PSEUDOSPECTRAL APPROXIMATION OF THE UNSTEADY STOKES EQUATION

by

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Abstract. We consider a pseudospectral (or collocation) approximation of the unsteady Stokes equation. Using the Uzawa algorithm the spectral system is decoupled into a Helmholtz equation and an equation with a pseudo-Laplace operator. We prove that the eigenvalues of the pseudo-Laplacian are real and negative (except one zero eigenvalue belonging to the constant mode). Our spectral method which only employs the Chebyshev Gauss-Lobatto nodes (no staggered grids!) does not introduce spurious modes. Finally we present effective finite difference preconditioners for the spectral pseudo-Laplacian.

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1. INTRODUCTION

We consider the unsteady Stokes equation which is discretized by a Chebyshev pseudospectral (or collocation) method. This means that the solution is approximated by global Chebyshev polynomials (see, e.g., Canuto et al. [6]).

Our approach to solve the spectral system is to use a global iterative decoupling procedure which is an extension of the classical Uzawa algorithm. The Uzawa algorithm was already extensively used in the finite element context (see Arrow, Hurwitz & Uzawa [1], Glowinski [9], Temam [23], Girault & Raviart [8], Bristeau, Glowinski & Periaux [5]). This approach was also chosen by Ronquist [21] in the context of spectral element methods. The Uzawa method is more efficient in terms of computational complexity than a direct approach (see Yamaguchi, Cheng & Brown [24], Bathe & Dong [2]). In particular, this becomes obvious for large three-dimensional systems. It requires no pressure boundary conditions or rediscretizations of the original problem. In so far it is more attractive than the Poisson equation approach (Glowinski & Pironneau [10], Kleiser & Schumann [19]).

Finally we present a decoupling where we reduce the problem to the solution of a Helmholtz equation and an other equation with a pseudo-Laplace operator. In many applications this pseudo-Laplace operator is also called energy. In the following we study the properties of this operator. For discretization we employ the Chebyshev Gauss-Lobatto nodes. We avoid the introduction of staggered grids with the Gauss nodes (see Bernardi & Maday [3]). We prove that the eigenvalues of the spectral pseudo-Laplacian are real and negative (except one eigenvalue which is zero and belongs to the constant mode). By using this result it is readily seen that our discretization does not introduce spurious modes for the pressure.

Since the spectral pseudo-Laplacian is very ill-conditioned an effective iterative solver requires a preconditioner. Hence we present suitable finite difference (FD) preconditioners.

2. SPLITTING OF THE UNSTEADY STOKES EQUATION

We consider the unsteady Stokes equation

$$\frac{\partial u}{\partial t} - \nabla^2 u + \nabla p = f \text{ in } \Omega = (-1,1)^2, \tag{2.1}$$

$$\nabla \cdot u = 0 \text{ in } \Omega, \tag{2.2}$$

$$u = 0 \text{ on } \partial\Omega, \tag{2.3}$$

where $u = (u_1, u_2)^t$ denotes the velocity and p the pressure. f is a giving force defined in Ω. Since the pressure is only determined up to a constant, we impose the average pressure to be zero, i.e. $\int_\Omega p dx = 0$.
The standard backward Euler time discretization of the equations (2.1)-(2.3) leads to the following scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} - \nabla^2 u^{n+1} + \nabla p^{n+1} = f^{n+1} \text{ in } \Omega, \tag{2.4}$$

$$\nabla \cdot u^{n+1} = 0 \text{ in } \Omega, \tag{2.5}$$

$$u^{n+1} = 0 \text{ on } \partial\Omega, \tag{2.6}$$

where Δt denotes the step size in t and the index n resp. $n+1$ indicates that the functions are evaluated at the time steps $t_n = n\Delta t$ resp. $t_{n+1} = (n+1)\Delta t$ for $n = 0, 1, 2, \ldots$.

As already proposed by Maday et al. [20] the following splitting method is useful:

$$\frac{\tilde{u}^{n+1} - u^n}{\Delta t} - \nabla^2 \tilde{u}^{n+1} + \nabla \overline{p}^{n+1} = f^{n+1} \text{ in } \Omega, \tag{2.7}$$

$$\tilde{u}^{n+1} = 0 \text{ on } \partial\Omega \tag{2.8}$$

and

$$\frac{u^{n+1} - \tilde{u}^{n+1}}{\Delta t} + \nabla p^{n+1} = 0 \text{ in } \Omega, \tag{2.9}$$

$$\nabla \cdot u^{n+1} = 0 \text{ in } \Omega, \tag{2.10}$$

$$u^{n+1} = 0 \text{ on } \partial\Omega. \tag{2.11}$$

Here $\tilde{u}^{n+1}$ denotes some intermediate velocity and $\overline{p}^{n+1}$ defines some extrapolant for the pressure obtained from previous time steps. A more detailed analysis concerning the extrapolant $\overline{p}^{n+1}$ is given in [20]. In this paper this problem is no more considered. The system (2.7)-(2.8) is essentially equivalent to the Helmholtz problem written as:

$$\begin{aligned} (-\nabla^2 + \frac{1}{\Delta t} I)\tilde{u}^{n+1} &= \tilde{f}^{n+1} \text{ in } \Omega, & (2.12)\\ \tilde{u}^{n+1} &= 0 \text{ on } \partial\Omega, & (2.13) \end{aligned}$$

where

$$\tilde{f}^{n+1} = f^{n+1} + \frac{1}{\Delta t} u^n - \nabla \overline{p}^{n+1}.$$

For the Helmholtz problem (2.12)-(2.13) many effective solvers are already available. In particular, we recommend some spectral multigrid (SMG) techniques. SMG was introduced by Zang et al. [25], [26] and further improvements have been obtained by Brandt et al. [4] in the Fourier case and by Heinrichs [15], [16] in the Dirichlet case. Here we especially recommend the use of algebraic SMG methods (see [18], [17]).
The system (2.9)-(2.11) is equivalent to:

$$\begin{aligned} \frac{1}{\Delta t} u^{n+1} + \nabla p^{n+1} &= \frac{1}{\Delta t}\tilde{u}^{n+1} \text{ in } \Omega, & (2.14)\\ \nabla \cdot u^{n+1} &= 0 \text{ in } \Omega, & (2.15)\\ u^{n+1} &= 0 \text{ on } \partial\Omega. & (2.16) \end{aligned}$$

In this paper we are mainly interested in the study of the system (2.14)-(2.16). For the effective solution we recommend an Uzawa-like (see [1]) decoupling procedure. But before we describe the Uzawa algroithm let us first introduce the pseudospectral discretization of the system.

3. PSEUDOSPECTRAL DISCRETIZATION

In order to give the pseudospectral discretizations of the Stokes systems we have to define spectral derivative operators for the velocity and the pressure. We first introduce these operators in the one-dimensional case. Then the two-dimensional spectral operators can easily be derived by using tensor product representation.

