,N})(0, y_j^4) = 0 \ for \ j = 1, \ldots, N-1. \end{aligned}$$

In the cross point we require continuity, i.e.,

$$(u_{1,N} - u_{2,N})(0,0) = (u_{2,N} - u_{3,N})(0,0) = (u_{3,N} - u_{4,N})(0,0) = 0.$$

This also implies $(u_{4,N} - u_{1,N})(0,0) = 0$ and we further add the cross point condition (5). These are $4(N+1)^2$ conditions of collocation for determining the four unknown functions $u_{k,N}, \ k = 1, \ldots, 4$.

In the following we write the spectral derivatives in matrix notation. First one has to introduce the transformation matrices from physical space to coefficient space. Since we employ a Chebyshev expansion we obtain the following matrix:

$$T = \left(\cos(k\frac{i\pi}{N})\right), \ i, k = 0, \ldots, N.$$

Further we need the differentation matrix in the Chebyshev coefficient space which is explicitely given by $\hat{D} = (d_{i,j}) \in \mathbb{R}^{N+1,N+1}$ with

$$d_{i,j} = \begin{cases} \frac{2j}{c_i}, & j = i+1, i+3, \ldots, N \\ 0\,, & \text{else} \end{cases}$$

and

$$c_i = \begin{cases} 2, & i = 0, \\ 1, & \text{else.} \end{cases}$$

Now we are able to write the spectral derivative matrices $D1_{sp}$ and $D2_{sp}$ for the first and second derivatives. They are explicitely given by

$$D1_{sp} = T\hat{D}T^{-1}, \;\; D2_{sp} = T\hat{D}^2T^{-1}.$$

The spectral operators can be efficiently evaluated by Fast Fourier Transforms (FFTs) in $O(N \log N)$ arithmetic operations. We further introduce the identity matrix $I \in \mathbb{R}^{N+1,N+1}$. By tensor product representation

$$A \otimes B = (Ab_{i,j})_{i,j}$$

we are now able to write the spectral Laplacian

$$\Delta_{sp} = D2_{sp} \otimes I + I \otimes D2_{sp}.$$

Partial derivatives can be discretized as follows:

$$\frac{\partial}{\partial x} \cong D1_{sp} \otimes I, \;\; \frac{\partial}{\partial y} \cong I \otimes D1_{sp}.$$

The 2D spectral operators can be efficiently evaluated by FFTs in $O(N^2 \log N)$ arithmetic operations. Now it is an easy task to implement the above collocation scheme. We numerically calculated the discrete L^2 - error $E2_1$ on Ω_1^c. It is defined by

$$E2_1 = \left(\sum_{i,j=0}^{N} (u_1 - u_{1,N})^2 (x_i^1, y_j^1) \right)^{1/2} /N.$$

For reasons of symmetry the error on the other subdomains behaves similar. We consider an example with a smooth solution given by

$$u(x,y) = \sin(\pi x + \pi/4) \sin(\pi y + \pi/4). \tag{8}$$

Because of the shift $\pi/4$ we have nonhomogeneous boundary conditions. From table I we observe spectral accuracy for increasing N. For $N = 16$ the precision of our machine (double precision) is already achieved. Since we still have conforming elements this behaviour could be expected. These results are later on compared to the accuracy of the nonconforming collocation scheme.

2.2 Finite difference preconditioning

The spectral Laplacian has a condition number increasing as $O(N^4)$ (see [?]). Here we consider global finite difference preconditioning. For second-order derivatives it is well known that the preconditioned spectral operator has real eigenvalues (see [?]) confined to the interval $[1, \pi^2/4]$. Haldenwang et al. [?] have analytically calculated the eigenvalues and confirmed this behaviour. An alternative type of preconditioning is based on finite elements (see Deville and Mund [?]). The Laplace operator is discretized by the standard five-point-star on the Chebyshev-Gauss-Lobatto grid (see [?]). In the 1D-case the second derivative is given by

$$u''(x_i) \cong 2\delta_i \left(\gamma_{i-1}u(x_{i-1}) - (\gamma_i + \gamma_{i-1})u(x_i) + \gamma_i u(x_{i+1})\right)$$

where

$$\delta_i = \frac{1}{x_{i+1} - x_{i-1}},\ \gamma_i = \frac{1}{x_{i+1} - x_i},\ i = 1, \ldots, N-1.$$

