Problem 1:
A matrix $A$ is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Solution:
Suppose that $\lambda$ is an eigenvalue of $A$. Then there is an eigenvector $x$, such that $Ax = \lambda x$. We have
\[
\lambda x = A x \\
= A^2 x \quad \text{as } A \text{ is idempotent} \\
= A(A x) \\
= A(\lambda x) \quad \text{as } x \text{ is eigenvector of } A \\
= \lambda(A x) \\
= \lambda(\lambda x) \quad \text{as } x \text{ is eigenvector of } A \\
= \lambda^2 x
\]
From this we get:
\[
0 = \lambda^2 x - \lambda x \\
= (\lambda^2 - \lambda) x
\]
Since $x$ is an eigenvector, it is nonzero, we get the conclusion that $\lambda^2 - \lambda = 0$, and the solutions to this quadratic polynomial equation for $\lambda$ are $\lambda = 0$ and $\lambda = 1$. The matrix
\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\]
is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, $\lambda = 0$ and $\lambda = 1$, so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix.

Problem 2:
The Matrix $A_t$ is given:
\[
\begin{bmatrix}
-1 & t \\
-3 & 1
\end{bmatrix}
\]

a) For which $t \in \mathbb{R}$ there do not exist any real eigenvalues for $A_t$?

b) For which $t \in \mathbb{R}$ there does exist exactly one real eigenvalue for $A_t$? Give the eigenvalue!

c) For which $t \in \mathbb{R}$ there do exist two different real eigenvalues for $A_t$? Give the eigenvalues in dependence on $t$!

d) Give the second real eigenvalue for c) if $-1$ is the first real eigenvalue!
Solution:

\[
(-\frac{1}{3} - \lambda)(1 - \lambda) + 2\, t = 0 \\
\iff \quad -\frac{1}{3} - \frac{2}{3}\lambda + \lambda^2 + 2\, t = 0 \\
\iff \quad (\lambda - \frac{1}{3})^2 - \frac{4}{9} + 2t = 0
\]

a) no real eigenvalues:

\[-\frac{4}{9} + 2\, t > 0 \quad \implies \quad t > \frac{2}{9}\]

The Matrix \( A_t \) has no real eigenvalues for all \( t > \frac{2}{9} \)

b) exactly one real eigenvalue:

\[-\frac{4}{9} + 2\, t = 0 \quad \implies \quad t = \frac{2}{9}\]

The Matrix \( A_t \) has one real eigenvalue for \( t = \frac{2}{9} \) which is \( \lambda = \frac{1}{3} \)

c) two real eigenvalues:

\[-\frac{4}{9} + 2\, t < 0 \quad \implies \quad t < \frac{2}{9}\]

The Matrix \( A_t \) has two real eigenvalues for \( t < \frac{2}{9} \), which are:

\[\lambda = \frac{1}{3} + \sqrt{\frac{4}{9} - 2\, t}\] and \[\lambda = \frac{1}{3} - \sqrt{\frac{4}{9} - 2\, t}\]

d) First we have to find the \( t \) for which one real eigenvalue is \(-1\):

\[-1 = \frac{1}{3} - \sqrt{\frac{4}{9} - 2\, t}\] \[\iff \quad \frac{4}{3} = -\sqrt{\frac{4}{9} - 2\, t}\] \[\iff \quad \frac{16}{9} = \frac{4}{9} - 2\, t\] \[\iff \quad \frac{12}{9} = -2\, t\] \[\iff \quad -\frac{2}{3} = t\]

Then we can calculate the second real eigenvalue with the given \( t \):

\[\lambda = \frac{1}{3} + \sqrt{\frac{4}{9} - 2\, t (-\frac{2}{3})}\] \[\iff \quad \lambda = \frac{1}{3} + \sqrt{\frac{4}{9} + \frac{4}{3}}\] \[\iff \quad \lambda = \frac{1}{3} + \sqrt{\frac{16}{9}}\] \[\iff \quad \lambda = \frac{5}{3}\]

So finally the second real eigenvalue is \( \frac{5}{3} \).
Problem 3: Find the eigenvalues and eigenvectors of the Matrix $A$:

$$
\begin{bmatrix}
1 & -1 \\
3 & 5 \\
\end{bmatrix}
$$

With the characteristic equation $|det(A - I\lambda)| = 0$ you get:

$$(1 - \lambda)(5 - \lambda) + 3 = 0$$
$$\Leftrightarrow \quad \lambda^2 - 6\lambda + 5 + 3 = 0$$
$$\Leftrightarrow \quad (\lambda - 2)(\lambda - 4) = 0$$
$$\Leftrightarrow \quad \lambda = 2 \lor \lambda = 4$$

With the eigenvalues now the eigenvectors can be computed. Therefore you insert the eigenvalue into the Matrix and go backwards from the Matrix to the system of equations it represents:

So for $\lambda = 2$ you will get:

$$-v_1 - v_2 = 0$$
$$3v_1 + 3v_2 = 0$$

One solution of this system of equations is the vector $\vec{v} = \begin{pmatrix} k \\ -k \end{pmatrix}$. The eigenvector is every multiple of this vector, so $\vec{v} = \begin{pmatrix} k \\ -k \end{pmatrix}$

For the second eigenvalue $\lambda = 4$ we can find the corresponding eigenvector in the same way:

$$-3v_1 - v_2 = 0$$
$$3v_1 + v_2 = 0$$

One solution of this system of equations is the vector $\vec{u} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$. The eigenvector than is every multiple of this vector, so $\vec{u} = \begin{pmatrix} l \\ -3l \end{pmatrix}$