Let $M \in \mathbf{N}$ and let denote by $\mathbf{P}_M$ the space of polynomials of degree at most M in the coordinate variables. By $\mathbf{P}_M^0$ we denote the space of polynomials in $\mathbf{P}_M$ which fulfill the homogeneous Dirichlet boundary conditions. Actually we approximate the velocity u by polynomials in $\mathbf{P}_N^0$, $N \in 2\mathbf{N}$ and the pressure p by polynomials in $\mathbf{P}_{N-2}$. A similar treatment is possible for odd N. Here we use the pseudospectral Chebyshev discretization. The corresponding spectral derivative operator $D_{sp} \in \mathbf{R}^{N+1,N+1}$ for the velocity u is for instance given in [6]. The spectral derivative operator D_{sp} can also be written as

$$D_{sp} \;=\; T_{sp}\hat{D}_{sp}T_{sp}^{-1},$$

where T_{sp} performs the transformation from coefficient space into physical space, T_{sp}^{-1} performs the inverse transform and $\hat{D}_{sp}$ performs the differentiation in coefficient space. Explicit formulas for these operators are available and for instance given by Gottlieb & Orszag [13] or Heinrichs [15]. By using the above representation it is obvious that the evaluation of D_{sp} only requires $O(N\ln N)$ arithmetic operations if fast cosinus transforms are used.
By eliminating the boundary conditions we arrive at:

$$u'(x_j) \;\cong\; (D^0u)_j, \quad j = 1, \ldots, N-1,$$

where $D^0 \in \mathbf{R}^{N-1,N-1}$, $x_j = \cos\frac{j\pi}{N}$, $j = 1, \ldots, N-1$. The corresponding second derivative operator is given by D_{sp}^2. We once more eliminate the homogeneous Dirichlet boundary conditions and obtain:

$$u''(x_j) \;\cong\; ((D^2)^0u)_j, \quad j = 1, \ldots, N-1,$$

where $(D^2)^0 \in \mathbf{R}^{N-1,N-1}$.
In order to avoid spurious modes in pressure (see, e.g., Canuto et al. [6]) we approximate the pressure by polynomials in $\mathbf{P}_{N-2}$. In [21], [3] the discretization for the pressure was performed on a staggered grid induced by the Chebyshev Gauss nodes. The Gauss nodes were mainly chosen for reasons of accomplishing an analysis of the spectral method. A disadvantage of the Gauss nodes lies in the fact that one has to interpolate between the Gauss-Lobatto and Gauss nodes. Hence we prefer non-staggered grids with the Gauss-Lobatto nodes only. Here we also observed no spurious modes.

The derivative operator D for the pressure is now defined as follows:

$$p'(x_j) \;\cong\; (Dp)_j, \quad j = 1, \ldots, N-1,$$

where $D \in \mathbf{R}^{N-1,N-1}$ is given by

$$D \;=\; T\hat{D}T^{-1}$$

and $T = (t_{i,j})_{i,j=1,\ldots,N-1}$, $\hat{D} = (d_{i,j})_{i,j=1,\ldots,N-1}$ can explicitely be written as:

$$t_{i,j} \;=\; \cos(j-1)\frac{i\pi}{N}, \quad i,j = 1, \ldots, N-1,$$

$$d_{i,j} \;=\; \begin{cases} \frac{2(j-1)}{c_i}, & j = i+1, i+3, \ldots, N-1 \\ 0, & \text{else} \end{cases}$$

and

$$c_i \;=\; \begin{cases} 2, & i = 1, \\ 1, & \text{else.} \end{cases}$$

This means that T^{-1} performs the interpolation into $\mathbf{P}_{N-2}$ which is induced by the Chebyshev Gauss-Lobatto nodes x_j, $j = 1, \ldots, N-1$. $\hat{D}$ performs the differentiation in coefficient space and T transforms back into physical space.

In the two-dimensional case the spectral derivative operator can easily be constructed using the tensor product representation. For matrices A, $B \in \mathbf{R}^{N-1,N-1}$ with $B = (b_{i,j})_{i,j=1,\ldots,N-1}$ we write

$$A \otimes B \;=\; (Ab_{i,j})_{i,j=1,\ldots,N-1} \in \mathbf{R}^{(N-1)^2,(N-1)^2}.$$

Now the partial derivative operators for the velocity read as follows:

$$\begin{aligned}
\frac{\partial}{\partial x} &: \quad D_x^0 \;=\; D^0 \otimes I, \quad \frac{\partial}{\partial y}: \quad D_y^0 \;=\; I \otimes D^0, \\
\frac{\partial^2}{\partial x^2} &: \quad D_{xx}^0 \;=\; (D^2)^0 \otimes I, \quad \frac{\partial^2}{\partial y^2}: \quad D_{yy}^0 \;=\; I \otimes (D^2)^0, \\
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} &: \quad D_\Delta^0 \;=\; (D_{xx}^0 + D_{yy}^0),
\end{aligned}$$

where $I \in \mathbf{R}^{N-1,N-1}$ denotes the identity matrix.
For the pressure we define:

$$\frac{\partial}{\partial x}: \quad D_x \;=\; D \otimes I, \quad \frac{\partial}{\partial y}: \quad D_y \;=\; I \otimes D.$$

Now we are able to present the pseudospectral discretization of the Stokes system. We first consider the system (2.4)-(2.6):

$$\left(-D_\Delta^0 + \frac{1}{\Delta t} I\right) u_1^{n+1} \;+\; D_x p^{n+1} \;=\; \hat{f}_1^{n+1}, \tag{3.1}$$

$$\left(-D_\Delta^0 + \frac{1}{\Delta t} I\right) u_2^{n+1} \;+\; D_y p^{n+1} \;=\; \hat{f}_2^{n+1}, \tag{3.2}$$

$$\begin{aligned}
D_x^0 u_1^{n+1} \;+\; D_y^0 u_2^{n+1} &\;=\; 0, \\
u_1^{n+1} \;=\; u_2^{n+1} &\;=\; 0 \quad \text{on } \partial\Omega,
\end{aligned}$$

where

$$\hat{f}_i^{n+1} \;=\; f^{n+1} \;+\; \frac{1}{\Delta t} u_i^n, \; i = 1, 2.$$

By using the Uzawa decoupling we arrive at the following equation for the pressure:

$$A p^{n+1} \;=\; \hat{f}^{n+1},$$

where

$$A \;=\; D_x^0\left(-D_\Delta^0+\frac{1}{\Delta t}I\right)^{-1} D_x \;+\; D_y^0\left(-D_\Delta^0+\frac{1}{\Delta t}I\right)^{-1} D_y$$

and

$$\hat{f}^{n+1} \;=\; D_x^0\left(-D_\Delta^0+\frac{1}{\Delta t}I\right)^{-1} \hat{f}_1^{n+1} \;+\; D_y^0\left(-D_\Delta^0+\frac{1}{\Delta t}I\right)^{-1} \hat{f}_2^{n+1}.$$

After having solved this equation for the pressure the velocity u can be calculated by means of the equations (3.1),(3.2). This corresponds to the solution of two Helmholtz equations. Our main interest concerns the operator A. Here we consider two limit cases:

$$\begin{aligned} \Delta t \to \infty: \quad A \;&\cong\; A_{st} \;=\; -D_x^0(D_\Delta^0)^{-1}D_x \;-\; D_y^0(D_\Delta^0)^{-1}D_y, \\ \Delta t \to \;0: \quad A \;&\cong\; \Delta t B,\; B \;=\; D_x^0 D_x \;+\; D_y^0 D_y. \end{aligned}$$

Here A_{st} is identical to the spectral operator which occurs for the steady Stokes problem. By a numerical calculation of the eigenvalues of A_{st} we found that this operator has - besides one zero eigenvalue - only real, positive eigenvalues. We further observed that A_{st} is very well-conditioned. This can be motivated by noticing that A_{st} is essentially the product of two first-order gradient operators “divided” by the second-order Laplacian, suggesting that A_{st} is identity-like as regards its spectrum. In table I we present the minimal and maximal (non-zero) eigenvalues λ_{min} and λ_{max} of A_{st}. More precisely, we define:

$$\begin{aligned} \lambda_{min} \;&=\; \min\left\{\mid \lambda \mid:\; \lambda \neq 0 \text{ eigenvalue of } A_{st}\right\}, \\ \lambda_{max} \;&=\; \max\left\{\mid \lambda \mid:\; \lambda \neq 0 \text{ eigenvalue of } A_{st}\right\}. \end{aligned}$$

N	λ_{min}	λ_{max}	$\lambda_{max}/\lambda_{min}$
4	0.3158	1.1206	3.5485
8	0.2355	1.4132	6.0009
12	0.2264	1.3208	5.8339
16	0.2329	1.4739	6.3285

Table I. λ_{min} and λ_{max} of A_{st}.