This defines the 1D matrix $D2_{fd}$. By tensor product representation the Laplacian can be approximated as before. The discretization is second order accurate. The C^1 - interface conditions are discretized by one-sided finite differences on both sides, respectively. By using continuity we explicitely obtain

$$\frac{1}{h_1}\Big(u_1(x_i^1, h_1) - 2u_1(x_i^1, 0) + u_2(x_i^1, -h_1)\Big) = 0,\ i = 1, \ldots, N-1,$$
$$\frac{1}{h_1}\Big(u_2(h_1, y_j^2) - 2u_2(0, y_j^2) + u_3(-h_1, y_j^2)\Big) = 0,\ j = 1, \ldots, N-1,$$
$$\frac{1}{h_1}\Big(u_3(x_i^3, -h_1) - 2u_3(x_i^3, 0) + u_4(x_i^3, h_1)\Big) = 0,\ i = 1, \ldots, N-1,$$
$$\frac{1}{h_1}\Big(u_4(-h_1, y_j^4) - 2u_4(0, y_j^4) + u_1(h_1, y_j^4)\Big) = 0,\ j = 1, \ldots, N-1.$$

Here the step size h_1 is given by

$$h_1 = \frac{1}{2}\left(1 - \cos\frac{\pi}{N}\right).$$

In the cross point we obtain the condition

$$\frac{1}{h_1}\left(u_1(h_1,0) + u_2(0,-h_1) + u_3(-h_1,0) + u_4(0,h_1) - 4u_1(0,0)\right) = 0$$

which is (after multiplying by $1/h_1$) identical to the five-point formula for the Laplacian. We numerically calculated the eigenvalues of the preconditioned spectral operator. Most of the eigenvalues are real. Only a few have very small imaginary parts which is not relevant for the parameter choice of an iterative scheme. In particular, we were interested in the minimal and maximal eigenvalues λ_{min} and λ_{max}. More precisely, we define:

$$\begin{aligned} \lambda_{min} &= \min\{\mid\lambda\mid:\ \lambda \text{ eigenvalue}\}, \\ \lambda_{max} &= \max\{\mid\lambda\mid:\ \lambda \text{ eigenvalue}\}. \end{aligned}$$

In table I we present these values for $N = 4, 8, 16, 32$. Obviously, $\lambda_{min} = 1$ and λ_{max} approximates $\frac{\pi^2}{4}$ from below.. For the (stationary) Richardson relaxation (see [?], [?]) the optimal relaxation parameter ω_{opt} is now given by

$$\omega_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}} = \frac{2}{1 + \pi^2/4} \cong 0.5768.$$

The corresponding convergence rate ρ is given by

$$\rho = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = \frac{\pi^2/4 - 1}{\pi^2/4 + 1} \cong 0.4232.$$

Further improvements in convergence speed can be achieved by Chebyshev acceleration (see [?]).

3 Nonconforming elements

Now we add a further subdomain to Ω^c and consider elliptic problems on

$$\Omega = \Omega^c \cup \Omega_0,\ \ \Omega_0 = (1,3) \times (-1,1).$$

Between the refined subdomain and Ω_0 we introduce the new interface line

$$\Gamma_0 = \Gamma_{0,1} \cup \Gamma_{0,2} \cup \{(1,0)\}$$

where

$$\Gamma_{0,1} = \{1\} \times (0,1), \;\; \Gamma_{0,2} = \{1\} \times (-1,0).$$

Here $(1,0)$ defines a further cross point where the three subdomains $\Omega_1^c, \Omega_2^c, \Omega_0$ intersect. The equations (3),(4) are now added by an elliptic equation in Ω_0:

$$\begin{aligned} -\Delta u_0 &= f \; in \; \Omega_0, && (9) \\ u_0 &= g \; on \; \partial\Omega_0 - \Gamma_0. && (10) \end{aligned}$$

Here u_0 defines the restriction of u to $\overline{\Omega_0}$. For a smooth decomposition we further require C^1 - interface conditions:

$$\begin{aligned} u_1 - u_0 = 0, \;\; \frac{\partial}{\partial x}(u_1 - u_0) = 0 \; on \; \Gamma_{0,1}, \\ u_2 - u_0 = 0, \;\; \frac{\partial}{\partial x}(u_2 - u_0) = 0 \; on \; \Gamma_{0,2}. \end{aligned}$$