The above discussion motivates an approach to invert A_{st} based on a nested global inner/outer iterative procedure. The outer iteration could be a Richardson iteration based on a minimal residual strategy for determining the relaxation parameters. For the inner iteration which inverts D_{Δ}^{0} we recommend spectral multigrid techniques (see [15], [16]). In context of the spectral element method Ronquist [21] proposed a preconditioned conjugate gradient method for the outer iteration. For preconditioning he used the diagonal of the mass matrix. However, for the steady Stokes problem the situation is rather satisfactory.

For the unsteady Stokes problem the spectral operator B becomes predominant. A direct application of the Uzawa decoupling to the system (2.14)-(2.16) leads to the equation:

$$Bp^{n+1} \;=\; \tilde{u}_0^{n+1},$$

where

$$\tilde{u}_0^{n+1} \;=\; \frac{1}{\Delta t}\left(D_x^0 \tilde{u}_1^{n+1} \;+\; D_y^0 \tilde{u}_2^{n+1}\right).$$

4. PROPERTIES OF THE SPECTRAL OPERATOR

In this section we investigate the spectral operator

$$B = D_x^0 D_x + D_y^0 D_y.$$

Here we mainly consider the one-dimensional operator

$$A = D^0 D.$$

In order to get more informations about the operator A we first numerically calculated its eigenvalues. We observed that the eigenvalues are real, one eigenvalues is zero and all other eigenvalues are negative. In table II we present λ_{min} and λ_{max} for A. For increasing N the minimal eigenvalue λ_{min} approximates $\pi^2/4$. The maximal eigenvalue increases as $O(N^3)$ but asymptotically we may expect an $O(N^4)$ behaviour.

N	λ_{min}	λ_{max}	λ_{max}/N^3
4	2.0000	$1.2000 \cdot 10^1$	0.1875
8	2.4665	$2.0160 \cdot 10^2$	0.3938
16	2.4674	$3.1377 \cdot 10^3$	0.7660
32	2.4674	$4.9802 \cdot 10^4$	1.5198
64	2.4674	$7.9519 \cdot 10^5$	3.0334
128	2.4674	$1.2716 \cdot 10^7$	6.0637

Table II. λ_{min} and λ_{max} of A

Apart from the zero eigenvalue the eigenvalue spectrum is very similar to the spectrum of the standard second order spectral operator. This could be expected since A is essentially the product of two first order operators. For completeness we also calculated the eigenvalues of B. One eigenvalue is zero and the others are real and negative. In table III we present the values λ_{min} and λ_{max} for $N = 4, 8, 16, 32$. λ_{max} once more increases as $O(N^3)$ for the considered values of N.

N	λ_{min}	λ_{max}	λ_{max}/N^3
4	2.0000	$2.4000 \cdot 10^1$	0.3750
8	2.4665	$4.0321 \cdot 10^2$	0.7875
12	2.4674	$2.0015 \cdot 10^3$	1.1583
16	2.4674	$6.2755 \cdot 10^3$	1.5321

Table III. λ_{min} and λ_{max} of B

For an analysis of the operator A we introduce a somewhat different representation of A. If we apply A to a polynomial $p \in \mathbf{P}_{N-2}$ we first take the first derivative of p:

$$p'(x_j) \;=\; (Dp)_j, \quad j = 1, \ldots, N-1.$$

By writing Dp we consider p as a vector in $\mathbf{R}^{N-1}$ with components $p(x_j),\ j = 1, \ldots, N-1$. This notation is used in the following.
Afterwards we apply the operator D^0 which contains the homogeneous Dirichlet boundary conditions. Hence we have to map p' onto the polynomial $p'_0 \in \mathbf{P}^0_N$ which is defined by:

$$p'_0(x_j) \;=\; p'(x_j), \quad j = 1, \ldots, N-1.$$

Obviously, p'_0 can explicitely be written as:

$$p'_0 \;=\; p' \;+\; \alpha T'_N \;+\; \beta x T'_N,$$

where T_N denotes the N-th Chebyshev polynomial, i.e. $T_N(x) = \cos N \arccos x$, and $\alpha,\ \beta$ are determined such that $p'_0(-1) \;=\; p'_0(1) \;=\; 0$, i.e.

$$\begin{aligned}
\alpha &= \frac{1}{2N^2}\left(p'(-1) \;-\; p'(1)\right), \\
\beta &= -\frac{1}{2N^2}\left(p'(-1) \;+\; p'(1)\right).
\end{aligned}$$

Applying A to p now corresponds to take the first derivative of p_0', i.e.

$$p_0'' \; = \; p'' \; + \; \alpha T_N'' \; + \; \beta (x T_N')'.$$

By further using that $(xT_N')'(x_j) \; = \; (xT_N'')(x_j),\; j = 1, \ldots, N-1$ we obtain:

$$\begin{aligned}
(Ap)_j \quad &= \quad p''(x_j) \; + \; \frac{T_N''(x_j)}{2N^2}\,(p'(-1) - p'(1)) \; - \; \frac{x_j T_N''(x_j)}{2N^2}\,(p'(-1) + p'(1)) \\
&= \quad p''(x_j) \; + \; \frac{1}{2N^2}(1 - x_j) T_N''(x_j) p'(-1) \; - \; \frac{1}{2N^2}(1 + x_j) T_N''(x_j) p'(1).
\end{aligned}$$

Hence we obtain

$$(Ap)_j \; = \; p''(x_j) \; + \; \frac{(-1)^{j+1}}{2(1+x_j)} p'(-1) \; + \; \frac{(-1)^j}{2(1-x_j)} p'(1). \tag{4.1}$$

Formula (4.1) is due to the differential equation for the Chebyshev polynomials. In the Gauss-Lobatto nodes we have

$$(1 - x_j^2) T_N''(x_j) \; = \; -N^2 T_N(x_j) \; = \; (-1)^{j+1} N^2 \;\; \text{for } j = 1, \ldots, N-1.$$

In order to represent A in matrix notation we introduce the diagonal matrices $D_-, D_+ \in \mathbf{R}^{N-1,N-1}$:

$$\begin{aligned}
D_- \;&=\; \mathrm{diag}(d_{j,j}^-)_{j=1,\ldots,N-1} : \quad d_{j,j}^- \quad = \quad \frac{(-1)^{j+1}}{2(1+x_j)}, \quad j = 1, \ldots, N-1, \\
D_+ \;&=\; \mathrm{diag}(d_{j,j}^+)_{j=1,\ldots,N-1} : \quad d_{j,j}^+ \quad = \quad \frac{(-1)^{j}}{2(1-x_j)}, \quad j = 1, \ldots, N-1.
\end{aligned}$$

We further introduce $P_+,\; P_- \in \mathbf{R}^{N-1,N-1}$ which represent $p'(1),\; p'(-1)$, i.e.

$$\begin{aligned}
P_+ \quad &= \quad e_+ f_+^t, \quad f_+ \; = \; e_+^t \hat{D} T^{-1}, \\
P_- \quad &= \quad e_+ f_-^t, \quad f_- \; = \; e_-^t \hat{D} T^{-1},
\end{aligned}$$

where $e_+ = (1, 1, \ldots, 1, 1)^t,\; e_- = (1, -1, 1, \ldots, -1, 1)^t \in \mathbf{R}^{N-1}$. Here t indicates the transpose. Clearly, P_+ and P_- have rank 1.

By using these matrices we are now able to write an representation of A which is equivalent to (4.1):

$$A \;=\; D^2 \;+\; D_- P_- \;+\; D_+ P_+. \tag{4.2}$$

This means that A is equal to D^2 corrected by the matrix $D_- P_- \;+\; D_+ P_+$ which has rank 2.

By using the above considerations we are now able to derive some interesting results for the operator A.

We first study the eigenvalues of A. Here we prove that the eigenvalues of A are real. We employ a technique similar to that introduced by Gottlieb & Lustman [12] for the standard second order operator. The idea for proof is that the characteristic polynomial is related to two other polynomials forming a positive pair. These are the characteristic polynomials of some suitable first order problems. First we show how the characteristic polynomial can be determined.

Let $p \in \mathbf{P}_{N-2}$ be an eigenfunction of A with respect to the eigenvalue λ. Then we have

$$[p'' \;+\; \alpha T_N'' \;+\; \beta(xT_N')'] \;-\; \lambda p \;=\; \sigma T_N',$$

where $\sigma \in \mathbf{R}$ is determined such that

$$\sigma T_N' \;-\; \beta(xT_N')' \in \mathbf{P}_{N-2}.$$

Hence it is equivalent to choose $\sigma \in \mathbf{R}$ such that

$$\sigma T_N \;-\; \beta x T_N' \in \mathbf{P}_{N-1}.$$

By expanding the polynomial xT_N' in Chebyshev polynomials (see [13]) we derive that the coefficient with respect to T_N is equal to N. Hence we obtain $\sigma \;=\; \beta N$ and

$$S_{N-2} \;:=\; (xT_N')' \;-\; NT_N' \in \mathbf{P}_{N-2}.$$

This implies

$$\lambda p \;-\; p'' \;=\; \alpha T_N'' \;+\; \beta S_{N-2}. \tag{4.3}$$

Now we first consider the case $\lambda = 0$. It is readily seen that $p \equiv$ const. $\neq 0$ satisfies (4.3) and is an eigenfunction. Furthermore by using (4.3) it can be seen that there are no other eigenfunctions for $\lambda = 0$. Since $-p'' \in \mathbf{P}_{N-4}$ we conclude that

$$\alpha T_N'' \ + \ \beta S_{N-2} \in \mathbf{P}_{N-4}.$$

It is not difficult to check that the later condition implies $\alpha = \beta = 0$ and

$$p'(-1) \ = \ p'(1) \ = \ 0.$$

Furthermore we now have $p''(x_j) \ = \ 0, \ \ j = 1, \ldots, N-1$. These conditions immediately imply that p is identical to a constant. Therefore we also proved that our spectral method does not introduce spurious modes.

In the following we consider the case $\lambda \neq 0$, $p \ \not\equiv$ const. Now let us introduce $p_i \ \equiv \ p_i(x, \lambda) \in \mathbf{P}_{N-2}, \ \ i = 1, 2$, satisfying

$$\begin{aligned} \lambda p_1 \ - \ p_1'' \quad &= \quad T_N'', \\ \lambda p_2 \ - \ p_2'' \quad &= \quad S_{N-2}. \end{aligned}$$

Hence we obtain explicit representations for $p_1, \ p_2$ as follows (see [12]):

$$\begin{aligned} p_1 \quad &= \quad \sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial^{2k}}{\partial x^{2k}} T_N'', \\ p_2 \quad &= \quad \sum_{k=0}^{\infty} \lambda^{-k-1} \frac{\partial^{2k}}{\partial x^{2k}} S_{N-2}. \end{aligned}$$

Now p is given by

$$p \ = \ \alpha p_1 \ + \ \beta p_2.$$

Here $\alpha, \ \beta$ have to be chosen such that

$$(p_1'(1, \lambda) + T_N'(1))\alpha \ + \ (p_2'(1, \lambda) + (xT_N')(1))\beta \quad = \quad 0, \tag{4.4}$$

$$(p_1'(-1, \lambda) + T_N'(-1))\alpha \ + \ (p_2'(-1, \lambda) + (xT_N')(-1))\beta \quad = \quad 0. \tag{4.5}$$

In order to have a nontrivial solution for α and β the determinant in (4.4), (4.5) must vanish, i.e.

$$\begin{aligned} [p_1'(1,\lambda) + T_N'(1)]\,[p_2'(-1,\lambda) + (xT_N')(-1)] & \; - \\ - \; [p_2'(1,\lambda) + (xT_N')(1)]\,[p_1'(-1,\lambda) + T_N'(-1)] & \; = \; 0. \end{aligned}$$

By further using the identities

$$p_1'(-1,\lambda) \;=\; -p_1'(1,\lambda), \quad p_2'(-1,\lambda) \;=\; p_2'(1,\lambda)$$

and

$$(xT_N')(-1) \;=\; (xT_N')(1) \;=\; N^2, \quad T_N'(-1) \;=\; -T_N'(1) \;=\; -N^2$$

we finally arrive at the following two characteristic equations:

$$p_1'(1,\lambda) \;+\; N^2 \;=\; 0, \tag{4.6}$$

$$p_2'(1,\lambda) \;+\; N^2 \;=\; 0. \tag{4.7}$$

Since both p_1' and p_2' are polynomials of degree $\frac{N}{2}-1$ in λ^{-1} both equations determine the (non-zero) $N-2$ eigenvalues. For instance, for $N=4$, we obtain $S_2 = 32x$ and the two characteristic equations are

$$\begin{aligned} 192\lambda^{-1} \;+\; 16 &\;=\; 0, \\ 32\lambda^{-1} \;+\; 16 &\;=\; 0, \end{aligned}$$

which determine the two eigenvalues $\lambda = -2$ and $\lambda = -12$ (see also table II).

Let us now introduce

$$f_N(x,\mu) \;=\; \sum_{k=0}^{\infty} \mu^k \frac{\partial^{k+1}}{\partial x^{k+1}} T_N(x), \tag{4.8}$$

$$g_N(x,\mu) \;=\; \sum_{k=0}^{\infty} \mu^{k+1} \frac{\partial^k}{\partial x^k} S_{N-2}(x) \;+\; xT_N'(x). \tag{4.9}$$

The lemma we are about to prove will show that the roots of $f_N(1,\mu)$ and $g_N(1,\mu)$ have negative real parts.