In the cross point we require continuity of normal derivatives over all three intersecting subdomains:

$$\frac{\partial}{\partial x}(u_0 - u_1) + \frac{\partial}{\partial y}(u_1 - u_2) + \frac{\partial}{\partial x}(u_2 - u_0) = 0 \; in \; (1,0). \tag{11}$$

3.1 Spectral collocation

For the spectral approximation of equations (9),(10) we employ polynomials $u_{0,N_0} \in \mathbb{P}_{N_0}, N_0 \in \mathbb{N}$. It is reasonable that the polynomial degree on the refined elements is smaller or equal to the degree on Ω_0. $N_0/2$ should be a lower bound for the degree. Hence we require $N_0/2 \leq N \leq N_0$. Typical values are $N = N_0, N_0 - 2, N_0 - 4$ or $N = N_0/2$. The Laplacian can be approximated as before. The interface conditions are discretized by spectral collocation. We require

$$\begin{aligned} (u_{1,N} - u_{0,N_0})(1, y_j^1) &= 0, \; j = 1, \ldots, N, && (12) \\ (u_{2,N} - u_{0,N_0})(1, y_j^2) &= 0, \; j = 0, \ldots, N-1 && (13) \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial x}(u_{1,N} - u_{0,N_0})(1, y_j) &= 0,\ j = 1, \ldots, N_0/2 - 1, & (14)\\ \frac{\partial}{\partial x}(u_{2,N} - u_{0,N_0})(1, y_j) &= 0,\ j = N_0/2 + 1, \ldots, N_0 - 1. & (15)\end{aligned}$$

We further have to add the cross point condition (11) in $(1,0)$. These are $2N + N_0 - 1$ conditions of collocation for the same number of unknowns on the interface line Γ_0. Hence the spectral scheme is completely determined. For calculating $u_{0,N_0}(1, y_j^{1,2})$ spectral interpolation is necessary. Also the partial derivatives $\frac{\partial}{\partial x}$ of $u_{1,N}, u_{2,N}$ in y_j can be obtained by spectral interpolation. Here we consider the partial derivatives in more detail. We prefer matrix notation. Let us define matrices

$$T0 = (\cos(k \arccos(1))) = (1, 1, \ldots, 1, 1)\,,\ k = 0, \ldots, N$$

and

$$\begin{aligned} T1 &= (\cos(k \arccos(2y_j - 1)))\,,\ k = 0, \ldots, N,\ j = 1, \ldots, N_0/2 - 1\\ T2 &= (\cos(k \arccos(2y_j + 1)))\,,\ k = 0, \ldots, N,\ j = N_0/2 + 1, \ldots, N_0 - 1\end{aligned}$$

Then $\frac{\partial}{\partial x}u_{1,N}(1, y_j)$ and $\frac{\partial}{\partial x}u_{2,N}(1, y_j)$ can be spectrally approximated by the matrices

$$(T0 \otimes T1)\,(\hat{D} \otimes I)\left(T^{-1} \otimes T^{-1}\right)$$

and

$$(T0 \otimes T2)\,(\hat{D} \otimes I)\left(T^{-1} \otimes T^{-1}\right).$$

This treatment also works for the cross point. By our spectral collocation scheme we enforced contnuity at the interface. C^1 - continuity only holds when $N = N_0/2$ which is in general not true. Otherwise we only have some weak continuity condition in the collocation nodes $(1, y_j)$ of Ω_0. Finally we were interested in the approximation properties of our nonconforming collocation scheme for different N and N_0. We once more considered example (8) and calculated the discrete L^2 - errors on Ω_0 and Ω_1^c denoted by $E2_0$ and $E2_1$. We consider the cases $N = N_0, N_0 - 2, N_0 - 4$ and $N = N_0/2$. In order to check the accuracy of our nonconforming spectral scheme we first calculated the errors on Ω_0 without any coupling with Ω^c. Instead of the continuity conditions we enforce exact Neumann boundary conditions on

Γ_0. The numerical results are presented in table II. We once more observe spectral accuracy where the errror is somewhat larger than on Ω^c. This is due to the fact that the size of Ω_0 is the double of $\Omega_i^c,\ i = 1, \ldots, 4$. For our nonconforming methods (see tables III-VI) we observe the typical spectral accuracy. For $N = N_0/2$ (table III) the error is dominated by the error on Ω_1^c and we obtain similar results as in table I. For $N = N_0$ (table VI) the error is dominated by the error on Ω_0. A compare with the results in table II shows that there is no loss in spectral accuracy. We still observe exponential convergence although we are no more able to enforce an exact C^1 - interface condition.