Lemma 4.1. *Let f_N and g_N be defined by (4.8) and (4.9). Then $f_N(1,\mu)$ and $g_N(1,\mu)$ are Hurwitz polynomials, i.e. their roots have negative real parts.*

Proof. The proof for f_N has already been given in [12]. Here we consider g_N. It is easy to see that

$$\begin{aligned} g_N(x,\mu) \ - \ \frac{\partial}{\partial x} g_N(x,\mu) \ &= \ \mu S_{N-2}(x) \ + \ x T_N'(x) \ - \ \mu (x T_N')'(x) \\ &= \ (x-\mu N) T_N'(x). \end{aligned}$$

Let $w_N \ = \ e^{(1/\mu)t} g_N(x,\mu)$. Then

$$\frac{\partial w_N}{\partial t} \ - \ \frac{\partial w_N}{\partial x} \ = \ e^{(1/\mu)t}(x-\mu N) T_N'(x), \tag{4.10}$$

and if μ is a root of $g_N(1,\mu)$, then

$$w_N(1,t) \ = \ 0.$$

Since the right hand side in (4.10) vanishes in all the collocation points it can be shown (see [11]) that $w_N(x,t)$ decreases in time and therefore real $\mu \ < \ 0$. $\oplus$

Now let

$$\begin{aligned} f_N^1(\mu) \ &= \ \sum_{k=0}^{\infty} \mu^{k+1} \frac{\partial^{2k+3}}{\partial x^{2k+3}} T_N(1) \ + \ T_N'(1), \\ f_N^2(\mu) \ &= \ \sum_{k=0}^{\infty} \mu^{k} \frac{\partial^{2k+2}}{\partial x^{2k+2}} T_N(1) \end{aligned}$$

and

$$\begin{aligned} g_N^1(\mu) \ &= \ \sum_{k=0}^{\infty} \mu^{k+1} \frac{\partial^{2k+1}}{\partial x^{2k+1}} S_{N-2}(1) \ + \ (x T_N')(1), \\ g_N^2(\mu) \ &= \ \sum_{k=0}^{\infty} \mu^{k} \frac{\partial^{2k}}{\partial x^{2k}} S_{N-2}(1). \end{aligned}$$

In the following we show that $f_N^1(\mu)$ and $f_N^2(\mu)$, $g_N^1(\mu)$ and $g_N^2(\mu)$ form a positive pair.

Definition 4.1. Let $k^1(\mu)$ and $k^2(\mu)$ be real polynomials of degree $(m-1)$ (or m) respectively. Then $k^1(\mu)$ and $k^2(\mu)$ form a positive pair if:

1. the roots $\mu_1, \ldots, \mu_m$ of k^1 and the roots $\mu'_1, \ldots, \mu'_{m-1}$ (or $\mu'_1, \ldots, \mu'_m$) of k^2 are all distinct, real and negative,

2. the roots alternate as follows:

$$\begin{array}{rcl} \mu_1 < \mu'_1 < \ldots < \mu'_{m-1} < \mu_m & < & 0 \\ (\text{or } \mu'_1 < \mu_1 < \mu'_2 < \ldots < \mu'_m < \mu_m & < & 0), \end{array}$$

3. the highest coefficients of k^1 and k^2 are of like sign.

Now the following lemma characterizes such positive pairs.

Lemma 4.2. *The polynomial*

$$h(\mu) \;=\; k^1(\mu^2) \;+\; \mu k^2(\mu^2)$$

is a Hurwitz polynomial (i.e. all of its roots have negative real parts) if and only if $k^1(\mu)$ and $k^2(\mu)$ form a positive pair.

For a proof of this lemma we refer to Gantmacher [7, p. 228]. Now we are able to prove the following lemma:

Lemma 4.3 *The polynomials $f_N^1(\mu)$ and $f_N^2(\mu)$, $g_N^1(\mu)$ and $g_N^2(\mu)$ form a positive pair.*

Proof. One easily observes that

$$\begin{aligned} f_N^1(\mu^2) \ + \ \mu f_N^2(\mu^2) \ &= \ f_N(1,\mu), \\ g_N^1(\mu^2) \ + \ \mu g_N^2(\mu^2) \ &= \ g_N(1,\mu) \end{aligned}$$

which are Hurwitz polynomials (see lemma 4.1). This concludes the proof. $\oplus$

By means of lemma 4.3 we immediately obtain the following theorem.

Theorem 4.1 *The eigenvalues of the spectral operator A are real. One eigenvalue is equal to zero (belonging to the constant mode) and the remaining $N-2$ eigenvalues are all different and have negative sign.*

Remark 4.1. The results from theorem 4.1 implicitely contain that our discretization of the Stokes problem does not introduce spurious modes.

Proof. By setting $\mu = \lambda^{-1}$ it is readily seen that the polynomials $f_N^1(\lambda^{-1})$ and $g_N^1(\lambda^{-1})$ are identical to the characteristic polynomials given in (4.6) and (4.7). By using the results of lemma 4.3 we immediately obtain the desired results of our theorem. $\oplus$

We remark that a similar proof can be performed in the Legendre case. Also in this case it is not obvious that the eigenvalues of A are real.

We were also interested in the properties of the spectral operator

$$\tilde{A} \ = \ DD^0,$$

which is obtained from A by a similarity transform using D^0:

$$\tilde{A} \ = \ (D^0)^{-1} A D^0.$$

Hence $\tilde{A}$ has the same eigenvalue spectrum as A. We observed in the one-dimensional case that $\tilde{A}$ is essentially very similar to the standard second order operator. Now let $\tilde{p} \in \mathbf{P}_N^0$ be given, then applying $\tilde{A}$ to $\tilde{p}$ means that one first takes the derivative of $\tilde{p}$, i.e.

$$(D^0\tilde{p})_j \;\cong\; \tilde{p}'(x_j), \quad j = 1, \ldots, N-1.$$

Applying D to $\tilde{p}'$ means that we first map $\tilde{p}'$ onto the corresponding polynomial

$$\tilde{q} \;=\; \tilde{p}' \;+\; \gamma T_N',$$

where $\gamma \in \mathbf{R}$ is chosen such that $\tilde{q} \in \mathbf{P}_{N-2}$. Afterwards we take the derivative of $\tilde{q}$:

$$\tilde{q}' \;=\; \tilde{p}'' \;+\; \gamma T_N''.$$

Now γ is equal to the negative of the N-th Chebyshev coefficient in the Chebyshev series for $\tilde{p}$, i.e.

$$\gamma \;=\; -\frac{1}{N} t_N^t \tilde{p},$$

where $\tilde{p}$ is considered as a vector in $\mathbf{R}^{N-1}$ with components $\tilde{p}(x_j),\ j = 1, \ldots, N-1$ and $t_N \in \mathbf{R}^{N-1}$ with components $(t_N)_j = T_N(x_j),\ j = 1, \ldots, N-1$. Hence in matrix notation $\tilde{A}$ can now be written as

$$\tilde{A} \;=\; (D^2)^0 \;-\; \frac{1}{N}\left(t_N'' t_N^t\right),$$

where $t_N'' \in \mathbf{R}^{N-1}$ is given by $(t_N'')_j = T_N''(x_j),\ j = 1, \ldots, N-1$. Therefore $\tilde{A}$ is essentially the sum of the standard second operator (with homogeneous Dirichlet boundary conditions incorporated) and a matrix of rank 1. Furthermore it is readily seen that the eigenfunction belonging to the eigenvalue zero is now given by $t_N^0 \in \mathbf{R}^{N-1}$ where

$$(t_N^0)_j \;=\; T_N(x_j) \;-\; 1, \quad j = 1, \ldots, N-1.$$

From the above representation of $\tilde{A}$ it can also be seen that those eigenfunctions of $(D^2)^0$ which do not contain T_N in their Chebyshev series are also eigenfunctions of $\tilde{A}$ with the same eigenvalue. For $N = 4, 8, 16$ and 32 we numerically observed that $1, 3, 8$ and 19 eigenvalues of $(D^2)^0$ and $\tilde{A}$ (or A) are identical. In any case the second-largest eigenvalue of $(D^2)^0$ is identical to the largest eigenvalue of $\tilde{A}$ (or A). At the moment we do not see how to prove the numerical results analytically since the eigenfunctions of the spectral second order operator are not explicitly known.
The corresponding two-dimensional operator is given by

$$\tilde{B} \;=\; D_x D_x^0 \;+\; D_y D_y^0.$$

$\tilde{B}$ and B are related by the following similarity transform:

$$\tilde{B} \;=\; \left(D_x^0 D_y^0\right)^{-1} B \left(D_x^0 D_y^0\right).$$

Hence solving a system with B is equivalent to solving a system with $\tilde{B}$ where additionally the essentially one-dimensional operators D_x^0 and D_y^0 have to be inverted. Hence if we find a good solver for the system related to $\tilde{B}$ then we have also found a good solver for the system related to B. The inversion of D_x^0 resp. D_y^0 can be accomplished in $O(N^3)$ arithmetic operations.
In the next section we study preconditioning for A which is necessary for an effective iterative solver.

5. PRECONDITIONING

Here we present efficient finite difference preconditioners for the spectral operator. We first consider the one-dimensional case. For giving a finite difference (FD) preconditioner of A we make use of the representations (4.1) resp. (4.2). Hence in the FD case we have to find approximations for

$$p''(x_j),\ j=1,\dots,N-1,\ p'(-1),\ p'(1).$$

For $p''(x_j),\ j=2,\dots,N-2$ we use the standard central FD scheme (see [15]), i.e.

$$p''(x_j) \;\cong\; a_j p(x_{j-1}) \;-\; (a_j+c_j)p(x_j) \;+\; c_j p(x_{j+1}) \tag{5.1}$$

where

$$\begin{aligned} a_j &= \alpha s_j / s_{j-\frac{1}{2}}, \\ c_j &= \alpha s_j / s_{j+\frac{1}{2}} \end{aligned}$$

and

$$\alpha \;=\; \frac{1}{2 s_{\frac{1}{2}} s_1}, \quad s_j \;=\; \sin(\frac{j\pi}{N}), \quad s_{j\pm\frac{1}{2}} \;=\; \sin(\frac{(j\pm\frac{1}{2})\pi}{N}).$$

For $p''(x_1)$ resp. $p''(x_{N-1})$ we first tried one-sided first order approximations, i.e.

$$\begin{aligned} p''(x_1) &\cong a_2 p(x_1) - (a_2+c_2)p(x_2) + c_2 p(x_3), && (5.2) \\ p''(x_{N-1}) &\cong a_{N-2} p(x_{N-3}) - (a_{N-2}+c_{N-2})p(x_{N-2}) && \\ &\quad + c_{N-2} p(x_{N-1}) && (5.3) \end{aligned}$$

For $p'(-1),\ p'(1)$ we first employed one-sided finite differences, i.e.

$$\begin{aligned} p'(-1) &\cong \frac{1}{x_1 - x_2}(p(x_1) - p(x_2)) && (5.4) \\ &= \frac{1}{2 s_{\frac{3}{2}} s_{\frac{1}{2}}}(p(x_1) - p(x_2)), && \\ p'(1) &\cong \frac{1}{x_{N-1} - x_{N-2}}(p(x_{N-1}) - p(x_{N-2})) && (5.5) \\ &= -\frac{1}{2 s_{\frac{3}{2}} s_{\frac{1}{2}}}(p(x_{N-1}) - p(x_{N-2})) && \end{aligned}$$

The FD scheme (5.1)-(5.5) is now associated with a FD operator, here called A^1_{FD}. Now we were interested in the preconditioning properties of A^1_{FD}. It is readily seen that any constant ($\neq 0$) is an eigenfunction of A^1_{FD} with eigenvalue 0. In order to check the preconditioning properties of A^1_{FD} we first eliminate the constant mode. For this purpose we introduce a transformation E given by

$$p \;=\; Eq,$$

where

$$E \;=\; \begin{bmatrix} 1,0,0,\ldots,0,0,1 \\ 0,1,0,\ldots,0,0,1 \\ 0,0,1,\ldots,0,0,1 \\ \cdot,\cdot,\cdot,\ldots,\cdot,\cdot,\cdot \\ \cdot,\cdot,\cdot,\ldots,\cdot,\cdot,\cdot \\ \cdot,\cdot,\cdot,\ldots,\cdot,\cdot,\cdot \\ 0,0,0,\ldots,1,0,1 \\ 0,0,0,\ldots,0,1,1 \\ 0,0,0,\ldots,0,0,1 \end{bmatrix}.$$

We now solve the problems in q and hence consider the modified operators

$$\hat{A} \;=\; AE \;\text{ resp. }\; \hat{A}^1_{FD} \;=\; A^1_{FD}E.$$

$\hat{A}$ and $\hat{A}^1_{FD}$ are identical to A and A^1_{FD} except on the last column which is now identical to the zero vector. This result is due to our choice of E where the last column corresponds to the constant. Furthermore the matrix E can easily be inverted and does not change the structure of the spectral or FD operators. In particular, the sparse structure of the FD operator is maintained. The singularity due to the constant mode is now localized on the last column. By eliminating the last column and one of the rows (we generally take the first or last row) we finally obtain a regular system in $\mathbf{R}^{N-2,N-2}$. The corresponding spectral or FD operators are now written as:

$$\hat{A}_0 \in \mathbf{R}^{N-2,N-2} \;\text{ resp. }\; \hat{A}^1_{FD,0} \in \mathbf{R}^{N-2,N-2}.$$

We numerically observed that the eigenvalues of $(\hat{A}^1_{FD,0})^{-1}\hat{A}_0$ are real. In table IV we present the minimal and maximal eigenvalues λ_{min} and λ_{max}. From the numerical results it becomes obvious that λ_{max} approximately behaves as $O(N)$.

N	λ_{min}	λ_{max}
4	1.00	1.4776
8	1.00	4.2980
16	1.00	9.8409
32	1.00	20.8401

Table IV. λ_{min} and λ_{max} of $(\hat{A}^1_{FD,0})^{-1}\hat{A}_0$

This could somewhat be reduced by approximating $p''(x_1),\ p''(x_{N-1})$ and $p'(-1),\ p'(1)$ by second order finite differences. The corresponding FD operator is called $\hat{A}^2_{FD,0}$. In table V we present the eigenvalues λ_{min} and λ_{max} of $(\hat{A}^2_{FD,0})^{-1}\hat{A}_0$. The eigenvalues are once more real. λ_{max} is somewhat smaller than in the previous case but it still behaves as $O(N)$.

N	λ_{min}	λ_{max}
4	1.00	1.0000
8	1.00	3.5295
16	1.00	8.3776
32	1.00	17.8697

Table V. λ_{min} and λ_{max} of $(\hat{A}^2_{FD,0})^{-1}\hat{A}_0$

Finally we employed the spectral scheme for the derivatives $p''(x_1),\ p''(x_{N-1})$ and $p'(-1),\ p'(1)$. $p''(x_j),\ j = 2, \ldots, N-2$ is still approximated by the standard central FD method. The corresponding FD operator is denoted by $\hat{A}^{sp}_{FD,0} \in \mathbf{R}^{N-2,N-2}$. Now the condition number of $(\hat{A}^{sp}_{FD,0})^{-1}\hat{A}_0$ seems to be bounded independently of N. We numerically observed that the eigenvalues are real. From table VI it can be seen that $\lambda_{min} = 1$ and λ_{max} approximates $\frac{\pi^2}{4}$ from below. This observation coincides with FD preconditioning properties for the Laplacian.

N	λ_{min}	λ_{max}
4	1.00	1.0000
8	1.00	1.3216
16	1.00	1.7719
32	1.00	2.0887
64	1.00	2.2717
128	1.00	2.3686
256	1.00	2.4237

Table VI. λ_{min} and λ_{max} of $(\hat{A}^{sp}_{FD,0})^{-1}\hat{A}_0$

An obvious disadvantage of using $(\hat{A}^{sp}_{FD,0})^{-1}$ for preconditioning is due to the fact $\hat{A}^{sp}_{FD,0}$ is no more sparse. The matrices D_-P_- and D_+P_+ are dense. Hence in an elimination procedure one first has to eliminate these matrices. Here one makes use of the fact that both matrices have rank 1. Here we propose the following procedure. One first eliminates D_-P_- by multiplying D_-^{-1} and then subtracting the second row of P_- from the first, the third from the second, and so on. Finally the eliminated matrix is identical to the zero matrix except for the last row. A similar procedure is then applied to D_+P_+. By this elimination process we maintain the sparse structure of the matrix belonging to p''. Only one non-zero diagonal is added. Furthermore in the eliminated matrix the first and last two rows are dense. Now a Gaussian elimination procedure specialized to this type of matrix can efficiently solve the system in $O(N)$ arithmet! ic operations.