3.2 Finite difference preconditioning

For preconditioning we once more use one-sided finite differences. For the continuity conditions on the interface once more interpolation techniques are necessary. Here we consider the partial derivatives $\frac{\partial}{\partial x}u_1(1, y_j)$ and $\frac{\partial}{\partial x}u_2(1, y_j)$ which are discretized by linear interpolation. We only consider Ω_1^c. A similar treatment works on Ω_2^c. Let $0 \leq j_1 \leq N$ and $1 \leq j \leq N_0/2 - 1$ be defined such that

$$y_{j_1+1}^1 \leq y_j \leq y_{j_1}^1.$$

Then we obtain

$$\frac{\partial}{\partial x}u_1(1, y_j) \cong w_1 \left(\frac{u_1(1, y_{j_1}^1) - u_1(x_1^1, y_{j_1}^1)}{h_1} \right) + w_2 \left(\frac{u_1(1, y_{j_1+1}^1) - u_1(x_1^1, y_{j_1+1}^1)}{h_1} \right)$$

where

$$w_1 = \frac{y_j - y_{j_1+1}^1}{y_{j_1}^1 - y_{j_1+1}^1},\ \ w_2 = \frac{y_{j_1}^1 - y_j}{y_{j_1}^1 - y_{j_1+1}^1}.$$

The other finite difference operators can be generated as before. We are interested in the preconditioning properties and hence calculated the eigenspectrum of the preconditioned spectral operator. Nearly all eigenvalues are real. Some are complex with fairly small imaginary parts. In particular, we were interested in the minimal and maximal eigenvalues. From table II we observe that without any coupling of Ω_0 the typical behaviour of the eigenvalues occurs. λ_{min} is equal to one and λ_{max} approaches $\frac{\pi^2}{4}$ from below. From the tables III-VI we observe that the maximal eigenvalues only dependent on N_0 and are independent of the specific choice of N. Now the minimal eigenvalues are slightly decreasing for increasing N.

4 Domain decomposition

We are now interested in an interface relaxation between the subdomains Ω^c and Ω_0. At the interface Γ_0 information is exchanged and the iteration is continued until continuity is reached. In one direction Dirichlet information is transferred and in the other direction Neumann information. We use an interface relaxation as proposed in [**?**], i.e., we solve a sequence of Dirichlet - Neumann problems. We iterate until the error at the nonconforming interface Γ_0 is smaller than 10^{-14}. The number of iterations required is denoted by IT_{DD}. We start the iteration with initial approximations $u_0^0 = u_1^0 = u_2^0 = 0$ and iterate as follows:

$$\begin{aligned} -\Delta u_k^m &= f \ in \ \Omega_k^c, \\ u_k^m &= g \ on \ \partial\Omega_k^c \cap \Gamma^c, \\ u_k^m &= \delta^{m-1} u_0^{m-1} + (1-\delta^{m-1}) u_k^{m-1} \ on \ \Gamma_{0,k} \ for \ k = 1,2 \end{aligned}$$

and

$$\begin{aligned} -\Delta u_0^m &= f \ in \ \Omega_0, \\ u_0^m &= g \ on \ \partial\Omega_0 - \Gamma_0 \\ \frac{\partial}{\partial x} u_0^m &= \frac{\partial}{\partial x} u_k^m \ on \ \Gamma_{0,k} \ for \ k = 1,2 \end{aligned}$$

for $m = 1, 2, \ldots$. In the cross point we enforce condition (11). The spectral discretizatzion is performed as before. In particular, at the interface we require collocation conditions as in (12),(13) and (14),(15). δ^m denotes a relaxation parameter which is chosen in order to accelerate the convergence. The dynamical choice based on a minimization of the L^2 - error at the interface was diverging. Hence we fix $\delta^m = \delta$ independent of m. By numerical experiments we found that $\delta = 0.60 - 0.67$ yields good results. We choose $\delta = 0.60$ for $N = N_0$, $\delta = 0.62$ for $N = N_0 - 2$ and $\delta = 0.67$ for $N = N_0 - 4, N_0/2$. We once more consider example (8). In the tables III-VI we list the number of iterations IT_{DD}. In 31-73 iterations we always obtained convergence up to an error of 10^{-14}.

Conclusion

A highly accurate spectral collocation scheme for nonconforming elements is proposed. On the interface line continuity is enforced. Continuity of normal derivatives is enforced in the collocation nodes of the not-refined element. By numerical experiments it is shown that the high spectral accuracy is not disturbed. By global preconditioning with finite differences the condition number is strongly improved. Finally a domain decomposition technique based on a Dirichlet-Neumann iteration is successfully implemented.

FIGURE

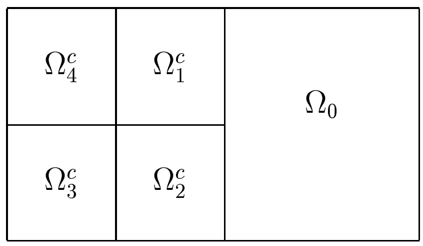

Figure I. The domains Ω^c and Ω_0.

TABLES

N	$E2_1$	λ_{min}	λ_{max}
4	$2.04{\cdot}10^{-2}$	1.00	1.7574
8	$4.13{\cdot}10^{-6}$	1.00	2.1314
16	$9.39{\cdot}10^{-15}$	1.00	2.3058
32	$1.71{\cdot}10^{-14}$	1.00	2.3884

Table I. Numerical results on Ω^c.

N_0	$E2_0$	λ_{min}	λ_{max}
4	$1.57{\cdot}10^{-1}$	1.00	1.7574
8	$3.95{\cdot}10^{-4}$	1.00	2.1314
16	$2.63{\cdot}10^{-11}$	1.00	2.3058
32	$1.12{\cdot}10^{-14}$	1.00	2.3884

Table II. Numerical results on Ω_0.

N_0	N	$E2_0$	$E2_1$	λ_{min}	λ_{max}	IT_{DD}
4	2	$2.66{\cdot}10^{-1}$	$6.70{\cdot}10^{-1}$	1.0000	1.7680	31
8	4	$3.62{\cdot}10^{-3}$	$2.00{\cdot}10^{-2}$	1.0000	2.1314	31
16	8	$6.12{\cdot}10^{-7}$	$4.05{\cdot}10^{-6}$	0.9081	2.3058	34
32	16	$1.16{\cdot}10^{-14}$	$2.19{\cdot}10^{-14}$	0.8132	2.3884	37

Table III. Nonconforming elements, $N = N_0/2$.

N_0	N	$E2_0$	$E2_1$	λ_{min}	λ_{max}	IT_{DD}
8	4	$3.62{\cdot}10^{-3}$	$2.00{\cdot}10^{-2}$	1.0000	2.1314	31
16	12	$3.05{\cdot}10^{-11}$	$1.25{\cdot}10^{-10}$	0.8105	2.3058	45
32	28	$2.48{\cdot}10^{-14}$	$4.86{\cdot}10^{-14}$	0.7273	2.3884	73

Table IV. Nonconforming elements, $N = N_0 - 4$.

N_0	N	$E2_0$	$E2_1$	λ_{min}	λ_{max}	IT_{DD}
4	2	$2.66{\cdot}10^{-1}$	$6.70{\cdot}10^{-1}$	1.0000	1.7680	34
8	6	$1.60{\cdot}10^{-4}$	$5.51{\cdot}10^{-4}$	0.8611	2.1314	34
16	14	$1.30{\cdot}10^{-11}$	$2.22{\cdot}10^{-11}$	0.7812	2.3058	34
32	30	$3.14{\cdot}10^{-14}$	$7.59{\cdot}10^{-14}$	0.7262	2.3884	51

Table V. Nonconforming elements, $N = N_0 - 2$.

N_0	N	$E2_0$	$E2_1$	λ_{min}	λ_{max}	IT_{DD}
4	4	$8.84{\cdot}10^{-2}$	$1.27{\cdot}10^{0}$	0.8513	1.7700	36
8	8	$1.96{\cdot}10^{-4}$	$3.18{\cdot}10^{-4}$	0.8128	2.1314	35
16	16	$1.30{\cdot}10^{-11}$	$2.19{\cdot}10^{-11}$	0.7710	2.3058	36
32	32	$1.07{\cdot}10^{-14}$	$2.81{\cdot}10^{-14}$	0.7169	2.3884	49

Table VI. Nonconforming elements, $N = N_0$.

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