However, the above elimination procedure is not easy to implement. Therefore we finally introduce a sparse preconditioner which requires the solution of only a tridiagonal system. Here we get rid of the terms $p'(-1)$, $p'(1)$ by setting them to zero. Hence we obtain the Neumann condition

$$p'(-1) \;=\; p'(1) \;=\; 0.$$

By using the Neumann condition the FD approximations for $p'(x_1),\ p'(x_{N-1})$ now read as follows:

$$
\begin{aligned}
p'(x_1) &\cong \alpha_1 p(x_1) \ - \ \alpha_1 p(x_2), \\
p'(x_{N-1}) &\cong \alpha_{N-1} p(x_{N-1}) \ - \ \alpha_{N-1} p(x_{N-2}),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_1 &= -\frac{1}{(\frac{1}{2}(x_2+x_1)-1)(x_2-x_1)} \ , \\
\alpha_{N-1} &= -\frac{1}{(\frac{1}{2}(x_{N-2}+x_{N-1})+1)(x_{N-2}-x_{N-1})} \ .
\end{aligned}
$$

The corresponding FD operator is called $\hat{A}^{NE}_{FD,0}$. Once more the eigenvalues of $(\hat{A}^{NE}_{FD,0})^{-1}\hat{A}_0$ are real. λ_{min} approximates 1 and λ_{max} increases as $O(N)$. However, this preconditioner has some advantages due to its tridiagonal structure. In particular, we recommend this preconditioner for iterative methods.

N	λ_{min}	λ_{max}
4	0.9142	2.0000
8	1.0353	5.2823
16	1.0055	11.8729
32	1.0010	25.0386

Table VI. λ_{min} and λ_{max} of $(\hat{A}^{NE}_{FD,0})^{-1}\hat{A}_0$

Similar results can be obtained in the two-dimensional case. The FD operators are constructed analogous to the spectral operators by means of tensor products. Here we introduce:

$$
\hat{B} \ = \ B(E \otimes E)
$$

and

$$
\begin{aligned}
\hat{B}^{sp}_{FD} &= \hat{A}^{sp}_{FD} \otimes E + \ E \otimes \hat{A}^{sp}_{FD}, \\
\hat{B}^{NE}_{FD} &= \hat{A}^{NE}_{FD} \otimes E + \ E \otimes \hat{A}^{NE}_{FD}.
\end{aligned}
$$

The operators $\hat{B}_0$, $\hat{B}^{sp}_{FD,0}$, $\hat{B}^{NE}_{FD,0}$ are obtained by eliminating the last column and row. Hence they are regular and we are able to study the preconditioning properties of the FD operators. We calculated the eigenvalues of $(\hat{B}^{sp}_{FD,0})^{-1}\hat{B}_0$ resp. $(\hat{B}^{NE}_{FD,0})^{-1}\hat{B}_0$. Once more the eigenvalues are real. λ_{min} is equal to one in the first case resp. approximates one in the second case. The maximal eigenvalue of $(\hat{B}^{sp}_{FD,0})^{-1}\hat{B}_0$ approximates $\frac{\pi^2}{4}$ from below whereas the maximal eigenvalue of $(\hat{B}^{NE}_{FD,0})^{-1}\hat{B}_0$ behaves as $O(N)$ for increasing N (see tables VII,VIII). Hence we numerically observe a similar behaviour as in the one-dimensional case. For the operator $\hat{B}^{sp}_{FD,0}$ we propose a similar elimination procedure as introduced in the one-dimensional case. Now everything has to be performed blockwise. By using an efficient band Gauss elimination procedure the system can be solved in $O(N^4)$ arithmetic operations.

N	λ_{min}	λ_{max}
4	1.0000	1.0000
8	1.0000	1.3216
12	1.0000	1.5962
16	1.0000	1.7719

Table VII. λ_{min} and λ_{max} of $(\hat{B}^{sp}_{FD,0})^{-1}\hat{B}_0$

N	λ_{min}	λ_{max}
4	0.9142	2.0000
8	1.0353	5.2823
12	1.0116	8.5789
16	1.0055	11.8729

Table VIII. λ_{min} and λ_{max} of $(\hat{B}^{NE}_{FD,0})^{-1}\hat{B}_0$

Finally we were also interested in finding good preconditioners for the operators $\tilde{A}$ and $\tilde{A}_0$. We first considered the one-dimensional case. In $\tilde{A}$ we approximate the second derivative $\tilde{p}''$, $\tilde{p} \in \mathbf{P}_N^0$ by central differences. After eliminating the homogeneous boundary conditions we obtain a FD operator $L_{FD} \in \mathbf{R}^{N-1,N-1}$. Furthermore we now write L_{sp} instead of $(D^2)^0$. A straightforward preconditioner for $\tilde{A}$ is given by:

$$\tilde{A}_{FD} \;=\; L_{FD} \;-\; \frac{1}{N} t_N'' t_N^t.$$

Unfortunately, t_N^0 is no more an eigenfunction of $\tilde{A}_{FD}$ with eigenvalue zero. We obtain

$$\tilde{A}_{FD} t_N^0 \;=\; L_{FD} t_N^0 \;-\; t_N'' \;\neq\; 0,$$

since the FD operator does not produce the exact derivative of t_N^0. Furthermore from the representation of $\tilde{A}_{FD} t_N^0$ it can be expected that $\tilde{A}_{FD}$ has one positive eigenvalue. The reason is that the eigenvalues of the spectral operator are (absolutely) greater than the eigenvalues of the FD operator. This expectation is confirmed by numerical computations. The eigenvalues of $\tilde{A}_{FD}$ are real, one eigenvalue is positive and the others are all negative. In table IX we give the positive eigenvalue λ_{pos} and the minimal and maximal eigenvalues λ_{min}, λ_{max}.

N	λ_{pos}	λ_{min}	λ_{max}
4	$8.7164 \cdot 10^0$	2.3733	$9.6569 \cdot 10^0$
8	$7.1642 \cdot 10^1$	2.4629	$1.2672 \cdot 10^2$
16	$3.9123 \cdot 10^2$	2.4662	$1.9010 \cdot 10^3$
32	$1.7985 \cdot 10^3$	2.4671	$2.9939 \cdot 10^4$

Table IX. Eigenvalues of $\tilde{A}_{FD}$

λ_{min} approximates $\frac{\pi^2}{4}$ from below. λ_{max} behaves as $O(N^3)$ for increasing N. Now we consoder the preconditioning properties of $\tilde{A}_{FD}$. First we prove the following interesting result.

Lemma 5.1. *For $N \in \mathbf{N}$, $N > 2$ the following equality holds:*

$$\tilde{A}_{FD}^{-1}\tilde{A} = L_{FD}^{-1}\tilde{A}.$$

Proof. For proof we first introduce the following abbreviations: $u := t_N''$, $v := -\frac{1}{N}t_N$. Hence $\tilde{A}$ can also be written as

$$\tilde{A} = L_{sp} + uv^t.$$

Now we first prove that

$$v^t L_{FD}^{-1} u \neq -1. \tag{5.6}$$

By introducing the spectral operator L_{sp} we obtain:

$$\begin{aligned} v^t L_{FD}^{-1} u &= (v^t L_{FD}^{-1} L_{sp}) L_{sp}^{-1} u \\ &= \lambda_N (v^t L_{sp}^{-1} u) \\ &= -\lambda_N \neq -1 \text{ for } N \geq 3. \end{aligned}$$

Here we used that $v^t L_{FD}^{-1} L_{sp} = \lambda_N v^t$, where $\lambda_N = N(N-1)\sin^2 \frac{\pi}{2N}$ (see Haldenwang et al. [14]). For $N \geq 3$ we have $\lambda_N \neq 1$ and for increasing N the value λ_N approximates $\frac{\pi^2}{4}$ from below. The equality

$$v^t L_{sp}^{-1} u = -1$$

is due to the fact that $L_{sp}^{-1} u = t_N^0$ and $v^t t_N^0 = -1$. Using (5.6) it can easily be shown that $\tilde{A}_{FD}$ is regular and the inverse can explicitly be written as (see, e.g., Stoer [22]):

$$(L_{FD} + uv^t)^{-1} = L_{FD}^{-1} - \frac{1}{1 + v^t L_{FD}^{-1} u} L_{FD}^{-1} u v^t L_{FD}^{-1} .$$

Hence we derive

$$\tilde{A}_{FD}^{-1}\tilde{A} \;=\; L_{FD}^{-1}\tilde{A} \;-\; \frac{1}{1+v^t L_{FD}^{-1}u} L_{FD}^{-1}uv^t L_{FD}^{-1}\tilde{A}.$$

We further observe that

$$\begin{aligned} v^t L_{FD}^{-1}\tilde{A} \;&=\; \lambda_N v^t \;+\; (v^t L_{FD}^{-1}u)v^t \\ &=\; \lambda_N v^t \;-\; \lambda_N v^t \;=\; 0, \end{aligned}$$

which concludes the proof. $\oplus$

Remark 5.1 For $N = 2$ one immediately obtains $\tilde{A}_{FD} \;=\; \tilde{A} \;=\; 0$, which corresponds to the fact that the pressure is only determined up to a constant.

By using lemma 5.1 we are now able to derive the following result concerning the eigenspectrum of the preconditioned operator.

Theorem 5.1. *The non-zero eigenvalues of $\tilde{A}_{FD}^{-1}\tilde{A}$ are equal to the $N-2$ eigenvalues of $L_{FD}^{-1}L_{sp}$, explicitly given by:*

$$\lambda_n \;=\; \frac{n(n-1)\sin^2\frac{\pi}{2N}\cos\frac{\pi}{2N}}{\sin n\frac{\pi}{2N}\sin(n-1)\frac{\pi}{2N}}, \quad 2 \le n \le N-1.$$

The maximal eigenvalue λ_N of $L_{FD}^{-1}L_{sp}$ now corresponds to the zero eigenvalue. The eigenfunction is equal to t_N^0.

Proof. In order to prove our theorem we write all operators mapping from $\mathbf{R}^{N+1}$ in $\mathbf{R}^{N+1}$. Due to the Dirichlet boundary conditions the first resp. last row is then equal to e_1^t resp. e_{N+1}^t where e_1 resp. e_{N+1} denote the 1. resp. (N+1). unit vector. The corresponding FD resp. spectral operators are written as $\overline{L}_{FD} \in \mathbf{R}^{N+1,N+1}$ resp. $\overline{L}_{sp} \in \mathbf{R}^{N+1,N+1}$. We further introduce $\overline{u},\ \overline{v},\ \overline{t}_N^0 \in \mathbf{R}^{N+1}$, where

$$\overline{u}_j \;=\; T_N''(x_j),\ \overline{v}_j \;=\; -\frac{1}{Nc_j}T_N(x_j),\ (\overline{t}_N^0)_j \;=\; T_N(x_j)-1, \quad j=0,\ldots,N.$$

Let $T \in \mathbf{R}^{N+1,N+1}$ denote the Chebyshev transformation matrix given by $T = (\cos \frac{ij\pi}{N})_{i,j=0,\ldots,N}$. T^{-1} denotes its inverse. From the proof in [14] it is immediately seen that

$$T^{-1}\overline{L}_{FD}^{-1}\overline{L}_{sp}T \in \mathbf{R}^{N+1,N+1}$$

is an upper triangular matrix containing its eigenvalues on the diagonal. Now we consider

$$T^{-1}\overline{L}_{FD}^{-1}\overline{u}\overline{v}^t T.$$

It is readily seen that

$$\begin{aligned} T^{-1}\overline{L}_{FD}^{-1}\overline{u} &= T^{-1}\overline{L}_{FD}^{-1}\overline{L}_{sp}\overline{L}_{sp}^{-1}\overline{u} \\ &= T^{-1}\overline{L}_{FD}^{-1}\overline{L}_{sp}\overline{t}_N^0 \\ &= \lambda_N T^{-1}\overline{t}_N^0 \\ &= \lambda_N(e_{N+1} - e_1). \end{aligned}$$

Hence we obtain

$$T^{-1}\overline{L}_{FD}^{-1}\overline{u}\overline{v}^t T = -\lambda_N(e_{N+1} - e_1)e_{N+1}^t,$$

which is identical to the zero matrix except for the last column which is equal to $-\lambda(e_{N+1} - e_1)$. In particular, it is an upper triangular matrix. $N-2$ eigenvalues are equal to zero and one is equal to $-\lambda_N$. This concludes the proof of our theorem. $\oplus$

Hence the eigenvalues of the preconditioned operator are all lying in between 1 and $\frac{\pi^2}{4}$.

Unfortunately, this result is no more true in the two-dimensional case. In the one-dimensional case the positive eigenvalue of the FD operator is set to zero since the highest frequency falls into the zero space of the spectral operator. In the two-dimensional case we have high frequencies given by

$$(T_N(x)-1)(T_m(y)-1), \quad m = 0, 2, \ldots, N-2$$

resp.

$$(T_n(x)-1)(T_N(y)-1), \quad n = 0, 2, \ldots, N-2,$$

which vanish by applying $D_x D_x^0$ resp. $D_y D_y^0$ but not by applying $D_y D_y^0$ resp. $D_x D_x^0$. Obviously, they are not eigenfunctions to the eigenvalue zero. The FD operator in two dimensions can easily be written as:

$$\tilde{B}_{FD} \;=\; L_{FD} \otimes I \;+\; I \otimes L_{FD}.$$

Since $\tilde{B}_{FD}$ does not cancel the above functions (as the spectral operator does) we may expect a bad performance of $\tilde{B}_{FD}^{-1}$ for preconditioning. This is fully confirmed by the numerical results presented in table X. The minimal eigenvalue λ_{min} tends to zero for increasing N. The preconditioning properties of $\tilde{B}_{FD}$ where L_{FD} is replaced by $\tilde{A}_{FD}$ are even worse since $\tilde{A}_{FD}$ has one positive eigenvalue. Therefore the preconditioned operator in the two-dimensional case becomes indefinite. At the moment it is an open question how to construct an efficient preconditioner for $\tilde{B}$.

N	λ_{min}	λ_{max}
4	$1.4640 \cdot 10^{-1}$	1.2426
8	$4.4137 \cdot 10^{-2}$	1.7302
12	$2.0460 \cdot 10^{-2}$	1.9402
16	$1.1677 \cdot 10^{-2}$	2.0571

Table X. λ_{min} and λ_{max} of $\tilde{B}_{FD}^{-1}\tilde{B}$

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